Characterization of g-Riesz Basis and their Dual
in Hilbert $\mathcal{A}$-module

Sayyed Mehrab Ramezani
Department of Mathematics
Shahid Bahonar University
Kerman 7616914111, Iran

Akbar Nazari
Department of Mathematics
Shahid Bahonar University
Kerman 7616914111, Iran

Abstract
We investigate dual of g-frames and g-Riesz bases in Hilbert $\mathcal{A}$-module. The purpose of this paper is to investigate some basic properties about g-Riesz basis and their duals. We obtain a necessary and sufficient condition for a dual of a g-Riesz basis to be again a g-Riesz basis.

Mathematics Subject Classification: 42C15, 46L08

Keywords: g-frame; g-Riesz basis; dual g-sequence; Hilbert $\mathcal{A}$-module

1 Introduction
Frame for Hilbert space were formally defined by Duffin and Schaeffer [5] in 1952 for studying some problem in non harmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossman and Meyer [4].
In [11] Wenchang Sun introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

The concept of frames especially the g-frame was introduced in Hilbert $\mathcal{A}$-module, and some of their properties were investigated in [6] and [8]. A g-Riesz basis always has a canonical dual which is necessary a g-Riesz basis. Khosravi in [9] introduced the notion of modular g-Riesz basis in Hilbert $\mathcal{A}$-module and he shares many properties with Riesz basis and g-Riesz basis in Hilbert spaces. Also in [1] Alijani gave a characterization of dual g-frame in terms of right invers of the synthesis operators. However we characterize those g-Riesz basis that have unique duals. The characterization is given in terms of the rang spaces of the analysis operators. At the end this paper we obtain a necessary and sufficient condition for a dual of a g-Riesz basis to be again a g-Riesz basis.

2 Preliminary Notes

In the following we briefly recall some definitions and basic properties of Hilbert $\mathcal{A}$-module and g-frames in Hilbert $\mathcal{A}$-module. We first give some notations which we need later. Throughout this paper $J$ is finite or countably index set. $\mathcal{A}$ is unital $C^*$-algebra with identity $1_\mathcal{A}$, $\mathcal{U}$ and $\mathcal{V}$ are finitely or countably generated Hilbert $\mathcal{A}$-module and $\{\mathcal{V}_j\}_{j \in J}$ be a sequence of closed Hilbert submodules of $\mathcal{V}$. For each $j \in J$, $\text{End}_\mathcal{A}(\mathcal{U}, \mathcal{V}_j)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ to $\mathcal{V}_j$.

We also denote

$$\bigoplus_{j \in J} V_j = \{g = \{g_j\} : g_j \in \mathcal{V}_j \text{ and } \sum_{j \in J} \langle g_j, g_j \rangle \text{ is norm convergent in } \mathcal{A}\}$$

For any $f = \{f_j\}$ and $g = \{g_j\}$, if the $\mathcal{A}$-valued inner product is define by $\langle f, g \rangle = \sum_{j \in J} \langle f_j, g_j \rangle$ and the norm is defined by $\|f\| = \|\langle f, f \rangle\|^{\frac{1}{2}}$, then $\bigoplus_{j \in J} \mathcal{V}_j$ is a Hilbert $\mathcal{A}$-module (see [10]).

Definition 2.1. Let $\mathcal{H}$ be a left $\mathcal{A}$-module such that the linear structure of $\mathcal{A}$ and $\mathcal{H}$ are compatible, $\mathcal{H}$ is called a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that

(i) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \ (f, g, h \in \mathcal{H}, \alpha, \beta \in \mathbb{C})$;

(ii) $\langle af, g \rangle = a \langle f, g \rangle \ (f, g \in \mathcal{H}, a \in \mathcal{A})$;

(iii) $\langle f, g \rangle = \langle g, f \rangle^* \ (f, g \in \mathcal{H})$;

(iv) $\langle f, f \rangle \geq 0$; if $\langle f, f \rangle = 0$ then $f = 0$. 

For every \( f \in \mathcal{H} \), we define \( \|f\| = \|\langle f, f \rangle\|^\frac{1}{2} \) and \( |f| = \langle f, f \rangle^\frac{1}{2} \). If \( \mathcal{H} \) is complete with respect to the norm, it is called a Hilbert \( A \)-module or a Hilbert \( C^* \)-module over \( A \).

**Definition 2.2.** Let \( \mathcal{H} \) be a Hilbert \( A \)-module. A collection \( \{f_j\}_{j \in J} \subset \mathcal{H} \) is called a frame for \( \mathcal{H} \), if there exist two positive constant \( c, d \), such that:

\[
c\langle f, f \rangle \leq \sum_{j \in J} \langle f, f_j \rangle f_j, f \leq d\langle f, f \rangle \quad \forall f \in \mathcal{H} \tag{1}
\]

We call \( c \) and \( d \) the lower and upper frame bounds, respectively. If only the right-hand inequality of (1) is satisfied, we call \( \{f_j\}_{j \in J} \) the Bessel sequence for \( \mathcal{H} \) with Bessel bound \( d \). If \( c = d = \lambda \), we call \( \{f_j\}_{j \in J} \) the \( \lambda \)-tight frame. Moreover, if \( \lambda = 1 \) we call \( \{f_j\}_{j \in J} \) the Parseval frame. The frame is standard if for every \( f \in \mathcal{H} \), the sum in (1) converges in norm.

**Definition 2.3.** We call a sequence \( \{\Lambda_j \in \text{End}_A^*(U, \mathcal{V}_j) : j \in J\} \) a \( g \)-frame in Hilbert \( A \)-module \( U \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \), if there exist two positive constant \( C, D \) such that:

\[
C\langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D\langle f, f \rangle \quad \forall f \in U \tag{2}
\]

We call \( C \) and \( D \) the lower and upper \( g \)-frame bounds, respectively. If only the right-hand inequality of (2) is satisfied, we call \( \{\Lambda_j\}_{j \in J} \) the \( g \)-Bessel sequence for \( U \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \) with \( g \)-Bessel bound \( D \). If \( C = D = \lambda \), we call \( \{\Lambda_j\}_{j \in J} \) the \( \lambda \)-tight \( g \)-frame. Moreover, if \( \lambda = 1 \) we call \( \{\Lambda_j\}_{j \in J} \) the Parseval \( g \)-frame. The \( g \)-frame is standard if for every \( f \in U \), the sum in (2) converges in norm.

**Definition 2.4.** Suppose that \( \{\Lambda_j \in \text{End}_A^*(U, \mathcal{V}_j) : j \in J\} \) is a \( g \)-sequence for \( U \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \), the pre-\( g \)-frame or analysis operator

\[
T_\Lambda : U \rightarrow \bigoplus_{j \in J} \mathcal{V}_j
\]

is defined by \( T_\Lambda f = \{\Lambda_j f\}_{j \in J} \).

If \( \{\Lambda_j \in \text{End}_A^*(U, \mathcal{V}_j) : j \in J\} \) be a \( g \)-Bessel sequence with bound \( B \) then analysis operator \( T_\Lambda \) is adjointable and \( \|T_\Lambda\| \leq \sqrt{B} \).

Moreover \( T_\Lambda^* \{g_j\}_{j \in J} = \sum_{j \in J} \Lambda_j^* g_j \) for all \( \{g_j\}_{j \in J} \subset \bigoplus_{j \in J} \mathcal{V}_j \). The adjointable map \( T_\Lambda^* \) is called synthesis operator of \( \{\Lambda_j\}_{j \in J} \).

In addition if \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-frame for \( U \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \), the \( g \)-frame operator \( S_\Lambda \), then \( S_\Lambda = T_\Lambda^* T_\Lambda \).

**Remark 2.5.** For any \( j \in J \), if we let \( \mathcal{V}_j = \mathcal{A} \) and \( \Lambda_j f = \langle f, f_j \rangle \) for any \( f \in U \), in this case the \( g \)-frame is just a frame for \( U \).

**Definition 2.6.** A \( g \)-frame \( \{\Lambda_j \in \text{End}_A^*(U, \mathcal{V}_j) : j \in J\} \) in Hilbert \( A \)-module \( U \) with respect to \( \{\mathcal{V}_j\} \) is said to be a \( g \)-Riesz basis if it satisfies
(i) \( \Lambda_j \neq 0 \);
(ii) If an \( A \)-linear combination \( \sum_{j \in K} \Lambda_j^* g_j \) is equal to zero, then every summand \( \Lambda_j^* g_j \) equal to zero, where \( g_j \in V_j \) and \( K \subseteq J \).

Definition 2.7. suppose that \( \{\Lambda_j\}_{j \in J} \) be a (standard) \( g \)-frame and \( \{\Gamma_j\}_{j \in J} \) a \( g \)-sequence for \( U \) with respect to \( \{V_j\} \). Then \( \{\Gamma_j\}_{j \in J} \) is said to be a (standard) dual \( g \)-sequence of \( \{\Lambda_j\}_{j \in J} \) if

\[
f = \sum_{j \in J} \Lambda_j^* \Gamma_j f
\]

holds for all \( f \in U \), where the sum in (3) converges in norm.

The pair \( \{\Lambda_j\}_{j \in J} \) and \( \{\Gamma_j\}_{j \in J} \) are called a dual \( g \)-frame pair when \( \{\Gamma_j\}_{j \in J} \) is also a \( g \)-frame.

Lemma 2.8. Suppose that \( \{f_j\}_{j \in J} \) and \( \{g_j\}_{j \in J} \) are two sequence of \( \bigoplus_{j \in J} V_j \) such that both \( \sum_{j \in J} \langle f_j, f_j \rangle \) and \( \sum_{j \in J} \langle g_j, g_j \rangle \) converge in \( A \), then

\[
\sum_{j \in J} \langle f_j + g_j, f_j + g_j \rangle \leq 2(\sum_{j \in J} \langle f_j, f_j \rangle + \sum_{j \in J} \langle g_j, g_j \rangle)
\]

3 Main Results

In this section we study the relations between \( g \)-Riesz basis and their dual \( g \)-frame in Hilbert \( A \)-module. We begin this section with a characterization of \( g \)-Riesz basis. Let \( P_n \) be the projection on \( \bigoplus_{j \in J} V_j \) that maps each element to its component, i.e.

\[
P_n(\{g_j\}_{j \in J}) = \{f_j\}_{j \in J}, \text{ where } f_j = \begin{cases} g_n & j = n \\ 0 & j \neq n \end{cases} \text{ for each } \{g_j\}_{j \in J} \in \bigoplus_{j \in J} V_j.
\]

Theorem 3.1. Suppose that \( \{\Lambda_j\}_{j \in J} \) be a \( g \)-frame of a finitely or countably generated Hilbert \( A \)-module \( U \) with respect to \( \{V_j\} \). Let \( T_\Lambda : U \rightarrow \bigoplus_{j \in J} V_j \) is the analysis operator of \( \{\Lambda_j\}_{j \in J} \). Then \( \{\Lambda_j\}_{j \in J} \) is \( g \)-Riesz basis if and only if \( \Lambda_n \neq 0 \) and \( P_n(\text{Rang} T_\Lambda) \subseteq \text{Rang} T_\Lambda \).

Proof. Suppose first \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-Riesz basis. Let \( g = \{g_j\}_{j \in J} \in (\text{Rang} T_\Lambda)^\perp \) then for all \( f \in U \), we have

\[
0 = \langle \{g_j\}_{j \in J}, T_\Lambda f \rangle = \langle T_\Lambda^* \{g_j\}_{j \in J}, f \rangle = \langle \sum_{j \in J} \Lambda_j^* g_j, f \rangle.
\]

So \( \sum_{j \in J} \Lambda_j^* g_j = 0 \) then \( \Lambda_j^* g_j = 0 \) hold for all \( j \in J \).
Now let \( f \in \mathcal{U} \) we have
\[
\langle \{g_j\}_{j \in J}, P_n(T_{\Lambda} f) \rangle = \langle \{g_j\}_{j \in J}, P_n(\{\Lambda_j f\}_{j \in J}) \rangle = \langle g_n, \Lambda_n f \rangle = \langle \Lambda_n^* g_n, f \rangle = 0
\]

So \((\text{Rang}T_{\Lambda})^\perp \subseteq P_n(\text{Rang}T_{\Lambda})^\perp\). Consequently \(P_n(\text{Rang}T_{\Lambda}) \subseteq \text{Rang}T_{\Lambda}\).

Suppose now that \(P_n(\text{Rang}T_{\Lambda}) \subseteq \text{Rang}T_{\Lambda}\) for each \(n\). Suppose that \(\sum_{j \in J} \Lambda_j^* g_j = 0\) and fix \(n \in J\), then \(P_n T_{\Lambda} f \in \text{Rang}T_{\Lambda}\), so there exist \(f_n \in \mathcal{U}\) such that \(P_n T_{\Lambda} f = T_{\Lambda} f_n\).

Now for any \(f \in \mathcal{U}\) we have
\[
\langle f, \Lambda_n^* g_n \rangle = \langle \Lambda_n f, g_n \rangle = \sum_{j \in J} \langle \Lambda_j f_n, g_j \rangle = \langle f_n, \sum_{j \in J} \Lambda_j^* g_j \rangle = 0
\]
then \(\Lambda_j^* g_j = 0\) hold for all \(n \in J\). \(\square\)

Not that in Hilbert space if \(\{\Lambda_j\}_{j \in J}\) is g-Riesz basis then \(\sum_{j \in J} \Lambda_j^* g_j\) converges for a sequence \(\{g_j\}\) if and only if \(\{g_j\} \in \bigoplus_{j \in J} \mathcal{V}_j\), but this is not the case in setting of Hilbert \(\mathcal{A}\)-module. We have the following example.

**Example 3.2.** Let \(\ell^\infty\) be the set of all bounded complex-valued sequences. For any \(f = \{f_i\}_{i \in \mathbb{N}}\) and \(g = \{g_i\}_{i \in \mathbb{N}}\) in \(\ell^\infty\) we define \(fg = \{f_i g_i\}_{i \in \mathbb{N}}\) and \(\|f\| = \max \{f_i\}_{i \in \mathbb{N}}\) and \(f^* = \{\overline{f_i}\}_{i \in \mathbb{N}}\) then \(\mathcal{A} = \{\ell^\infty, \|\cdot\|\}\) is a C*-algebra.

Let \(\mathcal{H} = C_0\) be the set of all sequence converging to zero. For any \(f, g \in \mathcal{H}\) we define \(\langle f, g \rangle = fg^* = (f, \overline{g_i})_{i \in \mathbb{N}}\) then \(\mathcal{H}\) is Hilbert \(\mathcal{A}\)-module.

Now let \(j \in J = \mathbb{N}\) and set \(\Lambda_j : \mathcal{H} \rightarrow \mathcal{H}\) by \(\Lambda_j(f_i)_{i \in \mathbb{N}} = (\delta_{ij} f_i)_{i \in \mathbb{N}}\) then \(\Lambda_j = \Lambda_j^*\) and \(\Lambda_j \neq 0\) for all \(j \in J\).

Obviously \(\{\Lambda_j\}_{j \in J}\) is a g-frame for \(\mathcal{H}\) with respect to \(\mathcal{H}\).

Also if \(g(j) = (g_i(j))_{i \in \mathbb{N}} \in \mathcal{H}\) and \(\sum_{j \in J} \Lambda_j^* g(j) = 0\) then
\[
0 = \sum_{j \in J} \Lambda_j^* g(j) = \sum_{j \in J} \Lambda_j g(j) = \sum_{j \in J} (\delta_{ij} g_i(j))_{i \in \mathbb{N}} = (\sum_{j \in J} \delta_{ij} g_i(j))_{i \in \mathbb{N}}
\]

In particular \(g_i(i) = 0\) for all \(i \in \mathbb{N}\) thus \(\Lambda_j^* g(j) = \Lambda_j g(j) = (\delta_{ij} g_i(j))_{i \in \mathbb{N}} = 0\), therefore \(\{\Lambda_j\}_{j \in J}\) is g-Riesz basis.

Now for each \(j \in J\) we let \(g(j) = \sqrt{\mathcal{J}}(\beta_{ij} f_i)_{i \in \mathbb{N}}\) where
\[
\beta_{ij} = \begin{cases} 
1 & i \neq j \\
0 & i = j 
\end{cases} \quad \text{and} \quad (f_i)_{i \in \mathbb{N}} \neq 0 \text{ is an arbitrary element of } \mathcal{H} \text{ then}
\]
\[\Lambda_j^* g(j) = \Lambda_j g(j) = (\delta_{ij} \beta_{ij} f_i)_{i \in \mathbb{N}} = 0 \text{ so } \sum_{j \in J} \Lambda_j^* g(j) \text{ is convergent but } \{g(j)\}_{j \in J} \text{ is not in } \ell^2(\mathcal{H}) \text{ because:}
\]
\[\|\sum_{j \in J} \langle g(j), g(j) \rangle\| = \|\sum_{j \in J} j (\beta_{ij} f_i \bar{f}_j)_{i \in \mathbb{N}}\| = \|\sum_{j \in J} j |\beta_{ij}| |f_i|^2\| = \max_{i \in \mathbb{N}} |\sum_{j \neq i} j |f_j|^2| = \infty
\]
So \(\sum_{j \in J} \langle g(j), g(j) \rangle\) is not convergent in \(\mathcal{A}\) and therefore \(\{g(j)\}_{j \in J} \notin \ell^2(\mathcal{H})\).

**Corollary 3.3.** Let \(\{\Lambda_j\}_{j \in J}\) is a g-frame of \(\mathcal{U}\) with respect to \(\{\mathcal{V}_j\}_{j \in J}\), then \(\{\Lambda_j\}_{j \in J}\) is a g-Riesz basis if and only if
(i) \(\Lambda_j \neq 0\);
(ii) if \(\sum_{j \in J} \Lambda_j^* g_j = 0\) for some sequence \(\{g_j\} \in \bigoplus_{j \in J} \mathcal{V}_j\), then \(\Lambda_j^* g_j = 0\) for each \(j \in J\).

**Proof.** See the proof of theorem 3.1.

The following example shows that the dual of g-Riesz basis of Hilbert \(\mathcal{A}\)-module are quite different from the Hilbert space cases, and shows that even the dual g-sequence of g-Riesz basis in Hilbert \(\mathcal{A}\)-module is g-Bessel sequence it still has the chance not to be a g-Riesz basis.

**Example 3.4.** Suppose that \(\mathcal{H}\) and \(\mathcal{A}\) are the same as in example 3.2.
Now let \(\Gamma_j = \begin{cases} 
\Lambda_1 + \Lambda_2 & j = 1, 2 \\
\Lambda_j & j \geq 3 
\end{cases}\) Then \(\{\Gamma_j\}_{j \in J}\) is a g-frame of \(\mathcal{H}\) with respect to \(\mathcal{H}\) with bounds 1, 2.
Obviously \(\{\Lambda_j\}\) is a dual of itself, and
\[
\sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j (f_i)_{i \in \mathbb{N}} = \Lambda_1 (\Lambda_1 + \Lambda_2) (f_i)_{i \in \mathbb{N}} + \Lambda_2 (\Lambda_1 + \Lambda_2) (f_i)_{i \in \mathbb{N}} + \sum_{j=3}^{\infty} \Lambda_j^* \Lambda_j (f_i)_{i \in \mathbb{N}}
\]
\[= \sum_{j=1}^{\infty} \Lambda_j \Lambda_j (f_i)_{i \in \mathbb{N}}
\]
So \(\{\Gamma_j\}_{j \in J}\) is a dual g-sequence of \(\{\Lambda_j\}_{j \in J}\) but is not g-Riesz basis.
Set \(g(1) = (f_i)_{i \in \mathbb{N}}, f_1 \neq 0\) and \(g(2) = -g(1)\) and \(g(j) = 0\) for all \(j \geq 3\).

We have the following equivalent definition for g-frame in Hilbert \(\mathcal{A}\)-module.
Proposition 3.5. Let \( U \) be a finitely or countably generated Hilbert \( A \)-module over a unital \( C^* \)-algebra \( A \) and \( \{\Lambda_j\}_{j \in J} \) be a \( g \)-sequence. Then \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-frame of \( U \) with respect to \( \{V_j\}_{j \in J} \) if and only if there are positive constants \( C, D \) such that

\[
C\|f\|^2 \leq \left\| \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \right\| \leq D\|f\|^2 \quad \forall f \in U. \quad (4)
\]

Proof. \( \Rightarrow \) Obvious.

\( \Leftarrow \) Suppose that \( \{\Lambda_j\}_{j \in J} \) fulfills (4). For any \( f \in U \), we define

\[
Tf = \sum_{j \in J} \Lambda_j^* \Lambda_j f
\]

then

\[
\|Tf\|^4 = \|\langle Tf, Tf \rangle\|^2 = \|\langle Tf, \sum_{j \in J} \Lambda_j^* \Lambda_j f \rangle\|^2 = \left\| \sum_{j \in J} \langle \Lambda_j Tf, \Lambda_j f \rangle \right\|^2 \leq \left\| \sum_{j \in J} \langle \Lambda_j Tf, \Lambda_j f \rangle \right\| \left\| \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \right\| \leq D^2\|Tf\|^2\|f\|^2
\]

So \( \|Tf\|^2 \leq D^2\|f\|^2 \).

It is easy to check that \( \langle Tf, g \rangle = \langle f, Tg \rangle \) for all \( f, g \in U \), so \( T \) is bounded and \( T = T^* \). From \( \langle Tf, f \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \) \( \geq 0 \) for all \( f \in U \), it follow that \( T \geq 0 \).

Now \( \langle T^{1/2} f, T^{1/2} f \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \) and (4) imply that \( \sqrt{C}\|f\| \leq \|T^{1/2} f\| \leq \sqrt{D}\|f\| \) for all \( f \in U \).

By corollary 2.2 from [2] there are positive constants \( C', D' \) such that

\[
C' \langle f, f \rangle \leq \langle T^{1/2} f, T^{1/2} f \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D'\langle f, f \rangle \quad (5)
\]

This prove that \( \{\Lambda_j\}_{j \in J} \) is a \( g \)-frame for \( U \) with respect to \( \{V_j\}_{j \in J} \). \( \square \)

Proposition 3.6. Suppose that \( U \) is a finitely or countably generated Hilbert \( A \)-module over a unital \( C^* \)-algebra \( A \). Let \( \{\Lambda_j\}_{j \in J} \) and \( \{\Gamma_j\}_{j \in J} \) be two \( g \)-Bessel sequence of \( U \) with respect to \( \{V_j\}_{j \in J} \).

If \( f = \sum_{j \in J} \Lambda_j^* \Gamma_j f \) hold for any \( f \in U \), then both \( \{\Lambda_j\}_{j \in J} \) and \( \{\Gamma_j\}_{j \in J} \) are \( g \)-frame of \( U \) with respect to \( \{V_j\}_{j \in J} \) and \( f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \) hold for all \( f \in U \).

Proof. Let us denote the \( g \)-Bessel bound of \( \{\Gamma_j\}_{j \in J} \) by \( D \).
For all \( f \in \mathcal{U} \) we have:

\[
\|f\|^4 = \|(f, f)\|^2 \\
= \|(\Sigma_{j \in J} \Lambda_j^* \Gamma_j f, f)\|^2 \\
= \|(\Sigma_{j \in J} (\Gamma_j f, \Lambda_j f))\|^2 \\
\leq \|(\Sigma_{j \in J} (\Gamma_j f, \Gamma_j f))\| \|\Sigma_{j \in J} (\Lambda_j f, \Lambda_j f)\| \\
\leq D \Gamma \|f\|^2 \|\Sigma_{j \in J} (\Lambda_j f, \Lambda_j f)\|
\]

It follow that \( \frac{1}{D \Gamma} \|f\|^2 \leq \|\Sigma_{j \in J} (\Lambda_j f, \Lambda_j f)\| \).

This implies that \( \{\Lambda_j\}_{j \in J} \) is g-frame.
Similarly we can show that \( \{\Gamma_j\}_{j \in J} \) is also a g-frame of \( \mathcal{U} \) with respect to \( \{V_j\}_{j \in J} \).

Now \( f = \sum_{j \in J} \Lambda_j^* \Gamma_j f \) and theorem 3.4 in [1] imply that \( T\Lambda^* T\Gamma = I \) so \( T\Gamma^* T\Lambda = I \) on the other hand

\[
\langle \Sigma_{j \in J} \Gamma_j^* \Lambda_j f, g \rangle = \sum_{j \in J} \langle \Lambda_j f, \Gamma_j g \rangle \\
= \langle T\Lambda f, T\Gamma g \rangle \\
= \langle T\Gamma^* T\Lambda f, g \rangle \\
= \langle f, g \rangle
\]

This demonstrates the second assertion. \( \square \)

To prove our main result, we also need the following lemma.

**Lemma 3.7.** Let \( \{\Lambda_j\}_{j \in J} \) be a g-frame of \( \mathcal{U} \) with respect to \( \{V_j\}_{j \in J} \) over unital \( C^* \)-algebra \( \mathcal{A} \). Suppose that \( \{\Gamma_j\}_{j \in J} \) and \( \{\Theta_j\}_{j \in J} \) are dual g-frame of \( \{\Lambda_j\}_{j \in J} \) with property that either \( \text{Rang}(T\Gamma) \subset \text{Rang}(T\Theta) \) or \( \text{Rang}(T\Theta) \subset \text{Rang}(T\Gamma) \), where \( T\Gamma \) are \( T\Theta \) the analysis operator of \( \{\Gamma_j\}_{j \in J} \) and \( \{\Theta_j\}_{j \in J} \) respectively.

then \( \Gamma_j = \Theta_j \) for all \( j \in J \).

**Proof.** Suppose that \( \text{Rang}(T\Theta) \subset \text{Rang}(T\Gamma) \). Then for each \( f \in \mathcal{U} \) there exist \( g_f \in \mathcal{U} \) such that:

\[
T\Theta(f) = T\Gamma(g_f) \tag{6}
\]

Applying \( T\Lambda^* \) on both sides (6), we have

\[
g_f = \sum_{j \in J} \Lambda_j^* \Gamma_j g_f = T\Lambda^* T\Gamma(g_f) = T\Lambda^* T\Theta(f) = \sum_{j \in J} \Lambda_j^* \Theta_j f = f
\]

So we have \( g_f = f \) therefore \( T\Theta(f) = T\Gamma(f) \), for all \( f \in \mathcal{U} \).

Equivalently

\( \{\Gamma_j f\}_{j \in J} = \{\Theta_j f\}_{j \in J} \) Hence \( \Gamma_j f = \Theta_j f \), for all \( j \in J \).

So we have \( \Gamma_j = \Theta_j \), for all \( j \in J \). \( \square \)
We now give a necessary and sufficient condition about the uniqueness of dual g-frame in Hilbert $C^*$-module. We also prove that if a g-frame has unique dual g-frame, then it must be a g-Riesz basis.

**Theorem 3.8.** Suppose that $\{\Lambda_j\}_{j \in J}$ is a g-frame for $U$ with respect to $\{V_j\}_{j \in J}$, then the following statements are equivalent

(i) $\{\Lambda_j\}_{j \in J}$ has a unique dual g-frame;

(ii) $\text{Rang} T_{\Lambda} = \bigoplus_{j \in J} V_j$;

**Proof.**

2 $\Rightarrow$ 1 Let $\{\tilde{\Lambda}_j\}_{j \in J}$ be the canonical dual of $\{\Lambda_j\}_{j \in J}$ with analysis operator $T_{\tilde{\Lambda}_j}$. Then $\tilde{\Lambda}_j = \Lambda_j S_{\Lambda}^{-1}$ where $S_{\Lambda}$ is the g-frame operator of $\{\Lambda_j\}_{j \in J}$.

Let $\{\Gamma_j\}_{j \in J}$ be any dual g-frame of $\{\Lambda_j\}_{j \in J}$ with analysis operator $T_{\Gamma}$, then

$$\text{Rang} T_{\Gamma} \subseteq \bigoplus_{j \in J} V_j = \text{Rang} T_{\Lambda} = \text{Rang} T_{\tilde{\Lambda}_j}$$  (7)

By lemma 3.7 $\Gamma_j = \tilde{\Lambda}_j$ for all $j \in J$.

1 $\Rightarrow$ 2 Assume on the contrary that $\text{Rang} T_{\Lambda} \neq \bigoplus_{j \in J} V_j$

By theorem 15.3.8 in [12], we have

$$\bigoplus_{j \in J} V_j = \text{Rang} T_{\Lambda} + \text{Ker} T_{\Lambda}^*$$  (8)

Let $P_{\Lambda}$ be the orthogonal projection from $\bigoplus_{j \in J} V_j$ onto $\text{Rang} T_{\Lambda}$, then

$$\bigoplus_{j \in J} V_j = P_{\Lambda}(\bigoplus_{j \in J} V_j) + P_{\Lambda}(\bigoplus_{j \in J} V_j)^\perp$$  (9)

Therefore $P_{\Lambda}(\bigoplus_{j \in J} V_j)^\perp = \text{Ker} T_{\Lambda}^* \neq 0$

Define $\xi_k : \bigoplus_{j \in J} V_j \longrightarrow V_j$ by $\xi_k \{g_j\}_{j \in J} = g_k$. Now choose $\mathcal{E}_{j_0} \in \{\mathcal{E}_j\}_{j \in J}$ such that $P_{\Lambda}^\perp \mathcal{E}_{j_0} \neq 0$ and we define an operator

$U : (\bigoplus_{j \in J} V_j) \longrightarrow U$ by $U g = \Lambda_{j_0}^* \mathcal{E}_{j_0} P_{\Lambda}^\perp g$ then $U$ is an adjointable linear operator.

Now let $\{\tilde{\Lambda}_j\}$ be the canonical dual of $\{\Lambda_j\}$ with upper bound $D_{\tilde{\Lambda}_j}$ and set $\Gamma_j f = \tilde{\Lambda}_j f + \mathcal{E}_j P_{\Lambda}^\perp U^* f$ so we have:

$$\sum_{j \in J} \langle \Gamma_j f, \Gamma_j f \rangle = \sum_{j \in J} \langle \tilde{\Lambda}_j f + \mathcal{E}_j P_{\Lambda}^\perp U^* f, \tilde{\Lambda}_j f + \mathcal{E}_j P_{\Lambda}^\perp U^* f \rangle$$  (10)

$$\leq 2 \left( \sum_{j \in J} \langle \tilde{\Lambda}_j f, \tilde{\Lambda}_j f \rangle + \sum_{j \in J} \langle \mathcal{E}_j P_{\Lambda}^\perp U^* f, \mathcal{E}_j P_{\Lambda}^\perp U^* f \rangle \right)$$  (11)

$$\leq 2 \left( D_{\tilde{\Lambda}_j} \langle f, f \rangle + \langle P_{\Lambda}^\perp U^* f, P_{\Lambda}^\perp U^* f \rangle \right)$$  (12)
Which implies that \( \{\Gamma_j\}_{j \in J} \) is a g-Bessel sequence. Not that in inequality (11) we apply lemma 2.8.

Now for any \( f \in \mathcal{U} \),
\[
\sum_{j \in J} A_j^* \mathcal{E}_j P_{\Lambda}^\perp U^* f = T^* \sum_{j \in J} \mathcal{E}_j P_{\Lambda}^\perp U^* f \\
= T^* P_{\Lambda}^\perp U^* f = 0
\]

So this yield that \( f = \sum_{j \in J} A_j^* \Gamma_j f \) for all \( f \in \mathcal{U} \)

By proposition 3.6 \( \{\Gamma_j\}_{j \in J} \) is dual g-frame of \( \{\Lambda_j\}_{j \in J} \) and is different from \( \{\tilde{\Lambda}_j\}_{j \in J} \), which contradiction with the uniqueness of dual g-frame of \( \{\Lambda_j\}_{j \in J} \).

**Theorem 3.9.** Let \( \{\Lambda_j\}_{j \in J} \) be a g-Riesz basis and \( \{\Gamma_j\}_{j \in J} \) be a g-sequence of \( \mathcal{U} \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \) over a unital C*-algebra \( \mathcal{A} \). Then \( \{\Gamma_j\}_{j \in J} \) is dual g-Riesz basis of \( \{\Lambda_j\}_{j \in J} \) if and only if \( \Gamma_j = \tilde{\Lambda}_j + \Delta_j \), for each \( j \in J \), where \( \{\Delta_j\}_{j \in J} \) is g-Bessel sequence of \( \mathcal{U} \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \) with the property that for each \( j \in J \) there exist a \( \xi_j \in \text{End}^*_\mathcal{A}(\mathcal{V}_j) \) such that \( \Delta_j = \xi_j \tilde{\Lambda}_j \) and \( \Lambda_j^* \xi_j \Lambda_j = 0 \).

**Proof.** \( \Rightarrow \) Suppose that \( \{\Gamma_j\}_{j \in J} \) is dual g-Riesz basis of \( \{\Lambda_j\}_{j \in J} \) and set \( \Delta_j = \Gamma_j - \tilde{\Lambda}_j \)

Then it is easy to see that \( \{\Delta_j\}_{j \in J} \) is g-Bessel sequence of \( \mathcal{U} \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \).

Now fix an \( n \in J \). From proposition 3.6 we have \( f = \sum_{j \in J} A_j^* \Gamma_j f = \sum_{j \in J} \Gamma_j^* A_j f \) so
\[
\Gamma_n^*(f) = \sum_{j \in J} \Gamma_j^* A_j (\Gamma_n^* f) \implies \sum_{j \neq n} \Gamma_j^* A_j (\Gamma_n^* f) = (\Gamma_n^* A_n (\Gamma_n^* f) - \Gamma_n^* (f)) = 0 \\
\implies \sum_{j \neq n} \Gamma_j^* A_j (\Gamma_n^* f) + \Gamma_n^* (A_n (\Gamma_n^* f) - f) = 0
\]

\( \{\Gamma_j\}_{j \in J} \) is g-Riesz basis so \( \Gamma_n^* (A_n (\Gamma_n^* f) - f) = 0 \) therefore \( \Gamma_n^* = \Gamma_n^* A_n \Gamma_n^* \) we can infer that \( \Gamma_n = \Gamma_n^* A_n \Gamma_n^* \).

On the other hand we have
\[
\tilde{\Lambda}_j + \Delta_j = \tilde{\Lambda}_j + \Delta_j \left( \Lambda_j^* (\tilde{\Lambda}_j + \Delta_j) \right) \\
= \tilde{\Lambda}_j + \Delta_j \left( \Lambda_j^* \tilde{\Lambda}_j + \Lambda_j^* \Delta_j \right) \\
= \Delta_j \Lambda_j^* \tilde{\Lambda}_j + \Delta_j \Lambda_j^* \Delta_j + \tilde{\Lambda}_j \Lambda_j^* \Delta_j + \tilde{\Lambda}_j \Lambda_j^* \Delta_j \\
\implies \Delta_j = \Delta_j \Lambda_j^* \tilde{\Lambda}_j + \Delta_j \Lambda_j^* \Delta_j + \tilde{\Lambda}_j \Lambda_j^* \Delta_j + \tilde{\Lambda}_j \Lambda_j^* \Delta_j - \tilde{\Lambda}_j
\]
To show that $\Delta_j \Lambda^*_j \Delta_j + \tilde{\Lambda}_j \Lambda^*_j \Delta_j = 0$, I have to show that $\Delta_j \Lambda^*_j \Delta_j(f) + \tilde{\Lambda}_j \Lambda^*_j \Delta_j(f) = 0$ hold for all $f \in U$. Not that

$$f = \sum_{j \in J} \Lambda^*_j \Gamma_j(f)$$

$$= \sum_{j \in J} \Lambda^*_j (\tilde{\Lambda}_j + \Delta_j)(f)$$

$$= \sum_{j \in J} \Lambda^*_j \tilde{\Lambda}_j(f) + \sum_{j \in J} \Lambda^*_j \Delta_j(f)$$

$$= \sum_{j \in J} \Lambda^*_j \Delta_j(f) + f$$

Which implies that

$$\sum_{j \in J} \Lambda^*_j \Delta_j(f) = 0$$

so $\Lambda^*_j \Delta_j(f) = 0$ for all $j \in J$ and $f \in U$. Particularly we have $\Lambda^*_n \Delta_n(f) = 0$ for all $f \in U$.

This yield that $\Delta_n \Lambda^*_n \Delta_n = 0$ and $\tilde{\Lambda}_n \Lambda^*_n \Delta_n = 0$, therefore we have $\Delta_n = \Delta_n \Lambda^*_n \tilde{\Lambda}_n = \tilde{\Lambda}_n \Lambda^*_n \Delta_n - \tilde{\Lambda}_n$. So $\Delta_n = \xi_n \tilde{\Lambda}_n$, where $\xi_n = \Delta_n \Lambda^*_n + \tilde{\Lambda}_n \Lambda^*_n - I_n$ and $\xi_n \in End^*_A(V_n)$.

On the other hand

$$0 = \Lambda^*_j \Delta_j(f) = \Lambda^*_j \xi_j \tilde{\Lambda}_j(f)$$

for all $j \in J$ and $f \in U$.

Therefore $\Lambda^*_j \xi_j \Lambda_j = 0$ hold for all $j \in J$.

$\Leftarrow$ Suppose now that $\Gamma_j = \tilde{\Lambda}_j + \Delta_j$, for each $j \in J$, where $\{\Delta_j\}_{j \in J}$ is $g$-Bessel sequence of $U$ with respect to $\{V_j\}_{j \in J}$ and for each $j \in J$ there exist $\xi_j \in End^*_A(V_j)$ such that $\Delta_j(f) = \xi_j \tilde{\Lambda}_j(f)$ and $\Lambda^*_j \xi_j \Lambda_j(f) = 0$ hold for all $f \in U$.

Now for arbitrary $f \in U$,

$$\sum_{j \in J} \Lambda^*_j \Gamma_j(f) = \sum_{j \in J} \Lambda^*_j (\tilde{\Lambda}_j + \Delta_j)(f)$$

$$= \sum_{j \in J} \Lambda^*_j \tilde{\Lambda}_j(f) + \sum_{j \in J} \Lambda^*_j \Delta_j(f)$$

$$= f + \sum_{j \in J} \Lambda^*_j \Delta_j(f)$$

$$= f + \sum_{j \in J} \Lambda^*_j \xi_j \tilde{\Lambda}_j(f)$$

$$= f$$
Which implies that \( \{\Gamma_j\}_{j \in J} \) is dual g-sequence of \( \{\Lambda_j\}_{j \in J} \).

With similar argument in (10)-(12) one can easily see that \( \{\Gamma_j\}_{j \in J} \) is g-Bessel sequence. It follow from proposition 3.6 that \( \{\Gamma_j\}_{j \in J} \) is a dual g-frame of \( \{\Lambda_j\}_{j \in J} \).

To complet the proof, we need to show that \( \{\Gamma_j\}_{j \in J} \) is a g-Riesz basis of \( U \) with respect to \( \{\mathcal{V}_j\}_{j \in J} \).

Suppose that \( \sum_{j \in J} \Gamma_j^* g_j = 0 \). then we have:

\[
0 = \sum_{j \in J} \Gamma_j^* g_j \\
= \sum_{j \in J} \Delta_j^* (g_j) + \Lambda_j^*(g_j) \\
= \sum_{j \in J} \tilde{\Lambda}_j^* \xi_j^*(g_j) + \Lambda_j^*(g_j) \\
= \sum_{j \in J} \tilde{\Lambda}_j^* (\xi_j^*(g_j) + g_j)
\]

Therefore \( \tilde{\Lambda}_j^* (\xi_j^*(g_j) + g_j) = 0 \) for all \( j \in J \)

i.e \( \Gamma_j^* g_j = 0 \) for all \( j \in J \).

We now show that \( \Gamma_j \neq 0 \) for each \( j \in J \).

Assume that on the contrary that \( \Gamma_n = 0 \) for some \( n \in J \).

Then \( \Delta_n = -\tilde{\Lambda}_n \) it follow that \( \Delta_n = -\Lambda_n S^{-1} \) so we have:

\[
0 = \Lambda_n^* \xi_n \Lambda_n(f) \\
= \Lambda_n^* \xi_n \tilde{\Lambda}_n S(f) \\
= \Lambda_n^* \Delta_n S(f) \\
= -\Lambda_n^* \Lambda_n(f)
\]

Hold for all \( f \in U \).

Therefore \( 0 = \langle \Lambda_n^* \Lambda_n f, f \rangle = \langle \Lambda_n f, \Lambda_n f \rangle \) so \( \Lambda_n = 0 \) a contradiction. \( \square \)

References


Characterization of g-Riesz basis and their dual in Hilbert $A$-module


Received: November 11, 2013