2-edge-Hamiltonian-connectedness of 4-connected plane graphs

Kenta Ozeki
National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku,
Tokyo 101-8430, Japan
and
JST, ERATO, Kawarabayashi Large Graph Project
e-mail: ozeki@nii.ac.jp

Petr Vrána*
Department of Mathematics,
University of West Bohemia,
Univerzitní 8 306 14, Plzeň, Czech Republic
e-mail: vranap@kma.zcu.cz

Abstract

A graph G is called 2-*edge-Hamiltonian-connected* if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G)\}$ with $1 \leq |X| \leq 2$, $G \cup X$ has a Hamiltonian cycle containing all edges in X, where $G \cup X$ is the graph obtained from G by adding all edges in X. In this paper, we show that every 4-connected plane graph is 2-edge-Hamiltonian-connected. This result is best possible in many senses and an extension of several known results on Hamiltonicity of 4-connected plane graphs, for example, Tutte's result saying that every 4-connected plane graph is Hamiltonian, and Thomassen's result saying that every 4-connected plane graph is Hamiltonian-connected. We also show that although the problem of deciding whether a given graph is 2-edge-Hamiltonian-connected is NP-complete, there exists a polynomial time algorithm to solve the problem if we restrict the input to plane graphs.

Keywords: Hamiltonian cycles, Hamiltonian-connected, 2-edge-Hamiltonian-connected, plane graphs

1 Introduction

In 1931, Whitney [27] proved that every 4-connected triangulation of the plane contains a Hamiltonian cycle. Tutte [26] generalized this result to 4-connected plane graphs. Extending the technique of Tutte, Thomassen [24] proved that every 4-connected plane graph is *Hamiltonian-connected*, i.e. there is a Hamiltonian path between any given pair of distinct vertices.

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In this paper, we consider the following condition, which appeared in several papers as the *scattering number*, see, for example [6, 7, 11]. Let G be a connected graph and ε be an integer. We denote the number of components of G by $\omega(G)$.

A(ε): For every $S \subset V(G)$ with $\omega(G-S) \geq 2$, $\omega(G-S) \leq |S| + \varepsilon$.

Note that every graph that is Hamiltonian (resp. Hamiltonian-connected) satisfies condition A(0) (resp. A(-1)), see [1, 6]. Hence conditions A(0) and A(-1) are necessary conditions for the properties of "being Hamiltonian" and "being Hamiltonian-connected", respectively. On the other hand, the following proposition is easily shown by Euler formula, see, for example, [20].

Proposition 1 *Let G be a 4-connected plane graph. Then for every $S \subset V(G)$ with $\omega(G-S) \geq 2$, we have that $\omega(G-S) \leq |S| - 2$.*

By Proposition 1, every 4-connected plane graph satisfies condition A(-2). In the sense of condition A(ε), a 4-connected plane graph satisfies a stronger condition than the necessary condition for the properties of both being Hamiltonian and being Hamiltonian-connected. Therefore, one can expect that every 4-connected plane graph also satisfies stronger properties than the property of being Hamiltonian-connected.

For a positive integer k, a graph G is called *k-Hamiltonian* (resp. *k-Hamiltonian-connected*), if G has at least $k+3$ vertices and for every k vertices $v_1, v_2, \ldots, v_k$ in G, $G - \{v_i : 1 \leq i \leq k\}$ is Hamiltonian (resp. Hamiltonian-connected). These properties have also been considered, for example see [10, 12, 13, 16, 28]. In Proposition 4 (ii) in Section 2, we will show that condition A($-k$) is also a necessary condition for the properties of both being k-Hamiltonian and being $(k-1)$-Hamiltonian-connected, see Proposition 4 (ii). In particular, condition A(-2) is a necessary condition for the properties of both being 2-Hamiltonian and being 1-Hamiltonian-connected.

For such properties for 4-connected plane graphs, some researchers have shown the following. Note that (I) was conjectured by Plummer [17] and shown by Thomas and Yu [21], and (II) and (III) are corollaries of a result due to Sanders [19].

Theorem 2 (Thomas and Yu [21], Sanders [19]) *Every 4-connected plane graph G satisfies all of the following properties.*

(I) *G is 2-Hamiltonian.*

(II) *G is 1-Hamiltonian-connected.*

(III) *For every pair of vertices x, y and every edge e in G, G has a Hamiltonian path between x and y through e.*

A graph with property (III) does not necessarily satisfy condition A(-2) but at least property (III) is stronger than the property of being Hamiltonian-connected. As other properties stronger than the property of being Hamiltonian-connected, Thomas and Yu [21] and Sanders [18] independently proved that for every 4-connected plane graph and every three edges incident with a common non-triangular face, there exists a Hamiltonian cycle passing through all the three edges.

In this paper, we concentrate on the following property, which was introduced in [12] for the case where $k = 2$. A graph G is called *k-edge-Hamiltonian-connected* if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G),\ x_1 \neq x_2\}$ such that $1 \leq |X| \leq k$ and the graph induced by X on $V(G)$ is a forest in which each component is a path, $G \cup X$ has a

Hamiltonian cycle containing all edges in X, where $G \cup X$ is the graph obtained from G by adding all edges in X (so $G \cup X$ might have parallel edges). It is easy to see that if a graph G is $(k+1)$-edge-Hamiltonian-connected, then G is k-edge-Hamiltonian-connected. Note that the property of being 1-edge-Hamiltonian-connected is actually equivalent to the property of being Hamiltonian-connected, see [12]. Thus, combining these two facts, we obtain that the property of being 2-edge-Hamiltonian-connected is stronger than the property of being Hamiltonian-connected. Notice also that as in Proposition 4 (ii) in Section 2, condition A(-2) is a necessary condition for the property of being 2-edge-Hamiltonian-connected. The following is the main theorem of this paper.

Theorem 3 *Every 4-connected plane graph is 2-edge-Hamiltonian-connected.*

As we will show in Section 2, if a graph is 2-edge-Hamiltonian-connected, then G is also 1-Hamiltonian-connected and satisfies property (III) in Theorem 2. Thus, Theorem 3 is stronger than Theorems 2 (II) and (III). Although there exist infinitely many graphs which are 2-edge-Hamiltonian-connected but not 2-Hamiltonian, there also exist infinitely many graphs which are 2-Hamiltonian but not 2-edge-Hamiltonian-connected, see Propositions 6 and 7 in Section 2. Thus, Theorem 3 is not weaker than Theorem 2 (I).

Let us observe that Theorem 3 is best possible in many senses. First we cannot improve it to graphs on other surfaces. Indeed, there are 4-connected graphs on the projective plane which do not satisfy condition A(-2), consider the face subdivisions of quadrangulations of the projective plane. Since condition A(-2) is a necessary condition for the property of being 2-edge-Hamiltonian-connected, such graphs cannot be 2-edge-Hamiltonian-connected. Secondly, for any ε, there are infinitely many 3-connected plane graphs which do not satisfy condition A(ε), for example, the face subdivisions of plane triangulations of large order. So, 4-connectedness in Theorem 3 is definitely necessary. Finally, there are infinitely many 4-connected plane graphs which are not 3-edge-Hamiltonian-connected. Actually, condition A(-3) is a necessary condition for the property of being 3-edge-Hamiltonian-connected (see Proposition 4 (ii)), but there are infinitely many 4-connected plane graphs which do not satisfy condition A(-3), consider the face subdivisions of quadrangulations of the plane.

Notice that in all the above situations, condition A(ε) plays a crucial role.

For the proof of Theorem 3, we will consider a powerful method, called the *Tutte path (or cycle)* method. (See Section 4 for the definition of a Tutte path and cycle.) The Tutte path or cycle method was used to show many results on Hamiltonicity of plane graphs and graphs on a surface, some of which were already mentioned, see [18, 19, 21, 24, 26]. We now mention other results on a Tutte path or cycle.

Thomas and Yu [21] improved the result by Tutte and proved that every 4-connected graph on the projective plane is Hamiltonian. Recently, Kawarabayashi and Ozeki [9] further improved it and proved that every 4-connected graph on the projective plane is Hamiltonian-connected. For graphs on the torus, Grünbaum [5] and Nash-Williams [15] independently conjectured that every 4-connected graph on the torus is Hamiltonian. This conjecture has been unsolved for 40 years, but some partial solutions were obtained. Thomas and Yu [22] showed that every 5-connected graph on the torus is Hamiltonian, and Thomas, Yu and Zang [23] showed that every

4-connected graph on the torus has a Hamiltonian path. On the other hand, little is known for graphs on the Klein bottle. It is known that there exist 4-connected non-Hamiltonian graphs on surfaces other than the plane, the projective plane, the torus and the Klein bottle. Yu [29] gave some positive results on the Hamiltonicity of 5-connected triangulations of a surface. All the positive results mentioned above were proven using Tutte cycles or paths.

In the next section, we will show the relationships among the properties mentioned in this section and condition A(ε). After that, in Section 3, we will prove a corollary of Theorem 3 concerning the complexity of the decision problem on 2-edge-Hamiltonian-connectedness. In order to show Theorem 3 in Section 5, we use a technical theorem (Theorem 11) using Tutte paths and cycles, which will be introduced in Section 4 and shown in Section 6.

2 The properties in Section 1 and condition A(ε)

In this section, we will consider the properties mentioned in the previous section, in particular, we focus on the relationship among them and condition A(ε). First we show the following.

Proposition 4 *Let k be a positive integer. Suppose that a graph G is k-Hamiltonian, or $(k-1)$-Hamiltonian-connected, or k-edge-Hamiltonian-connected. (However, we do not consider the case where $k = 1$ and G is $(k-1)$-Hamiltonian-connected, since 0-Hamiltonian-connectedness has not been defined.) Then*

(i) *G is $(k+2)$-connected unless G is isomorphic to K_{k+2}, and*

(ii) *G satisfies condition $A(-k)$.*

Proof. (i) Suppose that G is neither $(k+2)$-connected nor isomorphic to K_{k+2}. Then G has an l-cut, say $\{v_1, v_2, \ldots, v_l\}$, for some integer $l \le k+1$. However, $G - \{v_1, v_2, \ldots, v_{l-1}\}$ has no Hamiltonian cycle since it has a cut vertex v_l. Similarly $G - \{v_1, \ldots, v_{l-2}\}$ has no Hamiltonian path between v_{l-1} and v_l, and for $X = \{v_1v_2, v_2v_3, \ldots, v_{l-1}v_l\}$, $G \cup X$ has no Hamiltonian cycle containing all edges in X. Hence G is neither k-Hamiltonian, $(k-1)$-Hamiltonian-connected, nor k-edge-Hamiltonian-connected. This completes the proof of (i).

(ii) Let G be a graph that is k-Hamiltonian, or $(k-1)$-Hamiltonian-connected, or k-edge-Hamiltonian-connected. Let $S \subset V(G)$ be any set with $\omega(G - S) \ge 2$. By (i), $|S| \ge k+2$. If G is k-Hamiltonian, then let $S' \subset S$ with $|S'| = k$, and find a Hamiltonian cycle T in $G - S'$, which implies that

$$\omega(G - S) \le \omega\big(T - (S - S')\big) \le |S| - |S'| = |S| - k.$$

So condition A($-k$) holds. When G is $(k-1)$-Hamiltonian-connected, or G is k-edge-Hamiltonian-connected, then letting $S' \subset S$ with $|S'| = k$ or $X \subset \{x_1x_2 : x_1, x_2 \in S\}$ such that $1 \le |X| \le k$ and the graph induced by X on S is a forest in which each component is a path, respectively, we can show that condition A($-k$) holds in the same way. This completes the proof of (ii). $\square$

Next we will consider the relationship between the properties of being 2-Hamiltonian and being 2-edge-Hamiltonian-connected. Before that, we show the following. For a path P and two vertices $x, y \in V(P)$, $P[x, y]$ denotes the subpath of P between x and y.

Proposition 5 *Let G be a graph and G' be the graph obtained from G by joining a new vertex to all vertices in G. Then G is Hamiltonian-connected if and only if G' is 2-edge-Hamiltonian-connected.*

Proof. Suppose that G is Hamiltonian-connected. We will show that for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G')\}$ with $1 \le |X| \le 2$, $G' \cup X$ has a Hamiltonian cycle containing all edges in X. Let w be the new vertex in G'. (So $w \notin V(G)$.)

Let $x_1, x_2, y_1, y_2 \in V(G')$ so that $X = \{x_1x_2, y_1y_2\}$ if $|X| = 2$; otherwise $X = \{x_1x_2\}$ and y_1y_2 be any edge in G' with $\{x_1, x_2\} \neq \{y_1, y_2\}$. We may assume that $x_1 \neq w$ and $y_1 \neq w$. If $x_2 = y_2 = w$, then taking a Hamiltonian path P in G from x_1 to y_1, $P \cup \{x_1w, wy_1\}$ is a Hamiltonian cycle in $G' \cup X$ through x_1x_2 and y_1y_2. So, by symmetry, we may assume that $x_2 \neq w$. Let P be a Hamiltonian path in G from x_1 to x_2. If $y_2 = w$, then letting y_1^+ be the vertex next to y_1 in P,

$$P[x_1, y_1] \cup \{y_1w, wy_1^+\} \cup P[y_1^+, x_2] \cup \{x_1x_2\}$$

is a Hamiltonian cycle in $G' \cup X$ through x_1x_2 and y_1y_2. Thus, the case where none of x_1, x_2, y_1 and y_2 is w only remains. By symmetry, we may assume that $x_2 \neq y_1, y_2$. Then letting y_1^+ and y_2^+ be the vertices next to y_1 and y_2 in P, respectively,

$$P[x_1, y_1] \cup \{y_1y_2\} \cup P[y_2, y_1^+] \cup \{y_1^+w, wy_2^+\} \cup P[y_2^+, x_2] \cup \{x_1x_2\}$$

is a Hamiltonian cycle in $G' \cup X$ through x_1x_2 and y_1y_2. Hence G' is 2-edge-Hamiltonian-connected.

Conversely, suppose that G' is 2-edge-Hamiltonian-connected. Then for any $x, y \in V(G)$, letting $X = \{xw, yw\}$, $G' \cup X$ has a Hamiltonian cycle T containing all edges in X. Then $T - w$ is a Hamiltonian path in G from x to y. Hence G is Hamiltonian-connected. $\square$

We are now ready to show the following two propositions, which concern the relationship between the properties of being 2-Hamiltonian and being 2-edge-Hamiltonian-connected.

Proposition 6 *There exist infinitely many graphs G which are 2-edge-Hamiltonian-connected, but not 2-Hamiltonian.*

Proposition 7 *There exist infinitely many graphs G which are 2-Hamiltonian, but not 2-edge-Hamiltonian-connected.*

Proof of Proposition 6. Let G be a graph consisting of two disjoint cliques of order at least three and three vertex disjoint paths of length two connecting them. So G has exactly three vertices of degree 2. Note that G does not have a Hamiltonian cycle, since no cycle can pass through all the three paths. Let G' be the graph obtained from

G by adding two new vertices w_1, w_2, joining them to all vertices in G and adding an edge connecting w_1 and w_2.

It is clear that G' is not 2-Hamiltonian since $G' - \{w_1, w_2\}$ does not have a Hamiltonian cycle. On the other hand, it follows from Proposition 5 that G' is 2-edge-Hamiltonian-connected since $G' - w_1$ is Hamiltonian-connected and G' is obtained from $G' - w_1$ by joining w_1 to all vertices in $G' - w_1$. $\square$

Proof of Proposition 7. In the proof of Proposition 7, we use the results on the property of being *hypo-Hamiltonian*. A graph H is called *hypo-Hamiltonian* if H is not Hamiltonian, but for all $u \in V(H)$, $H - u$ is Hamiltonian. It is known that there exist infinitely many hypo-Hamiltonian graphs, see, for example, [14, 25]. Let H be a hypo-Hamiltonian graph and w be a vertex with $w \notin V(H)$. We obtain the graph G from H by joining w to all vertices of H. We show that G is 2-Hamiltonian, but not 2-edge-Hamiltonian-connected.

Let $u, v \in V(G)$. By symmetry, we may assume that $u \neq w$. Since H is hypo-Hamiltonian, $H - u$ has a Hamiltonian cycle, say T. If $v = w$, then T is also a Hamiltonian cycle of $G - \{u, v\}$. Otherwise, $v \in V(T)$ and let v^+ and v^- be the neighbors of v in T. Then $\big(T - \{v\}\big) \cup \{v^+w, wv^-\}$ is a Hamiltonian cycle of $G - \{u, v\}$. Hence G is 2-Hamiltonian.

On the other hand, let $X = \{wx, wy\}$ for some edge xy in H. If $G \cup X$ has a Hamiltonian cycle T containing all edges in X, then $\big(T - \{w\}\big) \cup \{xy\}$ is a Hamiltonian cycle of H, a contradiction. Thus G is not 2-edge-Hamiltonian-connected. $\square$

At the end of this section, we will consider the relationship between the property of being 2-edge-Hamiltonian-connected, the property of being 1-Hamiltonian-connected and property (III) in Theorem 2.

Proposition 8 *If a graph G is 2-edge-Hamiltonian-connected, then G is also 1-Hamiltonian-connected and satisfies property (III) in Theorem 2.*

Proof. Suppose that a graph G is 2-edge-Hamiltonian-connected. Take three arbitrary vertices v, x and y in G, and let $X = \{vx, vy\}$. Since G is 2-edge-Hamiltonian-connected, $G \cup X$ has a Hamiltonian cycle T containing all edges in X. Then G has a Hamiltonian path $T - v$ between x and y. Since we can choose any pair of vertices x, y in $G - v$, $G - v$ is Hamiltonian-connected, and hence G is also 1-Hamiltonian-connected.

Take two arbitrary vertices x and y and one edge e in G. Let $X = \{xy, x'y'\}$, where x' and y' are the end vertices of e. Since G is 2-edge-Hamiltonian-connected, $G \cup X$ has a Hamiltonian cycle T containing all edges in X. Then deleting the edge e and replacing $x'y'$ to the edge e, we obtain a Hamiltonian path between x and y through e, and hence G also satisfies property (III) in Theorem 2. $\square$

3 Computational complexity of problems on 2-edge-Hamiltonian-connectedness

The problem to decide whether a given graph G has a Hamiltonian cycle is one of the classical NP-complete problems (see [3]), and the Hamiltonian cycle problem remains NP-complete even when restricted to 3-connected cubic plane graphs [4]. The problem to decide whether a given graph G is Hamiltonian-connected is also known to be NP-complete [2]. The complexity of the corresponding Hamiltonian-connectedness problem in plane graphs is not known. In this section, we consider the following problems on 2-edge-Hamiltonian-connectedness.

2-E-HC
Instance: A graph G
Question: Is G 2-edge-Hamiltonian-connected?

P2-E-HC
Instance: A plane graph G
Question: Is G 2-edge-Hamiltonian-connected?

In this section, we will show that although 2-E-HC is NP-complete, there exists a polynomial time algorithm to solve P2-E-HC, which is a corollary of Theorem 3. For the proof of Theorem 9, we use the same construction as in [12].

Theorem 9 *2-E-HC is NP-complete.*

Theorem 10 *There exists a polynomial time algorithm to solve P2-E-HC.*

Proof of Theorem 9. It is clear that 2-E-HC $\in NP$. We shall reduce the Hamiltonian-connectedness problem to 2-E-HC. Given a graph G, let w be a vertex with $w \notin V(G)$. Let G' be the graph obtained from G by joining w to all vertices of G. By Proposition 5, G' is 2-edge-Hamiltonian-connected if and only if G is Hamiltonian-connected. Since the problem to decide whether a given graph G is Hamiltonian-connected is known to be NP-complete [2], 2-E-HC is also NP-complete. This completes the proof of Theorem 9. □

Proof of Theorem 10. Let G be a plane graph. If G is 4-connected, then by Theorem 3, G is 2-edge-Hamiltonian-connected. On the other hand, if G is not 4-connected, then by Proposition 4 (i), G is not 2-edge-Hamiltonian-connected. It is well-known that there exists a polynomial time algorithm to decide if a given graph is 4-connected or not (for example, see p. 332 in [8]). This algorithm is also a polynomial time algorithm to decide if a given plane graph is 2-edge-Hamiltonian-connected. □

4 Preliminaries and a technical statement

For a graph G, a pair (K_1, K_2) of subgraphs of G is called a *separation of* G if $V(G) = V(K_1) \cup V(K_2)$ and each edge of G are contained in exactly one of K_1 and K_2. A separation (K_1, K_2) of G is a *k-separation* if $|K_1|, |K_2| \geq k + 1$ and $\left|V(K_1 \cap K_2)\right| = k$. Note that G is k-connected if and only if G has no r-separation for each $r < k$.

Let T be a subgraph of a graph G. A *T-bridge* of G is either an edge of $G - E(T)$ with both ends on T or a subgraph of G induced by the edges in a component of $G - V(T)$ and all edges from that component to T. A T-bridge which is an edge is called *trivial*; otherwise it is *non-trivial.* For a T-bridge B of G, the vertices in

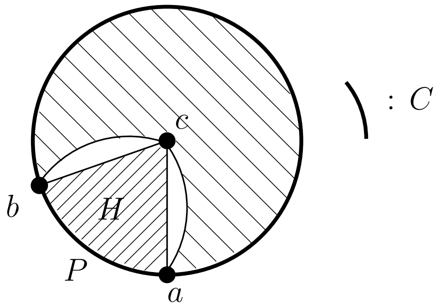

Figure 1: A C-flap H with attachments a, b, c and base path P.

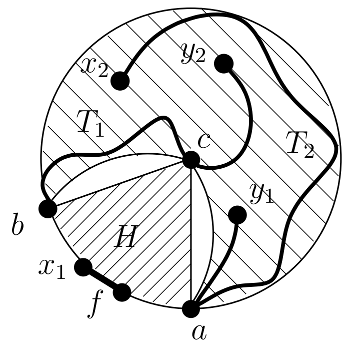

Figure 2: A C-flap H and a C-Tutte subgraph $T_1 \cup T_2$ in (T2) in Theorem 11.

$B \cap T$ are the *attachments* of B (on T). We say that T is a *Tutte subgraph* in G if every T-bridge of G has at most three attachments on T. For $C \subset G$, T is a *C-Tutte subgraph* in G if T is a Tutte subgraph in G and every T-bridge of G containing an edge of C has at most two attachments on T. A *Tutte path* (respectively, a *Tutte cycle*) in a graph is a path (respectively, a cycle) which is a Tutte subgraph.

Let G be a connected plane graph. Each face of G is bounded by a walk of G, called its *facial* walk. Note that if G is 2-connected, then every facial walk must be a cycle, which is called a *facial* cycle. The walk that bounds the outer face of G is called the *outer walk*, and the *outer cycle* if it is a cycle. For a cycle C of G and two vertices $x, y \in V(C)$, $C[x, y]$ denotes the subpath of C between x and y in the clockwise order. A *C-flap* is an $\{a, b, c\}$-bridge H of G for some set of three vertices a, b, c with $a, b \in V(C)$ such that

(i) $a, b \in V(C) \cap V(H)$ and $a \neq b$,

(ii) H contains a subpath P of C connecting a and b, and

(iii) H is embedded on the plane such that P and c appear in the outer walk.

We call the subpath P of C in (ii) the *base* path of H. Note that a, b, c are the *attachments of H*. See Figure 1. (In all figures of the present paper, the region represented by rising diagonal strokes from bottom left to top right indicates a C-flap, while the one represented by falling diagonal strokes from top left to bottom right indicates other parts of a given graph. White regions indicate faces. We usually draw figures so that the specified cycle C is the outer cycle.)

Remark: Notice that a C-flap was defined by Thomas and Yu [21], but they regard the empty graph also as a C-flap. In the present paper, we exclude the empty graph from the definition of a C-flap, since it seems easier to understand.

Two vertex disjoint paths T_1 and T_2 *connect* $\{x_1, x_2\}$ and $\{y_1, y_2\}$ if T_1 connects x_i and y_j and T_2 connects x_{3-i} and y_{3-j} for some $i, j = 1, 2$. Note that if $x_1 = y_1$, then one of the paths T_1 and T_2 must consist of only x_1.

In order to prove Theorem 3, we use a method that is similar to the one in [9]. We show the following technical theorem, which is used in the proof of Theorem 3 in Section 5. See Figure 2 for (T2).

Theorem 11 *Let G be a 2-connected plane graph, and let C be a facial cycle of G. Suppose that for every 2-separation (G_1, G_2) of G, $E(C) \cap E(G_1) \neq \emptyset$ and $E(C) \cap$*

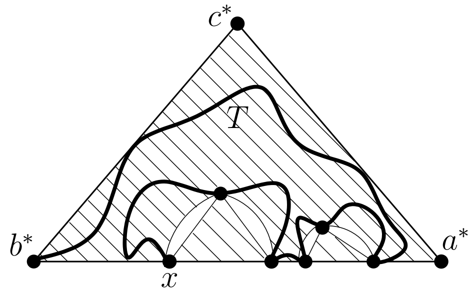

Figure 3: A $C[x, b^*]$-Tutte path T desired in Lemma 13.

$E(G_2) \neq \emptyset$. Let $x_1 \in V(C)$ and let $x_2, y_1, y_2 \in V(G)$ with $x_2 \neq x_1, y_1, y_2$ and $y_1 \neq y_2$. Let f be an edge of C that is incident with x_1. Then one of the following holds.

(T1) *There exists a C-Tutte subgraph in G consisting of two vertex disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{x_1, x_2\}$ and $\{y_1, y_2\}$.*

(T2) *There exist a C-flap H with attachments a, b, c and base path P, and a C-Tutte subgraph in $G - \big(V(H) - \{a, b, c\}\big)$ consisting of two vertex disjoint paths T_1 and T_2 such that T_1 and T_2 connect $\{b, x_2\}$ and $\{y_1, y_2\}$, $a, c \in V(T_1 \cup T_2)$, $x_1 \in \big(V(P) - \{a\}\big) \cup \{b\}$, P contains f, and a, f, x_1 appear in P in this order.*

Notice also that without the assumption on 2-separations of G, Theorem 11 can be shown, but we here assume that to simplify the proof.

Our proof of Theorem 11 depends on some known lemmas concerning Tutte cycles and paths. The first lemma was proved by Sanders [19]. See also the paper by Thomassen [24].

Theorem 12 *Let G be a connected plane graph and let C be a facial walk of G. Let $x, y \in V(G)$ with $x \neq y$ and let $e \in E(C)$. Assume that G contains a path from x to y through e. Then G has a C-Tutte path from x to y through e.*

Note that if G is 2-connected, then for any $x, y \in V(G)$ with $x \neq y$ and any $e \in E(C)$, G has a path from x to y through e.

The following lemma, which is a direct corollary of Theorem (2.5) in [21], appeared in [9]. By the following lemma, we can deal with a Tutte path inside of a C-flap. See Figure 3.

Lemma 13 *Let H be a connected graph embedded on the plane, and let C be a facial walk of H. Let $a^*, x, b^*, c^* \in V(C)$ be four distinct vertices of C that appear in C in this clockwise order. Suppose that $\widetilde{H}$ is 2-connected, where $\widetilde{H}$ is the graph obtained from H by adding an edge connecting b^* and c^* and an edge connecting c^* and a^*. Then there exists a $C[x, b^*]$-Tutte subgraph in H consisting of the two vertices a^* and c^* and a path T from b^* to x with $a^*, c^* \notin V(T)$.*

In the end of this section, we show the outline of the proof of Theorem 11. In the proof of Theorem 11, what we are trying to do is to delete the vertex x_1 and to use the induction hypothesis to the resultant graph $G - x_1$. To do that, first we have to show that $G - x_1$ is also 2-connected (Claim 1). Then after dealing with a special case, that is, the case where $x_1 = y_1$ or $x_1 = y_2$ (Claim 2), we will use the induction hypothesis to $G - x_1$. During those processes, we use the induction hypothesis to some graphs and obtain that such graphs satisfy either (T1) or (T2). For the graphs satisfying (T1), we can directly use two paths T_1^* and T_2^* satisfying the conditions in (T1) in the next setp. However, for the graphs satisfying (T2), together with T_1^* and T_2^*, we have to deal with the flap satisfying the conditions in (T2), and sometimes have to find a path inside of the flap using Lemma 13. Connecting T_1^*, T_2^*, and the path inside of the flap if necessary, we will obtain two vertex disjoint paths, together with a C-flap in some cases, which have required properties in (T1) or (T2).

5 Proof of Theorem 3

Let G be a 4-connected plane graph. Let $X = \{x_1x_2, y_1y_2\}$ if $|X| = 2$; otherwise let $X = \{x_1x_2\}$ and let y_1y_2 be any edge in G with $\{x_1, x_2\} \neq \{y_1, y_2\}$. We will find two vertex disjoint paths in G connecting $\{x_1, x_2\}$ and $\{y_1, y_2\}$ and containing all vertices in G.

Since $\{x_1, x_2\} \neq \{y_1, y_2\}$, without loss of generality, we may assume that $x_2 \neq y_1, y_2$. Let C^* be a facial cycle of G containing x_1, but not containing x_2. Since G is 4-connected, G has such a facial cycle. If $x_1 = y_1$ or $x_1 = y_2$, then by symmetry, we may assume that $x_1 = y_1$. Note that $x_1 \neq y_2$ in either case. Let f be an edge of C^* that is incident with x_1. If both y_1 and y_2 appear in C^*, we choose such an edge f so that y_1, y_2, f appear in C^* in this order. By symmetry, we may assume that y_1, y_2, f appear in C^* in this clockwise order.

Let G^* be the graph obtained from G by deleting the edge x_2y_2 if it exists. Since x_2 does not appear in C^*, C^* is also a facial cycle of G^*. Since G^* is 3-connected, G^* satisfies the conditions in Theorem 11. Hence by Theorem 11 with respect to G^*, C^*, x_1, x_2, y_1, y_2 and f, (which means that we use Theorem 11 for the graph G^*, the facial cycle C^* of G^*, and so on with a natural correspondence,) we obtain that G^* satisfies (T1) or (T2).

Suppose first that G^* satisfies (T2). Let T_1 and T_2 be the two vertex disjoint paths and H be the C^*-flap with attachments a, b, c and base path P satisfying the conditions in (T2). Notice that $x_2, y_2 \notin V(H) - \{a, b, c\}$, and hence $H - \{a, b, c\}$ is separated by $\{a, b, c\}$ from $G - V(H)$ in G. Since G is 4-connected, we have that $V(G) - V(H) = \emptyset$, that is, $x_2, y_1, y_2 \in V(T_1 \cup T_2) = \{a, b, c\}$. Since x_2 does not appear in C^*, $c = x_2$. Then $\{a, b\} = \{y_1, y_2\}$, and hence by the choice of f and the condition that a, f, x_1 appear in P in this order, we have that $a = y_2$ and $b = y_1$. Since T_1 and T_2 connects $\{b, x_2\}$ and $\{y_1, y_2\}$ in $G^* - \big(V(H) - \{a, b, c\}\big)$, one of T_1 and T_2 consists of only the vertex $b = y_1$, and the other consists of the two vertices x_2 and y_2, which contradicts that $x_2y_2 \notin E(G^*)$. Hence G^* does not satisfy (T2), and satisfies (T1).

Let T_1 and T_2 be the two vertex disjoint paths satisfying the conditions in (T1). If $x_1 = y_1$, since $x_2y_2 \notin E(G^*)$, one of the paths T_1 and T_2 has at least three vertices. On the other hand, if $x_1 \neq y_1$, then by the choice of x_1, we have that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$, and hence both T_1 and T_2 have at least two vertices. In either case, we have that $|T_1 \cup T_2| \geq 4$. Thus, if there exists a non-trivial $(T_1 \cup T_2)$-bridge B of G, then the attachments of B form a 3-cut of G, contradicting the assumption that G is 4-connected. (Recall that $x_2, y_2 \in V(T_1 \cup T_2)$.) This implies that there are no non-trivial $(T_1 \cup T_2)$-bridges of G, that is, $V\big(T_1 \cup T_2\big) = V(G)$. Hence $T_1 \cup T_2 \cup \{x_1x_2, y_1y_2\}$ is a Hamiltonian cycle in $G \cup X$ containing all edges in X. This completes the proof of

Theorem 3. $\square$

6 Proof of Theorem 11

We prove Theorem 11 by induction on $|G|$. If $|G| = 3$, it is clear that G satisfies (T1). So, we may assume that $|G| \geq 4$.

First we prove the following claim.

Claim 1 *If there exists a 2-separation (G_1, G_2) of G such that $x_1 \in V(G_1 \cap G_2)$, then G satisfies (T1) or (T2).*

Proof. Suppose that G has a 2-separation (G_1, G_2) such that $x_1 \in V(G_1 \cap G_2)$. Let $\{x_1, z\} = V(G_1 \cap G_2)$. By the assumption of Theorem 11, $E(G_1) \cap E(C) \neq \emptyset$ and $E(G_2) \cap E(C) \neq \emptyset$. Thus, $z \in V(C)$. By symmetry, we may assume that $x_2 \in V(G_1)$.

For $i = 1, 2$, let G_i^* be the graph obtained from G_i by adding an edge connecting x_1 and z so that the edge x_1z appears on the region bounded by C (and deleting the original edge connecting x_1 and z if it exists in G_i). In other words, G_i^* is obtained from G by replacing G_{3-i} with an edge. Note that both G_1^* and G_2^* are 2-connected plane graphs. Here we may assume that $E(C[x_1, z]) = E(C) \cap E(G_1)$ and $E(C[z, x_1]) = E(C) \cap E(G_2)$. Let $C_1^* = C[x_1, z] \cup \{x_1z\}$ and let $C_2^* = C[z, x_1] \cup \{x_1z\}$, which are facial cycles of G_1^* and G_2^*, respectively. We divide the proof of Claim 1 into three cases.

Case 1. $y_1, y_2 \in V(G_2)$.

By Theorem 12, G_1^* has a C_1^*-Tutte path T_1^* from x_1 to x_2 through x_1z, and G_2^* has a C_2^*-Tutte path T_2^* from y_1 to y_2 through x_1z. Then $\big(T_1^* - \{x_1z\}\big) \cup \big(T_2^* - \{x_1z\}\big)$ consists of two vertex disjoint paths in G, say T_1 and T_2. See Figure 4. Note that T_1 and T_2 connect $\{x_1, x_2\}$ and $\{y_1, y_2\}$.

We will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G. Let B be a $\big(T_1 \cup T_2\big)$-bridge of G. Then B is either (i) a T_1^*-bridge of G_1^*, or (ii) a T_2^*-bridge of G_2^*. Then it follows from the conditions of T_1^* and T_2^* that B has at most three attachments. Moreover, suppose that B contains an edge of C. Recall that $E(C) \cap E(G_1^*) \subset E(C_1^*)$, and $E(C) \cap E(G_2^*) \subset E(C_2^*)$. Thus, B contains an edge of C_1^* if B satisfies (i), or an edge of C_2^* if B satisfies (ii), and hence B has exactly two attachments. These imply that $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Case 2. $y_1 \in V(G_1) - V(G_2)$ and $y_2 \in V(G_2)$, or $y_1 \in V(G_2)$ and $y_2 \in V(G_1) - V(G_2)$.

By symmetry, we may assume that $y_1 \in V(G_1) - V(G_2)$ and $y_2 \in V(G_2)$. Let $y_2^* = z$ and $f_1^* = zx_1$, which is an edge of G_1^*. By induction hypothesis with respect to G_1^*, C_1^*, x_1, x_2, y_1, y_2^* and f_1^*, we obtain that G_1^* satisfies either (T1) or (T2).

Case 2.1. G_1^* satisfies (T1).

Let T_1^* and T_2^* be the two vertex disjoint paths satisfying the conditions in (T1) for G_1^*. By symmetry, we may assume that x_1 is an end vertex of T_1^*. Let $T_1 = T_1^*$. By Theorem 12, G_2^* has a C_2^*-Tutte path T' from x_1 to y_2 through x_1z. Let $T_2 = T_2^* \cup \big(T' - \{x_1\}\big)$. See Figure 5. Then T_1 and T_2 are two vertex disjoint paths in G connecting $\{x_1, x_2\}$ and $\{y_1, y_2\}$.

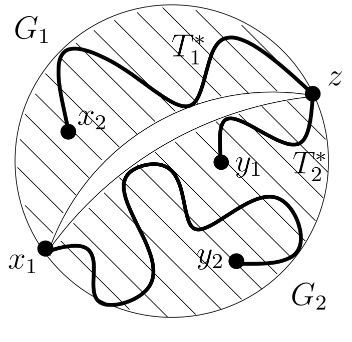

Figure 4: Case 1 in the proof of Claim 1.

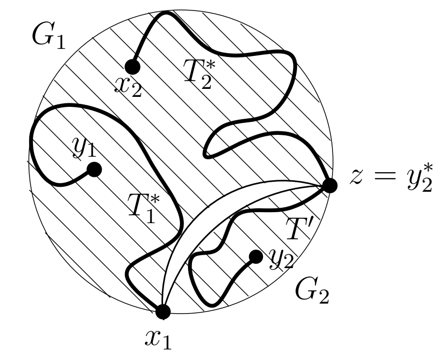

Figure 5: Case 2.1 in the proof of Claim 1.

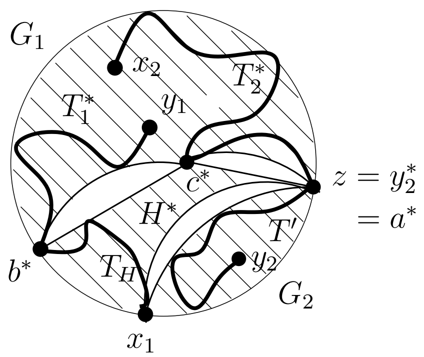

Figure 6: Case 2.2 in the proof of Claim 1.

We will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G. Let B be a $(T_1 \cup T_2)$-bridge of G. Note that B is either (i) a $(T_1^* \cup T_2^*)$-bridge of G_1^*, or (ii) a T'-bridge of G_2^*. Since $T_1^* \cup T_2^*$ is a C_1^*-Tutte subgraph in G_1^* and T' is a C_2^*-Tutte path in G_2^*, B has at most three attachments. Moreover, suppose that B contains an edge of C. Recall that $E(C) \cap E(G_1^*) \subset E(C_1^*)$, and $E(C) \cap E(G_2^*) \subset E(C_2^*)$. Thus, B contains an edge of C_1^* if B satisfies (i), or an edge of C_2^* if B satisfies (ii). This implies that B has exactly two attachments, since $T_1^* \cup T_2^*$ is a C_1^*-Tutte subgraph in G_1^* and T' is a C_2^*-Tutte path in G_2^*. Thus, $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Case 2.2. G_1^* satisfies (T2).

Let T_1^* and T_2^* be the two vertex disjoint paths and H^* be the C_1^*-flap with attachments a^*, b^*, c^* and base path P^* satisfying the conditions in (T2) for G_1^*. In this case, if $a^* \neq z$, then since $f_1^* = zx_1$ and a^*, f_1^*, x_1 appear in P^* in this order, z is contained in $P^* - \{a^*, x_1\}$, which contradicts that $z = y_2^* \in V(T_1^* \cup T_2^*)$. Thus, we have $a^* = z = y_2^*$.

By symmetry, we may assume that b^* is an end vertex of T_1^*. By Lemma 13, H^* has a path T_H from b^* to x_1 with $a^*, c^* \notin V(T_H)$ such that T_H together with a^* and c^* is a $C_1^*[x_1, b^*]$-Tutte subgraph in H^*. On the other hand, by Theorem 12, G_2^* has a C_2^*-Tutte path T' from x_1 to y_2 through x_1z. Let $T_1 = T_1^* \cup T_H$, and let $T_2 = T_2^* \cup (T' - \{x_1\})$. See Figure 6. Then T_1 and T_2 are two vertex disjoint paths in G connecting $\{x_1, x_2\}$ and $\{y_1, y_2\}$.

We will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G. Let B be a $(T_1 \cup T_2)$-bridge of G. Then B is either (i) a $(T_1^* \cup T_2^*)$-bridge of $G_1^* - (V(H^*) - \{a^*, b^*, c^*\})$, or (ii) a T'-bridge of G_2^*, or (iii) a $(T_H \cup \{a^*, c^*\})$-bridge of H^*. If B satisfies (i) or (ii), in the same way as Case 2.1, we can show that B has at most three attachments, and B has exactly two attachments if B contains an edge of C. Even in the case where B satisfies (iii), since T_H together with a^* and c^* is a $C_1^*[x_1, b^*]$-Tutte subgraph in H^* and $C \cap H^* = C_1^*[x_1, b^*]$, B has at most three attachments, and B has exactly two attachments if B contains an edge of C. These imply that $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1). This completes the proof of Case 2.

Case 3. $y_1, y_2 \in V(G_1) - V(G_2)$.

If $f \in E(G_1)$, then let $f_1^* = f$; otherwise, that is, if $f \in E(G_2)$, then let $f_1^* = x_1z \in E(G_1^*)$. By the induction hypothesis with respect to G_1^*, C_1^*, x_1, x_2, y_1, y_2 and

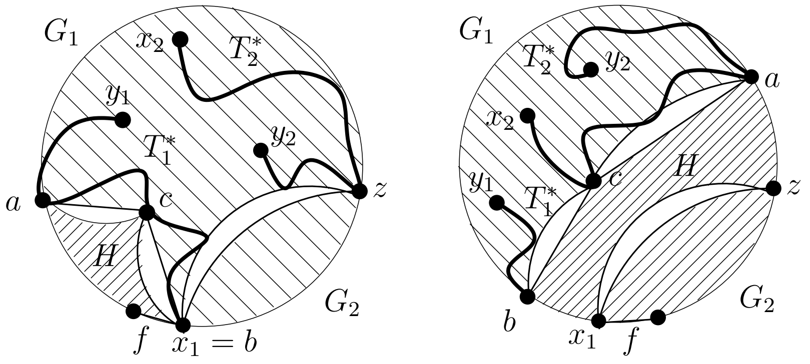

Figure 7: Case 3.1 in the proof of Claim 1.

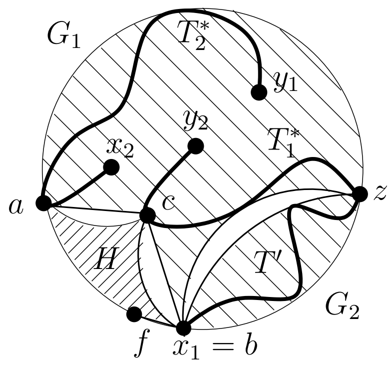

Figure 8: Case 3.2 in the proof of Claim 1.

f_1^*, we obtain that G_1^* satisfies either (T1) or (T2). When G_1^* satisfies (T1), let T_1^* and T_2^* be the two vertex disjoint paths satisfying the conditions in (T1). On the other hand, when G_1^* satisfies (T2), let T_1^* and T_2^* be the two vertex disjoint paths and H^* be the C_1^*-flap with attachments a^*, b^*, c^* and base path P^* satisfying the conditions in (T2). Now we consider two subcases.

Case 3.1 $x_1 z \notin E(T_1^* \cup T_2^*)$.

Let $T_1 = T_1^*$ and $T_2 = T_2^*$.

Suppose first that G_1^* satisfies (T1). Then the $(T_1 \cup T_2)$-bridge of G containing G_2 is the unique one which is not a $(T_1^* \cup T_2^*)$-bridge of G_1^*, but the attachments of it are not changed (one of them is x_1). Hence G also satisfies (T1).

Suppose next that G_1^* satisfies (T2). If $x_1 z \notin E(H^*)$, then let $H = H^*$ and $P = P^*$; otherwise let H be the subgraph of G induced by $\big(E(H^*) - \{x_1 z\}\big) \cup E(G_2)$ and $P = \big(P^* - \{x_1 z\}\big) \cup C[z, x_1]$. See Figure 7. (The left side of Figure 7 represents the former case, while the right side represents the latter. In the former case, $x_1 = b^*$, since $x_1 z \notin E(H^*)$, P^* contains f_1^*, and a^*, f_1^*, x_1 appear in P^* in this order.) In either case, H still has exactly three attachments a^*, b^*, c^*, and hence H is a C-flap in G with attachments a, b, c and base path P, where $a = a^*$, $b = b^*$ and $c = c^*$. Moreover, we can show that G satisfies (T2).

Case 3.2. $x_1 z \in E(T_1^* \cup T_2^*)$.

By symmetry, we may assume that $x_1 z \in E(T_1^*)$. Let $T_2 = T_2^*$. Using Theorem 12 with specifying an arbitrary edge in $C_2^* - \{x_1 z\}$ as e, we can show that there exists a C_2^*-Tutte path T' in G_2^* from x_1 to z with $x_1 z \notin E(T')$. Let $T_1 = \big(T_1^* - \{x_1 z\}\big) \cup T'$.

Suppose first that G_1^* satisfies (T1). Then every $(T_1 \cup T_2)$-bridge B of G is either a $(T_1^* \cup T_2^*)$-bridge of G_1^*, or a T'-bridge of G_2^*. By the conditions of T_1^*, T_2^* and T', B has at most three attachments and exactly two attachments if B contains an edge of C_1^* or C_2^*. Since $E(C) \subset E(C_1^*) \cup E(C_2^*)$, this implies that $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Suppose next that G_1^* satisfies (T2). Since $x_1 z \in E(T_1^* \cup T_2^*)$ and a^*, f_1^*, x_1 appear in P^* in this order, note that $b^* = x_1$. Let $H = H^*$, $a = a^*$, $b = b^*$, $c = c^*$, and $P = P^*$. See Figure 8. Then H is a C-flap with attachments a, b, c and base path P. By the same argument as above, we can show that $T_1 \cup T_2$ is a C-Tutte subgraph in $G - \big(V(H) - \{a, b, c\}\big)$. Hence G satisfies (T2).

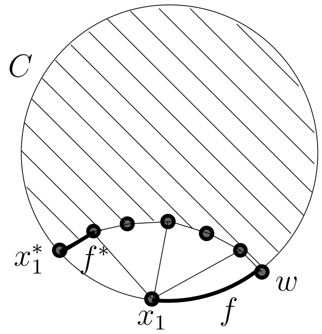

Figure 9: The vertex x_1^*, and the edge f^*

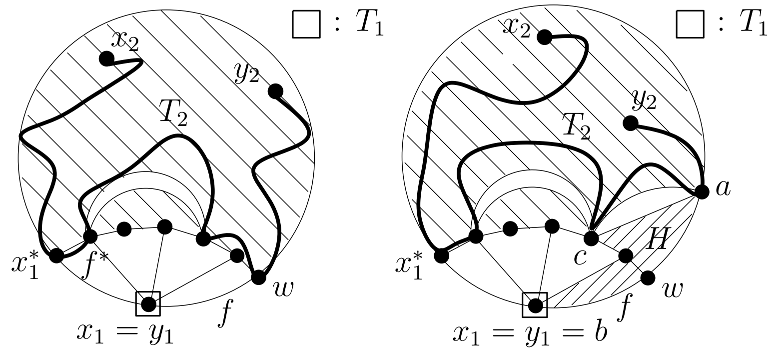

Figure 10: The proof of Claim 2.

This completes the proof of Case 3 and Claim 1. $\square$

Let $G^* = G - x_1$. By Claim 1, we may assume that G^* is a 2-connected plane graph. Let C^* be the (unique) facial cycle of G^* which is not facial in G. Let w and x_1^* be the neighbors of x_1 in C with $x_1w = f$ and $x_1^*x_1 \neq f$. By symmetry, we may assume that w, f, x_1, x_1^* appear in C in this clockwise order. This implies that $C^*[x_1^*, w] = C[x_1^*, w]$. If $V(C^*[w, x_1^*]) \cap V(C) - \{x_1^*, w\} \neq \emptyset$, say $z \in V(C^*[w, x_1^*]) \cap V(C) - \{x_1^*, w\}$, then x_1 and z form a 2-cut of G, and we are done by Claim 1. Thus, we may assume that $V(C^*[w, x_1^*]) \cap V(C) - \{x_1^*, w\} = \emptyset$. In particular, $E(C^*[w, x_1^*]) \cap E(C) = \emptyset$. Let f^* be the edge of C^* such that f^* is incident with x_1^* and $f^* \notin E(C)$. See Figure 9.

Claim 2 *If $x_1 = y_1$ or $x_1 = y_2$, then G satisfies (T1) or (T2).*

Proof. Suppose that $x_1 = y_1$ or $x_1 = y_2$, say $x_1 = y_1$ by symmetry. Let T_1 be the path in G consisting of only $x_1 = y_1$. By Theorem 12, G^* has a C^*-Tutte path T_2 from x_2 to y_2 through f^*. Note that T_1 and T_2 are two vertex disjoint paths in G connecting $\{x_1, x_2\}$ and $\{y_1, y_2\}$.

Suppose first that $w \in V(T_2)$. See the left side of Figure 10. In this case, we will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G, which shows that G satisfies (T1). Let B be a $(T_1 \cup T_2)$-bridge of G. If x_1 is not an attachment of B, then B is also a T_2-bridge of G^*. Hence B has at most three attachments and exactly two attachments if B contains an edge of C^*. On the other hand, suppose that x_1 is an attachment of B. Then $B - x_1$ is a T_2-bridge of G^* containing an edge of $C^*[w, x_1^*]$, and hence $B - x_1$ has exactly two attachments in G^*, in particular, in $C^*[w, x_1^*] - \{x_1^*\}$ since $f^* \in E(T_2)$ and $w \in V(T_2)$. Let z_1 and z_2 be the attachments of $B - x_1$, and by symmetry, we may assume that z_1 is closer to w in $C^*[w, x_1^*]$. Then it follows from the choice of z_2 and the fact $f^* \in E(T_2)$, $z_2 \in V(C^*[w, x_1^*]) - \{w, x_1^*\}$. Since $V(C^*[w, x_1^*]) \cap V(C) - \{x_1^*, w\} = \emptyset$, we have $z_2 \notin V(C)$. If $E(B - x_1) \cap E(C[x_1^*, w]) \neq \emptyset$, then since $B - x_1$ has only two attachments z_1 and z_2 in G^*, both of z_1 and z_2 must be contained in $C[x_1^*, w]$, contradicting that $z_2 \notin V(C)$. Hence $E(B - x_1) \cap E(C[x_1^*, w]) = \emptyset$. Since $E(C^*[w, x_1^*]) \cap E(C) = \emptyset$, $B - x_1$ contains no edge in C, so neither does B. Thus, $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Suppose next that $w \notin V(T_2)$. Let H be the (unique) $(T_1 \cup T_2)$-bridge of G containing w. See the right side of Figure 10. Note that x_1 is an attachment of H, and $H - x_1$ is a T_2-bridge of G^*. Since $H - x_1$ contains an edge of C^*, $H - x_1$ has exactly two attachments in G^*. In particular, both of the two attachments of $H - x_1$ are contained in C^*, since G^* is a 2-connected plane graph. Since $w \in V(H)$ and

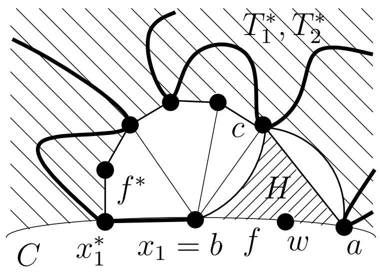

Figure 11: Case I-ii.

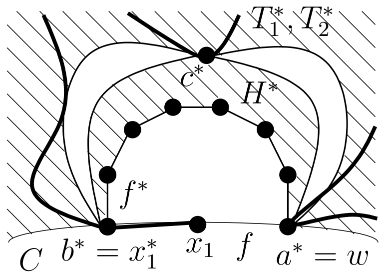

Figure 12: Case II-i.

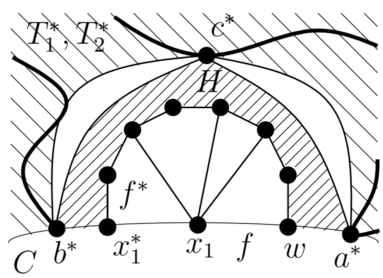

Figure 13: Case II-ii.

$f^* \notin E(H)$, one of the attachments of $H - x_1$ is contained in $V(C^*[x_1^*, w]) - \{w\}$, say a, and the other is contained in $V(C^*[w, x_1^*]) - \{w, x_1^*\}$, say c. Let $b = x_1$ and $P = C[a, b]$. Note that H is a C-flap of G with attachments a, b, c and base path P such that $a, c \in V(T_1 \cup T_2)$, $x_1 \in \big(V(P) - \{a\}\big) \cup \{b\}$, P contains $f = wx_1$, and a, f, x_1 appear in P in this order. By the same argument as above, $T_1 \cup T_2$ is a C-Tutte subgraph in $G - \big(V(H) - \{a, b, c\}\big)$. Hence G satisfies (T2).

This completes the proof of Claim 2. □

By Claim 2, we may assume that $x_1 \neq y_1$ and $x_1 \neq y_2$. By the induction hypothesis with respect to G^*, C^*, x_1^*, x_2, y_1, y_2 and f^*, we obtain that G^* satisfies either (T1) or (T2). We divide the rest of the proof into three cases.

Case I. G^* satisfies (T1).

Let T_1^* and T_2^* be the two vertex disjoint paths satisfying the conditions in (T1) for G^*. By symmetry, we may assume that x_2 is an end vertex of T_2^*. Let $T_2 = T_2^*$. In Case I, let $T_1 = T_1^* \cup \{x_1 x_1^*\}$. We also divide this case into two subcases, regarding the condition on w.

Case I-i. $w \in V(T_1 \cup T_2)$.

In this case, we will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G, which shows that G satisfies (T1). Let B be a $\big(T_1 \cup T_2\big)$-bridge of G. If x_1 is not an attachment of B, then B is also a $\big(T_1^* \cup T_2^*\big)$-bridge of G^*, and hence B has at most three attachments and exactly two attachments if B contains an edge of C^*. On the other hand, suppose that x_1 is an attachment of B. Then $B - x_1$ is a $\big(T_1^* \cup T_2^*\big)$-bridge of G^* containing an edge of $C^*[w, x_1^*]$, and hence $B - x_1$ has exactly two attachments in G^*, in particular, in $C^*[w, x_1^*]$ since $w, x_1^* \in V(T_1^* \cup T_2^*)$. Since $V(C^*[w, x_1^*]) \cap V(C) - \{x_1^*, w\} = \emptyset$, we have $E(B) \cap E(C^*[x_1^*, w]) = \emptyset$. Since $E(C^*[w, x_1^*]) \cap E(C) = \emptyset$, B contains no edge in C. Thus, $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Case I-ii. $w \notin V(T_1 \cup T_2)$.

Let H be the (unique) $\big(T_1 \cup T_2\big)$-bridge of G containing w. Note that x_1 is an attachment of H, and $H - x_1$ is a $\big(T_1^* \cup T_2^*\big)$-bridge of G^*. Since $H - x_1$ contains an edge of $C^*[w, x_1^*]$, $H - x_1$ has exactly two attachments in G^*. Since $w \in V(H)$ and $x_1^* \in V(T_1^*)$, one of the attachments of $H - x_1$ is contained in $V(C^*[x_1^*, w]) - \{w\}$, say a, and the other is contained in $V(C^*[w, x_1^*]) - \{w\}$, say c. Let $b = x_1$ and $P = C[a, b]$. Note that H is a C-flap of G with attachments a, b, c and base path P such that $a, c \in V(T_1 \cup T_2)$, $x_1 \in \big(V(P) - \{a\}\big) \cup \{b\}$, P contains $f = wx_1$, and a, f, x_1 appear in P in this order. By the same argument as above, $T_1 \cup T_2$ is a C-Tutte subgraph in $G - \big(V(H) - \{a,b,c\}\big)$. Hence G satisfies (T2). This completes the proof of Case I.

In the remaining two cases, we will deal with the case where G^* satisfies (T2). Let T_1^* and T_2^* be the two vertex disjoint paths and H^* be the C^*-flap with attachments a^*, b^*, c^* and base path P^* satisfying the conditions in (T2) for G^*. By symmetry, we may assume that x_2 is an end vertex of T_2^*. Let $T_2 = T_2^*$.

Since a^*, f^*, x_1^* appear in P^* in this order and we assumed that w, f, x_1, x_1^* appear in C in this clockwise order, we have that P^* connects clockwise a^* and b^*, that is, $P^* = C^*[a^*, b^*]$. Moreover, since f^* is incident with x_1^*, we have only two possible clockwise orders of a^*, f^*, x_1^* and w in C^*; a^*, w, f^*, x_1^* (Case II), and a^*, f^*, x_1^*, w (Case III). Here we regard the case $a^* = w$ as Case II.

Case II. G^* satisfies (T2), and a^*, w, f^*, x_1^* appear in C^* in this clockwise order or $a^* = w$.

Let H be the subgraph of G induced by the edges in H^* and all edges incident with x_1 in G. Let $a = a^*$, $b = b^*$, $c = c^*$ and $P = C[a,b]$. Note that H satisfies the conditions on a C-flap of G with attachments a, b, c and base path P, unless $H - \{a,b,c\}$ is not connected. (Indeed, in the exceptional case, H is not an $\{a,b,c\}$-bridge of G.) In particular, when $H - \{a,b,c\}$ is connected, H is a C-flap satisfying the conditions desired in (T2). See Figures 12 and 13. Thus, we further divide Case II into two subcases, to distinguish the above two cases.

Case II-i. $a^* = w$, $b^* = x_1^*$ and w and x_1^* are the only neighbors of x_1 in G.

In this case, $H - \{a^*, b^*, c^*\}$ is not connected. See Figure 12. Let $T_1 = T_1^* \cup \{x_1^* x_1\}$. Since $b^* = x_1^*$, T_1 is a path in G of which one of the end vertices is x_1. Since x_1 is of degree two, x_1 is not an attachment of H^*. Hence H^* is a $\big(T_1 \cup T_2\big)$-bridge of G having exactly three attachments. Notice that H^* does not contain an edge in C. Moreover, every non-trivial $\big(T_1 \cup T_2\big)$-bridge of G is a $\big(T_1^* \cup T_2^*\big)$-bridge of G^*. So, $T_1 \cup T_2$ is a C-Tutte subgraph in G, and hence G satisfies (T1).

Case II-ii. $a^* \neq w$ or $b^* \neq x_1^*$ or x_1 has a neighbor in G other than w and x_1^*.

In this case, let $T_1 = T_1^*$. See Figure 13. By the conditions of Case II-ii, $H - \{a^*, b^*, c^*\}$ is connected, and hence H is a C-flap with attachments a, b, c and base path P. Moreover, we can check that G satisfies (T2). This completes the proof of Case II.

Case III. G^* satisfies (T2) and a^*, f^*, x_1^*, w appear in C^* in this clockwise order.

In this case, depending on the place of the vertex b^*, we have two cases; b^* is placed in $C^*[x_1^*, w]$ or in $C^*[w, a^*] - \{w, a^*\}$. See the left side of Figure 14. (The top corresponds to the case where b^* is placed in $C^*[x_1^*, w]$, and the bottom one corresponds to the other case. In the latter case, H^* is connected through the two sides of the figure.)

Let $\widetilde{H}$ be the graph obtained from H^* by adding the vertex x_1, all the edges in G connecting x_1 and a vertex in $C^*[a^*, x_1^*]$ and edges b^*c^*, a^*c^* and $x_1 a^*$. Notice that when b^* is placed in $C^*[w, a^*] - \{w, a^*\}$, we do not add any edges of G connecting x_1 and a vertex in $C[w, b^*]$. Note also that $\widetilde{H}$ is 2-connected plane graph. Let $\widetilde{C}$ be the facial cycle of $\widetilde{H}$ consisting of the edges in $C^*[x_1^*, b^*] \cup \{b^*c^*, c^*a^*, a^*x_1, x_1x_1^*\}$. Then by

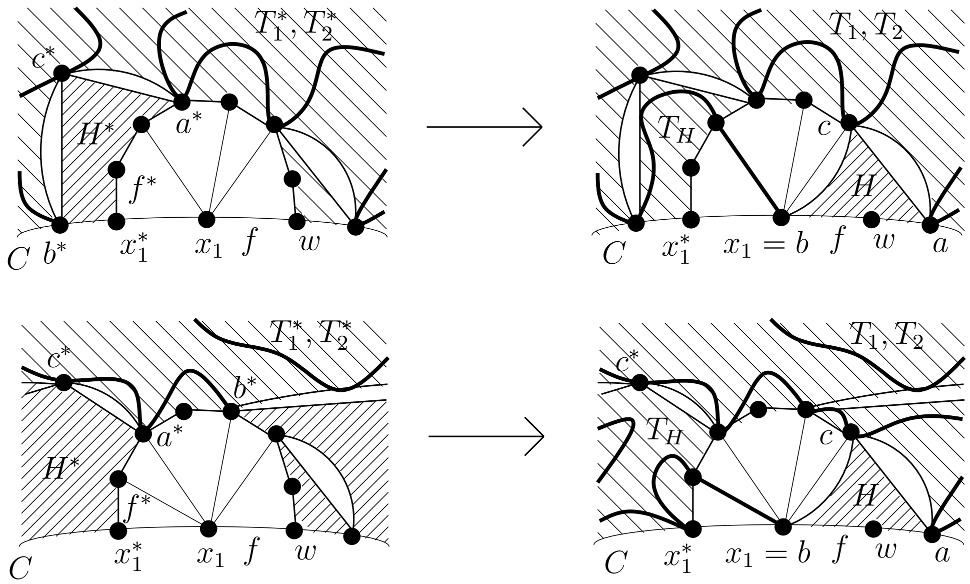

Figure 14: Case III-ii.

Lemma 13, there exists a $\widetilde{C}[x_1, b^*]$-Tutte subgraph in $\widetilde{H}$ consisting of the two vertices a^* and c^* and a path T_H from b^* to x_1 with $a^*, c^* \notin V(T_H)$. Let $T_1 = T_1^* \cup T_H$.

We will show that T_1 and T_2 have the properties required in (T1) or (T2).

Case III-i. $w \in V(T_1 \cup T_2)$.

In this case, we will show that $T_1 \cup T_2$ is a C-Tutte subgraph in G, which shows that G satisfies (T1). Let B be a $(T_1 \cup T_2)$-bridge of G. Note that B is either (i) a $(T_1^* \cup T_2^* \cup \{x_1\})$-bridge of $G - (V(H^*) - \{a^*, b^*, c^*\})$, or (ii) a $(T_H \cup \{a^*, c^*\})$-bridge of $G[V(H^*) \cup \{x_1\}]$, where $G[V(H^*) \cup \{x_1\}]$ is the subgraph of G induced by $V(H^*) \cup \{x_1\}$.

Suppose first that B satisfies (i). In the case where x_1 is not an attachment of B, B is also a $(T_1^* \cup T_2^*)$-bridge of $G^* - (V(H^*) - \{a^*, b^*, c^*\})$. Since $T_1^* \cup T_2^*$ is a C^*-Tutte subgraph in $G^* - (V(H^*) - \{a^*, b^*, c^*\})$, B has at most three attachments and two attachments if B contains an edge of $C - \{x_1 w, x_1 x_1^*\} \subset C^*$. On the other hand, in the case where x_1 is an attachment of B, B contains an edge of $C^*[w, x_1^*] \subset C^*$, so B has three attachments on $T_1 \cup T_2$, that is, two of them are on $T_1^* \cup T_2^*$ and the other is x_1. Since $E(C^*[w, x_1^*]) \cap E(C) = \emptyset$, B contains no edge in C.

Suppose next that B satisfies (ii). Since $T_H \cup \{a^*, c^*\}$ is a $\widetilde{C}[x_1, b^*]$-Tutte subgraph in $\widetilde{H}$, B has at most three attachments and at most two attachments if B contains an edge of $C \cap \widetilde{H} = \widetilde{C}[x_1, b^*]$. In either case, B has at most three attachments, and at most two attachments if B contains an edge of C. Thus, $T_1 \cup T_2$ is a C-Tutte subgraph of G. Thus, G satisfies (T1).

Case III-ii. $w \notin V(T_1 \cup T_2)$.

In this case, see Figure 14.

Let H be the $(T_1 \cup T_2)$-bridge of G containing w. Note that x_1 is an attachment of H. Then $H - x_1$ is either a $(T_1^* \cup T_2^*)$-bridge of $G^* - (V(H^*) - \{a^*, b^*, c^*\})$ containing an edge of C^* or a $(T_H \cup \{a^*, c^*\})$-bridge of $\widetilde{H}$ containing an edge of $\widetilde{C}[x_1, b^*]$. This implies that $H - x_1$ has exactly two attachments. In particular, since $x_1^* \in V\big(T_1 \cup T_2\big)$ and $w \notin V\big(T_1 \cup T_2\big)$, one of the attachments, say a, is contained in $C^*[x_1^*, w] - \{w\}$, and the other, say c, is contained in $C^*[w, x_1^*] - \{w\}$. Let $b = x_1$ and $P = C[a, b]$. Note that H is a C-flap of G with attachments a, b, c and base path P such that $x_1 \in \big(V(P) - \{a\}\big) \cup \{b\}$, P contains $f = wx_1$, and a, f, x_1 appear in P in this order.

By the same argument as in Case III-i, we can show that $T_1 \cup T_2$ is a C-Tutte subgraph of $G - \big(V(H) - \{a, b, c\}\big)$. Hence G satisfies (T2).

This completes the proof of Theorem 11. $\square$

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