

# Logarithmic Space and Permutations<sup>☆</sup>

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## Abstract

In a recent work, Girard [1] proposed a new and innovative approach to computational complexity based on the proofs-as-programs correspondence. In a previous paper [2], the authors showed how Girard proposal succeeds in obtaining a new characterization of co-NL languages as a set of operators acting on a Hilbert Space. In this paper, we extend this work by showing that it is also possible to define a set of operators characterizing the class L of logarithmic space languages.

*Keywords:* Linear Logic, Complexity, Geometry of Interaction, Pointer Machine, Finite Automata, Logarithmic Space

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## 1. Introduction

### 1.1. Linear Logic and Implicit Computational Complexity

Logic, and more precisely proof theory — the domain whose purpose is the formalization and study of mathematical proofs — recently yielded numerous developments in theoretical computer science. These developments are founded on a correspondence, often called Curry-Howard correspondence, between mathematical proofs and computer programs (usually formalized in lambda-calculus). The main interest of this correspondence lies in its dynamic nature: program execution corresponds to a procedure on mathematical proofs known as the cut-elimination procedure.

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In the eighties, Jean-Yves Girard discovered linear logic through a study of mathematical models of the lambda-calculus. This logical system, as a direct consequence of this correspondence between proofs and programs, is particularly interesting from the point of view of the mathematical foundations of computer science for its resource-awareness. In particular, it gave birth to a number of developments in the field of implicit computational complexity, for instance through the definition and study of restricted logical systems (sub-systems of linear logic) in which the set of representable functions captures a complexity class. For instance, elementary linear logic (ELL) restricts the rules governing the use of exponential connectives — the connectives dealing with the duplication of the arguments of a function — and the set of representable functions in ELL is exactly the set of elementary time functions [3]. It was also shown [4] that a characterization of logarithmic space computation can be obtained if one restricts both the rules of exponential connectives and the use of universal quantifiers. Finally, a variation on the notion of linear logic *proof nets* succeeded in characterizing the classes **NC** of problems that can be efficiently parallelized [5, 6, 7].

### 1.2. Geometry of Interaction

A deep study of the formalization of proofs in linear logic, in particular their formalization as proof nets, lead Jean-Yves Girard to initiate a program untitled *geometry of interaction* (GoI) [8]. This program, in a first approximation, intends to define a semantics of proofs that accounts for the dynamics of the cut-elimination procedure. Through the correspondence between proofs and programs, this would define a semantics of programs that accounts for the dynamics of their execution. However, the geometry of interaction program is more ambitious: beyond the mere interpretation of proofs, its purpose is to completely reconstruct logic around the dynamics of the cut-elimination procedure. This means reconstructing the logic of programs, where the notion of formula — or type — accounts for the behavior of algorithms.

Informally, a geometry of interaction is defined by a set of *paraproofs* together with a notion of interaction, in the same way one defines strategies and their composition in game semantics. An important tool in the construction is a binary function that measures the interaction between two paraproofs. With this function one defines a notion of orthogonality that corresponds to the negation of logic and reconstructs the formulas as sets of

paraproofs equal to a given set of paraproofs: a formula is therefore a set of “programs” that interact in a similar way to a given set of *tests*.

Since the introduction of this program Jean-Yves Girard proposed several constructions to realize it. These constructions share the notion of paraproofs: operators in a von Neumann algebra. They however differ on the notion of orthogonality they use: in the first constructions, this notion was founded on the nilpotency of the product of two operators, while the more recent construction [9] uses Fuglede-Kadison determinant — a generalization of the usual determinant of matrices that can be defined in type  $II_1$  factors.

Since the reconstruction of logic is based on the notion of execution, geometry of interaction constructions are particularly interesting for the study of computational complexity. It is worth noting that the first construction of GoI [10] allowed Abadi, Gonthier, and Levy [11] to explain the optimal reduction of  $\lambda$ -calculus defined by Lamping [12]. This first GoI construction was also used to obtain results in the field of implicit computational complexity [13].

### *1.3. A new approach of complexity*

Recently Jean-Yves Girard proposed a new approach for the study of complexity classes that was inspired by his latest construction of a geometry of interaction. Using the crossed product construction between a von Neumann algebra and a group acting on it, he proposed to characterize complexity classes as sets of operators obtained through the internalization of outer automorphisms of the type  $II_1$  hyperfinite factor. The authors showed in a recent paper [2] that this approach succeeds in defining a characterization of the set of **co-NL** languages as a set of operators in the type  $II_1$  hyperfinite factor. The proof of this result was obtained through the introduction of *non-deterministic pointer machines*, which are abstract machines designed to mimic the computational behavior of operators. The result was obtained by showing that a **co-NL** complete problem could be solved by these machines.

In this paper, we extend these results in two ways. The first important contribution is that we give an alternative proof of the fact that **co-NL** is indeed characterized by non-deterministic pointer machines. This new proof consists in showing that pointer machines can simulate the well-known and studied two-way multi-head finite automata [14, 15]. The second contribution of this paper consists in obtaining a characterization of the class **L** as a set of operators in the hyperfinite factor of type  $II_1$ . By studying the set of

operators that characterize the class **co-NL** and in particular the encoding of non-deterministic pointer machines as operators, we are able to show that the operators encoding a deterministic machine satisfy a condition expressed in terms of norm. We then manage to show that the language decided by an operator satisfying this norm condition is in the class **L**, showing that the set of all such operators characterizes **L**.

## 2. The Basic Picture

The construction uses an operator-theoretic construction known as the crossed product of an algebra by a group acting on it. The interested reader can find a quick overview of the theory of von Neumann algebras in the appendix of the second author's work on geometry of interaction [16], and a brief presentation of the crossed product construction in the authors' previous work [2] on the characterization of **co-NL**. For a more complete presentation of the theory of operators and the crossed product construction, we refer to the well-known series of Takesaki [17, 18, 19].

In a nutshell, the crossed product construction  $\mathfrak{A} \rtimes_{\alpha} G$  of a von Neumann algebra  $\mathfrak{A}$  and a group action  $\alpha : G \rightarrow \text{Aut}(\mathfrak{A})$  defines a von Neumann algebra containing  $\mathfrak{A}$  and unitaries that internalize the automorphisms  $\alpha(g)$  for  $g \in G$ . For this, one considers the Hilbert space<sup>1</sup>  $\mathbb{K} = L^2(G, \mathbb{H})$  where  $\mathbb{H}$  is the Hilbert space  $\mathfrak{A}$  is acting on, and one defines two families of unitary operators<sup>2</sup> in  $\mathcal{L}(\mathbb{K})$ :

- the family  $\{\pi(u) \mid u \in \mathfrak{A} \text{ unitary}\}$  which is a representation of  $\mathfrak{A}$  on  $\mathbb{K}$ ;
- the family  $\{\lambda(g) \mid g \in G\}$  which contains unitaries that internalize the automorphisms  $\lambda(g)$ .

Then the crossed product  $\mathfrak{A} \rtimes_{\alpha} G$  is the von Neumann algebra generated by these two families.

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<sup>1</sup>The construction  $L^2(G, \mathbb{H})$  is a generalization of the well-known construction of the Hilbert space of square-summable functions: in case  $G$  is considered with the discrete topology, the elements are functions  $f : G \rightarrow \mathbb{H}$  such that  $\sum_{g \in G} \|f(g)\|^2 < \infty$ .

<sup>2</sup>Recall that in the algebra  $\mathcal{L}(\mathbb{H})$  of bounded linear operators on the Hilbert space  $\mathbb{H}$  (we denote by  $\langle \cdot, \cdot \rangle$  its inner product), there exists an anti-linear involution  $(\cdot)^*$  such that for any  $\xi, \eta \in \mathbb{H}$  and  $A \in \mathcal{L}(\mathbb{H})$ ,  $\langle A\xi, \eta \rangle = \langle \xi, A^* \eta \rangle$ . This *adjoint operator* coincides with the conjugate-transpose in the algebras of square matrices. A *unitary operator*  $u$  is an operator such that  $uu^* = u^*u = 1$ .

As a by-product of Girard's work on Geometry of Interaction [9], we know<sup>3</sup> how to represent integers as operators in the type  $\text{II}_1$  hyperfinite factor  $\mathfrak{R}$ .

This representation comes from logical considerations, and it has some specificities, among which the fact that the integer is represented as a *circular* binary string, and the fact that every bit is considered as having an input and an output<sup>4</sup>. This representation arises from the interpretation of proofs of the type  $\forall X!(X \multimap X) \multimap !(X \multimap X) \multimap !(X \multimap X)$  (the type of binary lists in Elementary Linear Logic) in the setting of Geometry of Interaction (see [2] for a detailed explanation of the representation of integers). The obtained interpretation is the image of a matrix through an embedding of the matrix algebra in which it is contained in the hyperfinite factor. Since there are no natural choice of a particular embedding, a given integer does not have a unique representation in  $\mathfrak{R}$ , but a family of representations. Luckily, any two representations of a given integer are unitarily equivalent (we denote by  $\mathfrak{M}_k(\mathbf{C})$  the algebra of  $k \times k$  matrices over  $\mathbf{C}$ ):

**Proposition 1** ([2, Proposition 9]). *Let  $N, N' \in \mathfrak{M}_k(\mathbf{C}) \otimes \mathfrak{R}$  be two representations of a given integer. There exists a unitary  $u \in \mathfrak{R}$  such that:*

$$N = (1 \otimes u)^* N' (1 \otimes u)$$

The next step is then to define a representation of machines, or algorithms, as operators that do not discriminate two distinct representations of a given integer, i.e. operators that act uniformly on the set of representations of a given integer. As the authors shown in their previous work, the crossed product construction allows one to characterize an algebra of operators acting in such a uniform way. We recall the construction in the particular case of the group of finite permutations, which will be the only setting that will be studied in this work.

The algebra  $\mathfrak{R}$  can be embedded in the infinite tensor product algebra  $\mathfrak{T} = \bigotimes_{n \in \mathbf{N}} \mathfrak{R}$  through the morphism  $\iota_0 : x \mapsto x \otimes 1 \otimes 1 \otimes \dots$ . We will denote by  $\mathfrak{N}$  the image of  $\mathfrak{R}$  in  $\mathfrak{T}$ . Notice that this tensor product algebra is isomorphic

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<sup>3</sup>A detailed explanation of this representation can be found in the authors' previous work [2] or in the second author's PhD thesis [20].

<sup>4</sup>Something that will be proven useful in Section 5, for it will help us to determine in which direction the integer is read.

to  $\mathfrak{A}$ . The idea of Girard is then to use the action of the group  $\mathcal{S}$  of finite permutations of  $\mathbf{N}$  onto  $\mathfrak{T}$ , defined by:

$$\sigma.(x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes \dots) = x_{\sigma^{-1}(0)} \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(k)} \otimes \dots$$

This group action defines a sub-algebra  $\mathfrak{G}$  of the crossed product algebra  $\mathfrak{K} = (\otimes_{n \in \mathbf{N}} \mathfrak{A}) \rtimes \mathcal{S}$ . This algebra  $\mathfrak{G}$  is the algebra generated by the family of unitaries  $\lambda(\sigma)$  for  $\sigma \in \mathcal{S}$ , and we think of it as the algebra of machines, or algorithms. As it turns out, the elements of this algebra act uniformly on the set of representations of a given integer:

**Proposition 2** ([2, Proposition 11]). *Let  $N, N' \in \mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{T}$  be two representations of a given integer, and  $\phi \in \mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{G}$ . Then:*

$$\phi N \text{ is nilpotent iff } \phi N' \text{ is nilpotent}$$

To be a bit more precise, we will be interested in the elements of the algebras of operators  $\mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{G} \otimes \mathfrak{M}_k(\mathbf{C})$ . By tensoring with a matrix algebra, which we call the *algebra of states*, we will be able to develop more computational power. An element of one of these algebras will be called an *observation*, and it still acts uniformly on distinct representations  $N_n, N'_n$  of a given integer: if  $\phi$  is an observation,  $\phi(N_n \otimes 1_{\mathfrak{M}_k(\mathbf{C})})$  is nilpotent if and only if  $\phi(N'_n \otimes 1_{\mathfrak{M}_k(\mathbf{C})})$  is nilpotent<sup>5</sup>. From this proposition, one can justify that the following definition makes sense:

**Definition 3.** Let  $\phi \in \mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{G} \otimes \mathfrak{M}_k(\mathbf{C})$  be an observation. We define the language accepted by  $\phi$  by  $(N_n$  denotes any representation of the integer  $n$ ):

$$[\phi] = \{n \in \mathbf{N} \mid \phi(N_n \otimes 1) \text{ nilpotent}\}$$

By extension, if  $P$  is a set of observations, we denote by  $[P]$  the set  $\{[\phi] \mid \phi \in P\}$ .

To sum up the construction, one has the following objects:

- an algebra containing representations of integers: the hyperfinite factor  $\mathfrak{A}$  of type  $\text{II}_1$ , embedded in  $\mathfrak{K}$  through the morphism  $\pi \circ \iota_0$ ;

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<sup>5</sup>We will in the following simply write 1 for the identity element of  $\mathfrak{M}_k(\mathbf{C})$ .

- an algebra containing the representations of machines, or algorithms: the von Neumann sub-algebra  $\mathfrak{G}$  of  $\mathfrak{K}$  generated by the set of unitaries  $\{\lambda(\sigma) \mid \sigma \in \mathcal{S}\}$ ;
- a notion of acceptance: if  $N$  is a representation of a integer and  $\phi$  is a representation of a machine, we say that  $\phi$  accepts  $N$  when  $\phi N$  is nilpotent.

### 3. Pointer Machines

We previously introduced [2] non-deterministic pointers machines that were designed to mimic the computational behavior of operators: they do not have the ability to write, their input tape is cyclic, and they are “universally non-deterministic”, i.e. rejection is meaningful whereas acceptance is the default behavior. We exhibited a non-deterministic pointer machine that decides a **co-NL**-complete language and showed how a reduction could be performed with pointers. Here, we complete this result by simulating multi-head finite automata [21], a method that allows us to get a deeper understanding of pointer machines, in both their deterministic and non-deterministic versions.

A pointer machine is given by a set of pointers that can move back and forth on the input tape and read (but not write) the values it contains, together with a set of states. For  $1 \leq i \leq p$ , given a pointer  $p_i$ , only one of three different *instructions* can be performed at each step:  $p_i+$ , i.e. “move one step forward”,  $p_i-$ , i.e. “move one step backward” and  $\epsilon_i$ , i.e. “do not move”. In the following definition, we let  $I_{\{1, \dots, p\}} = \{p_i+, p_i-, \epsilon_i \mid i \in \{1, \dots, p\}\}$  be the set of instructions and  $\Sigma = \{0, 1, \star\}$  be the alphabet. We will denote by  $\#p_i$  the value of the pointer (the address it points at).

**Definition 4.** A non-deterministic pointer machine with  $p \in \mathbf{N}^*$  pointers is a couple  $M = \{Q, \rightarrow\}$  where  $Q$  is the set of *states* and  $\rightarrow \subseteq (\Sigma^p \times Q) \times ((I_{\{1, \dots, p\}}^p \times Q) \cup \{\mathbf{accept}, \mathbf{reject}\})$  is the *transition relation* and it is total. We write  $\text{NDPM}(p)$  the set of non-deterministic pointer machines with  $p$  pointers.

A configuration of  $M \in \text{NDPM}(p)$  is as usual a “snapshot” of  $M$  at a given time, and we define a *pseudo-configuration*  $c$  of  $M \in \text{NDPM}(p)$  as a “partial snapshot”:  $c \in \Sigma^p \times Q$  contains the last values read by the  $p$  pointers and the current state, *but does not contain the addresses of the  $p$  pointers*. It is therefore impossible to resume the computation from a pseudo-configuration

provided the description of  $M$  and the input, but one can know what would be the set of instructions and the new state resulting from the application of  $\rightarrow$ . The set of pseudo-configurations of a machine  $M$  is written  $C_M$  and it is the domain of the transition relation. If  $\rightarrow$  is functional,  $M$  is a *deterministic pointer machine*. We write  $\text{DPM}(p)$  the set of deterministic pointer machines with  $p$  pointers.

Let  $M \in \text{NDPM}(p)$ ,  $s \in C_M$  and  $n \in \mathbf{N}$  an input. We define  $M_s(n)$  as  $M$  with  $n$  encoded as a string on its circular input tape (as  $\star a_1 \dots a_k$  for  $a_1 \dots a_k$  the binary encoding of  $n$  and  $a_{k+1} = a_0 = \star$ ) starting in the pseudo-configuration  $s$  with  $\#p_i = 0$  for all  $1 \leq i \leq p$  (that is, the pointers are initialized with the address of the symbol  $\star$ ). The pointers may be considered as variables that have been declared but not initialized yet. They are associated with *memory slots* that store the values and are updated only when the pointer moves, so as the pointers did not moved yet, those memory slots haven't been initialized. The *initial pseudo-configuration*  $s$  initializes those  $p$  registers, not necessarily in a faithful way (it may not reflect the values contained at  $\#p_i$ ). The entry  $n$  is *accepted* (resp. *rejected*) by  $M$  with *initial pseudo-configuration*  $s \in C_M$  if after a finite number of transitions every branch of  $M_s(n)$  reaches **accept** (resp. at least a branch of  $M$  reaches **reject**). We say that  $M_s(n)$  halts if it accepts or rejects  $n$  and that  $M$  decides a set  $S$  if there exists a pseudo-configuration  $s \in C_M$  such that  $M_s(n)$  accepts if and only if  $n \in S$ .

As we will see, this notion of non-deterministic pointer machines is quite similar to the classical notion of multi-head finite automata:

**Definition 5.** [15, Definition 1] For  $k \in \mathbf{N}^*$ , a non-deterministic two-way  $k$ -head finite automaton is a tuple  $M = \{S, A, k, \triangleright, \triangleleft, s_0, F, \sigma\}$  where:

- $S$  is the finite set of *states*;
- $A$  is the *alphabet*<sup>6</sup>;
- $k$  is the number of heads;
- $\triangleright$  and  $\triangleleft$  are the *left and right endmarkers*;
- $s_0 \in S$  is the *initial state*;

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<sup>6</sup>We take it to be always equal to  $\{0, 1\}$ , as any alphabet is equivalent to this one modulo a reasonable translation.



- $F \subseteq S$  is the set of *accepting states*;
- $\sigma \subseteq (S \times (A \cup \{\triangleright, \triangleleft\})^k) \times (S \times \{-1, 0, 1\}^k)$  is the *transition relation*, where  $-1$  means to move the head one square to the left,  $0$  means to keep the head on the current square and  $1$  means to move it one square to the right.

Moreover, whenever<sup>7</sup>  $(s', (d_1, \dots, d_k)) \in \sigma(s, (a_1, \dots, a_k))$ , then  $a_i = \triangleright$  ( $1 \leq i \leq k$ ) implies  $d_i \in \{0, 1\}$ , and  $a_i = \triangleleft$  ( $1 \leq i \leq k$ ) implies  $d_i \in \{-1, 0\}$ .

In the following, we will denote by  $2\text{NDFFA}(k)$  the set of non-deterministic two-ways  $k$ -head finite automata.

One defines configurations and transitions in a classical way. One should however notice that the heads cannot move beyond the endmarkers because of the definition of the transition relation.

Let  $M \in 2\text{NDFFA}(k)$  and  $n \in \mathbf{N}$  an input whose binary writing is  $w$ . We say that  $M$  *accepts*  $n$  if  $M$  starts in state  $s_0$ , with  $\triangleright w \triangleleft$  written on its input tape and all of its heads on  $\triangleright$ , and if after a finite number of transitions at least one branch of  $M$  reaches a state belonging to  $F$ . We say that  $M$  *always halt* if for all input all branches of  $M$  reach after a finite number of transitions a configuration such that no transition may be applied. If  $\sigma$  is functional,  $M$  is said to be *deterministic*. We write  $2\text{DFA}(p)$  the set of deterministic two-way  $k$  heads automata.

#### 4. Simulation and first results

Multi-head finite automata were introduced in the early seventies [14] and provided many promising perspectives<sup>8</sup>. Their nice characterization of **L** and **NL** and the similarities they share with non-deterministic pointer machines will be the key to prove (anew) that non-deterministic pointer machines characterize **co-NL** and that their deterministic restriction characterizes **L**.

We will denote by **DPM**, **NDPM**, **2DFA** and **2NDFFA** the sets of languages decided by respectively the sets  $\cup_{k \in \mathbf{N}^*} \text{DPM}(k)$ ,  $\cup_{k \in \mathbf{N}^*} \text{NDPM}(k)$ ,  $\cup_{k \in \mathbf{N}^*} 2\text{DFA}(k)$  and  $\cup_{k \in \mathbf{N}^*} 2\text{NDFFA}(k)$ .

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<sup>7</sup>Of course, one may see  $\sigma$  as a partial function from  $S \times (A \cup \{\triangleright, \triangleleft\})^k$  to  $\mathcal{P}(S \times \{-1, 0, 1\}^k)$ , and that justifies this notation that we will use from now on.

<sup>8</sup>An overview of the main results and open problems of the theory was recently published [15].

**Theorem 6** ([14], p.338).  $L = 2DFA$  and  $NL = 2N DFA$ .

We will denote in the following by  $\mathcal{L}(X)$  (resp.  $\text{co} - \mathcal{L}(X)$ ) the language accepted (resp. rejected) by  $X$ .

**Proposition 7.** *For all  $M \in 2N DFA(k)$ , there exists  $M' \in 2N DFA(k')$  such that  $M'$  always halt and  $\mathcal{L}(M) = \mathcal{L}(M')$ .*

*Proof.* Given an input  $n \in \mathbf{N}$ , we know that the number of configurations of  $M$  is bounded by  $\text{Card}(S) \times (\text{Card}(A \cup \{\triangleright, \triangleleft\}))^k \times (\log_2(n) + 2)^k$ , that is to say by  $O(\log_2(n)^d)$  for  $d$  a constant fixed with  $M$ . We will construct  $M'$  so that it halts rejecting after performing more than this number of transitions.

We set  $k' = k + d + 1$ , and we construct  $M'$  so that its  $k$  first heads acts as the heads of  $M$ , and the  $d + 1$  heads act as the hands of a clock, going back and forth between the two endmarkers. For all  $s \in S$  of  $M$ ,  $S'$  contains the set of states  $s \times \{\rightarrow, \leftarrow\}^{d+1}$  that give the current direction of the  $d + 1$  heads. At every transition the  $k + 1$ -th head moves according to its direction. For  $k < i < k'$ , when the  $i$ -th head has made a round-trip on  $n$ , the  $i + 1$ -th head moves one square according to its direction. If the  $k'$ -th head has made a round-trip on  $n$  — that is, is back on  $\triangleright$  —  $M'$  halts rejecting<sup>9</sup>.

If  $M$  did not stop after  $\text{Card}(S) \times 2^k \times (\log_2(n) + 2)^k$  transitions, it means that it was stuck in a loop and therefore will never accept. By construction,  $M'$  always halts after  $\log_2(n)^{d+1}$  transitions and accepts exactly as  $M$  does, so we proved that  $\mathcal{L}(M) = \mathcal{L}(M')$  and that  $M'$  always halts. Notice that if  $M$  was deterministic, then  $M'$  is deterministic, so this result may be restricted to the deterministic case.  $\square$

**Proposition 8.** *For all  $k \in \mathbf{N}^*$ , for all  $M \in 2N DFA(k)$  (resp.  $M \in 2DFA(k)$ ) that always halts, there exists  $M' \in NDPM(k + c)$  (resp.  $M' \in 2DFA(k + c)$ ) such that  $\text{co} - \mathcal{L}(M) = \mathcal{L}(M')$ .*

*Proof.* The construction of  $M'$  is almost the same in the deterministic and the non-deterministic cases:  $M'$  takes as input the description of  $M$  as a binary list<sup>10</sup> encoding  $S \triangleright w \triangleleft \sigma s_0 F$ . Then  $M'$  is initialized by placing  $k$  pointers on  $\triangleright$ , a pointer on  $s_0$  and the last one on the first symbol of  $\sigma$ . This

<sup>9</sup>That can be simply done by halting the computation in a state not belonging to  $F$ .

<sup>10</sup>To handle this binary encoding require some additional pointers, but this is not what is at stake here.

last pointer scans  $\sigma$  to find a transition that can be applied. To simulate a transition,  $M'$  moves the  $k$  first pointers according to  $\sigma$  and the  $k + 1$ -th pointer is moved to a new state according to  $\sigma$ .

If  $M$  is non-deterministic, when a transition that may be applied is found, a non-deterministic transition takes place: on one way  $M'$  simulates this first transition, on the other way  $M'$  keeps scanning  $\sigma$  to look for another transition to apply. If  $M$  is deterministic,  $M'$  simply simulates the first transition it encounters. When no transition may be applied – and it always happens, as  $M$  always halts – ,  $M'$  checks if the current state belongs to  $S$ : if it does,  $M'$  rejects, elsewhere  $M'$  accepts.

It can be easily verified that  $M'$  accepts iff  $M$  rejects, that essentially  $k + 2q$  pointers are enough to perform this simulation, and that if  $M$  is deterministic, so is  $M'$ . Note moreover that  $M'$  always halts, as  $M$  does.  $\square$

This proposition has as an immediate corollary the inclusions **co-2DFA**  $\subseteq$  **DPM** and **co-2N DFA**  $\subseteq$  **NDPM**. By combining this result with Theorem 6 and the fact that **L** is closed under complementation<sup>11</sup>, we obtain the expected result:

**Corollary 9.** ***L**  $\subseteq$  **DPM** and **co-NL**  $\subseteq$  **NDPM**.*

The converse inclusion will be a by-product of the encoding of NDPMs into operators and won't be proved directly, but it is reasonable to think of NDPMs as “modified” 2NFAs. Those modifications are introduced to ease the simulation by operators: we already mentioned that the transition relation was total, the input circular, the importance of the initial pseudo-configuration to initialize the computation, but we should also mention that any NDPM can be modified so that it does not move more than one pointer at each transition.

## 5. Pointer Machines and Operators

### 5.1. Encoding NDPMs

In this section, we will briefly explain the encoding of non-deterministic pointer machines as operators, which was already defined in our previous

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<sup>11</sup>Using the fact that **NL** = **co-NL** [22], we could show that **NL**  $\subseteq$  **NDPM**. However, we chose not to use the well-known equality between **NL** and **co-NL** in the hope that this new approach to complexity may provide a new proof of this result. Moreover, it is easier to grasp the computation of observations as “universally non-deterministic”.

work [2].

We will begin by a definition of two families of observations.

**Definition 10.** Let  $\phi \in \mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{G} \otimes \mathfrak{M}_k(\mathbf{C})$  be an observation that we will write as a  $6k \times 6k$  matrix  $(a_{i,j})_{1 \leq i,j \leq 6k}$  with coefficients in  $\mathfrak{G}$ . We say that  $\phi$  is:

- A *positive observation* if for all  $1 \leq i, j \leq 6k$ , we have  $a_{i,j} = \sum_{m=0}^l \alpha_m^{i,j} \lambda(g_m^{i,j})$  where  $l$  is an integer, and  $\alpha_m^{i,j}$  are positive real numbers.
- A *boolean observation* if for all  $1 \leq i, j \leq 6k$ , we have  $a_{i,j} = \sum_{m=0}^l \lambda(g_m^{i,j})$  where  $l$  is an integer.

**Definition 11.** We define the following sets:

$$\begin{aligned} P_{\geq 0} &= \{\phi \mid \phi \text{ is a positive observation}\} \\ P_+ &= \{\phi \mid \phi \text{ is a boolean observation}\} \end{aligned}$$

We showed [2] that any non-deterministic pointer machine  $M$  can be encoded as an operator in  $P_+$ , implying that  $\mathbf{co-NL} \subseteq [P_+]$ .

The encoding of non-deterministic pointer machines was defined as follows: we encode each couple  $(c, t) \in \rightarrow$  by an operator  $\phi_{c,t}$ , and then the encoding of the transition relation corresponds to the sum:

$$\rightarrow^\bullet = \sum_{c \in C_M} \sum_{t \text{ s.t. } c \rightarrow t} \phi_{c,t}$$

To define the encoding correctly, we need to introduce a number of additional states. We will denote by  $Q^\dagger$  the extended set of states obtained from  $Q$ . The machine will then be encoded as an operator in  $\mathfrak{M}_6(\mathfrak{G}) \otimes \mathfrak{P}$ , where  $\mathfrak{P}$  is the algebra of *pseudo-states*, i.e. an algebra that will encode the memory slots that contain the last value read by the pointers, and the extended set of states  $Q^\dagger$ . That is:

$$\mathfrak{P} = \underbrace{\mathfrak{M}_6(\mathbf{C}) \otimes \mathfrak{M}_6(\mathbf{C}) \otimes \cdots \otimes \mathfrak{M}_6(\mathbf{C})}_{p \text{ times}} \otimes \mathfrak{M}_{\text{Card}(Q^\dagger)}(\mathbf{C})$$

The following encoding differs from the one used in our earlier work [2]. The reason for this (small) change is that the encoding of the “move forward” and “move backward” instructions were defined as operators whose norm was strictly greater than 1. Since we want the encoding of a deterministic

machine to be of norm at most 1 (see the next subsection), it is necessary to define a more convenient encoding. The encoding of acceptance and rejection are not modified, but we recall them anyway. The reader should also keep in mind that the encoding of the integer is somehow redundant, for every bit will have an input and an output: this will help us to discriminate between the movements of a pointer from right to left and the movements from left to right.

Before describing the encoding in details, we need to introduce several notations that will be used for the definition of operators. We recall that the integer is encoded as a circular string, whose start and end is  $\star$ , and that every bit has an output that is connected to the input of the following bit. So for instance “reading”  $0i$  is wondering what is the bit before the bit 0 under treatment, “reading”  $\star o$  (that we also denote  $s$ ) is wondering what is the first bit of the integer. The way the “pointer” will parse the entry is highly interactive, or dynamic, for it is always on the edge of reading the next bit. So we define the projections  $\pi_*$  of  $\mathfrak{M}_6(\mathbf{C})$  ( $*$   $\in \{0i, 0o, 1i, 1o, e, s\}$ ) as the projections on the subspace induced by the basis element<sup>12</sup>  $*$ . We will sometimes denote  $e$  (resp.  $s$ ) by  $\star i$  (resp.  $\star o$ ), and we also define the projections  $\pi_{\text{in}} = \pi_{0i} + \pi_{1i} + \pi_e$  and  $\pi_{\text{out}} = \pi_{0o} + \pi_{1o} + \pi_s$ .

For the sake of simplicity, we also define the following operators in  $\mathfrak{B}$ : if  $c = (a_1, \dots, a_p, \mathbf{q})$  and  $c' = (a'_1, \dots, a'_p, \mathbf{q}')$ , we define the partial isometry:

$$(c \rightarrow c') = (a_1 \rightarrow a'_1) \otimes \dots \otimes (a_p \rightarrow a'_p) \otimes (\mathbf{q} \rightarrow \mathbf{q}')$$

where  $(p \rightarrow p')$  (for  $p, p' \in \mathbb{Q}^\dagger$  or  $p, p' \in \{0i, 0o, 1, 1o, e, s\}$ ) is defined as:

$$p' \begin{matrix} p \\ \left( \begin{array}{ccccc} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{array} \right) \end{matrix}$$

For  $\mathbf{S}$  a set of states, we will use the notation  $(\mathbf{S} \rightarrow a'_i)$  (denoted  $(\rightarrow a'_i)$  when  $\mathbf{S}$  contains all possible states) for the sum of partial isometries  $\sum_{s \in \mathbf{S}} (s \rightarrow a'_i)$

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<sup>12</sup>To be coherent with the notations of our earlier paper, we will always denote by  $\{0i, 0o, 1i, 1o, e, s\}$  the basis of  $\mathfrak{M}_6(\mathbf{C})$  we are working with.

$$\begin{aligned}
\text{bf}_{j,\mathbf{q}}^c &= \pi_{\text{out}} \otimes \tau_{0,j} \otimes ([\{0o, 1o, \star o\} \rightarrow]_j; \mathbf{q} \rightarrow \mathbf{mov}_j^{\mathbf{q},\mathbf{q}'}) \\
\text{ff}_{j,\mathbf{q}}^c &= [\text{in} \rightarrow \text{out}] \otimes \tau_{0,j} \otimes ([\{0i, 1i, \star i\} \rightarrow]_j; \mathbf{q} \rightarrow \mathbf{mov}_j^{\mathbf{q},\mathbf{q}'}) \\
\text{fb}_{j,\mathbf{q}}^c &= \pi_{\text{in}} \otimes \tau_{0,j} \otimes ([\{0i, 1i, \star i\} \rightarrow]_j; \mathbf{q} \rightarrow \mathbf{mov}_j^{\mathbf{q},\mathbf{q}'}) \\
\text{bb}_{j,\mathbf{q}}^c &= [\text{out} \rightarrow \text{in}] \otimes \tau_{0,j} \otimes ([\{0o, 1o, \star o\} \rightarrow]_j; \mathbf{q} \rightarrow \mathbf{mov}_j^{\mathbf{q},\mathbf{q}'}) \\
\text{rec}_{j,\mathbf{q}}^c &= \sum_{\bullet \in \{0,1,\star\}} (\pi_{\bullet i} \otimes \tau_{0,j} \otimes ([\rightarrow \bullet i]_j; \mathbf{mov}_j^{\mathbf{q},\mathbf{q}'} \rightarrow \mathbf{q}'))
\end{aligned}$$

Figure 1: The operators encoding the forward and backward move instructions

that encodes a transition from any state in  $\mathbf{S}$  to  $a'_i$ . The notation  $(\mathbf{S} \rightarrow)$  will denote the projection on the subspace generated by  $\mathbf{S}$ .

A transition that impacts only on the values stored in the subset<sup>13</sup>  $p_{i_1}, \dots, p_{i_l}$  and the state  $\mathbf{q}$  will be denoted by

$$([\{a_{i_1} \rightarrow a'_{i_1}\}_{i_1}; \dots; \{a_{i_l} \rightarrow a'_{i_l}\}_{i_l}; \mathbf{q} \rightarrow \mathbf{q}')$$

This operator is defined as

$$u_1 \otimes u_2 \otimes \dots \otimes u_p \otimes (\mathbf{q} \rightarrow \mathbf{q}')$$

where  $u_i = (a_{i_j} \rightarrow a'_{i_j})$  if  $\exists j, i = i_j$ ,  $u_i = 1$  elsewhere.

Finally, we will denote by  $\tau_{i,j}$  the transposition exchanging the integers  $i$  and  $j$ .

Now, we can explain how the different transitions are encoded. Figures 1 and 2 give the definitions of the operators used in the following description.

*Move a pointer, read a value and change state.* We first explain how to encode the action “move forward the pointer  $j$ ”. When we are in the pseudo-configuration  $c = (a_1, \dots, a_p; \mathbf{q})$ , the machine, during this transition, moves the pointer  $j$  one step to the right, reads the value  $a'_j$  stored at  $\#p_j$ , updates

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<sup>13</sup>In fact this subset will always be a singleton, for our encoding corresponds to a NDPM that moves at most one pointer at each transition and any NDPM can be replaced by another NDPM that moves its pointers one at a time.

$$\begin{aligned}
\text{rm}_i &= 1 \otimes \tau_{0,i} \otimes (\mathbf{back}_i \rightarrow \mathbf{mov-back}_i) \\
\text{rr}_i^0 &= \pi_{0o} \otimes \tau_{0,i} \otimes ([\rightarrow 0o]_i; \mathbf{mov-back}_i \rightarrow \mathbf{back}_i) \\
\text{rr}_i^1 &= \pi_{1o} \otimes \tau_{0,i} \otimes ([\rightarrow 1o]_i; \mathbf{mov-back}_i \rightarrow \mathbf{back}_i) \\
\text{rc}_i &= \begin{cases} \pi_{\star o} \otimes \tau_{0,i} \otimes ([\rightarrow a_i]_i; \mathbf{mov-back}_i \rightarrow \mathbf{back}_{i+1}) & (1 \leq i < p) \\ \pi_{\star o} \otimes \tau_{0,p} \otimes ([\rightarrow a_p]_p; \mathbf{mov-back}_p \rightarrow \mathbf{q}_0) & (i = p) \end{cases}
\end{aligned}$$

The operator  $\text{rc}_i$  is parametric on an initial pseudo-configuration  $s = (a_1, \dots, a_p; \mathbf{q}_0)$ .

Figure 2: The operators encoding rejection

the memory slot in consequence and changes the state from  $\mathbf{q}$  to  $\mathbf{q}'$ . There are two cases: either the last movement of the pointer  $j$  was a forward move, either it was a backward move. The case we are dealing with is obtained from the value stored in memory slot of the  $j$ -th pointer: if the value is  $0i$ ,  $1i$  or  $\star i$ , then the last move was a forward move, if the value is  $0o$ ,  $1o$  or  $\star o$ , the last move was a backward move. In the first case, the operator  $\text{ff}_{j,\mathbf{q}}^c$  will be applied (notice the projection  $\pi_{\text{in}}$  on the  $j$ -th memory slot), and in the second the  $\text{bf}_{j,\mathbf{q}}^c$  operator will be applied. Both these operators somehow *activate* the  $j$ -th pointer by using the transposition  $\tau_{0,j}$  and prepare for reading the representation of the integer. This representation will then give the value of the next (when moving forward) digit of the input. The  $\text{rec}_{j,\mathbf{q}}^c$  operator is then applied in order to simultaneously update the value of the  $j$ -th memory slot and *deactivate* the  $j$ -th pointer.

The operator that will encode the moving forward instruction is then defined as  $\text{forward}_{j,\mathbf{q}}^c = \text{bf}_{j,\mathbf{q}}^c + \text{ff}_{j,\mathbf{q}}^c + \text{rec}_{j,\mathbf{q}}^c$ .

In case of a “move backwards pointer  $j$ ” instruction, the operator encoding the transition is  $\text{backward}_{j,\mathbf{q}}^c = \text{fb}_{j,\mathbf{q}}^c + \text{bb}_{j,\mathbf{q}}^c + \text{rec}_{j,\mathbf{q}}^c$ .

In the following, we will forget about the subscripts and superscripts of these operators and write  $\text{ff}$ ,  $\text{fb}$ ,  $\text{bf}$ ,  $\text{bb}$  and  $\text{rec}$  in order to simplify notations.

*Accept.* The case of acceptance is especially easy: we want to stop the computation<sup>14</sup>, so every transition  $(a_1, \dots, a_n; \mathbf{q}) \rightarrow \mathbf{accept}$  will be encoded by

<sup>14</sup>This is one of the reason we did not take **accept** as being a state: we want the computation to stop immediately, there is no need to provide “one last set of instructions”.

0.

*Reject.* The transitions of the form  $(a_1, \dots, a_n; \mathbf{q}) \rightarrow \mathbf{reject}$  are encoded by the operator that represents the “transition”  $(a_1, \dots, a_n; \mathbf{q}) \rightarrow (a_1, \dots, a_n; \mathbf{back}_1)$ .

This defines the operator  $\rightarrow^\bullet$  encoding the transition relation  $\rightarrow$ . But we need a bit more to interpret a machine. Indeed, as we already explained, the representation of an integer is accepted by an observation  $\phi$  if the product of the two operators is nilpotent. Hence, a rejection should lead to the creation of a loop. We therefore add two new states —  $\mathbf{back}_i$  and  $\mathbf{mov-back}_i$  — for each  $i = 1, \dots, p$ . We can then define  $\mathbf{reject}_s = \sum_{i=1}^p \mathbf{rm}_i + \mathbf{rr}_i^0 + \mathbf{rr}_i^1 + \mathbf{rc}_i$ . This operator actually encodes a reinitialization<sup>15</sup> of the machine that starts when the computation reaches the state  $\mathbf{back}_1$ : the pointers are moved back one after the other until they point on the symbol  $\star$ , and the memory slots and the state are initialized according to the given initial pseudo-configuration  $s = (a_1, \dots, a_p; \mathbf{q}_0)$ . The reason for this complicated way of encoding rejection comes from the fact that the computation simulated by the operators is massively parallel: the operator  $M_s^\bullet$  is a big sum that recollect every single evolution from a pseudo-configuration to any other “reachable” with  $\rightarrow$ , and it starts moreover in all possible initial pseudo-configurations at the same time.

Another complication due to this parallel computation and this particular way of defining rejection is that the operators simulate correctly only *acyclic machines*, i.e. machines that never enter a computational loop (whichever initial pseudo-configuration is chosen). This is necessary because we want the entering of a loop to be the result of rejection. Fortunately, it can be shown that:

**Proposition 12** ([2, Lemma 22]). *For all  $M \in \text{NDPM}(p)$  there exists  $M' \in \text{NDPM}(p')$  such that  $M'$  is acyclic and  $\mathcal{L}(M) = \mathcal{L}(M')$ .*

In the particular case of acyclic machines, one can show that the encoding is sound:

**Proposition 13** ([2, Lemma 25]). *Let  $M \in \text{NDPM}(p)$  be acyclic,  $c \in C_M$  and  $M_s^\bullet = \rightarrow^\bullet + \mathbf{reject}_s$ . For all  $n \in \mathbf{N}$  and every binary representation  $N_n \in \mathfrak{M}_6(\mathfrak{N}_0)$*

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<sup>15</sup>And this is why the encoding of a NDPM is done provided an initial pseudo-configuration  $s$ .



of the integer  $n$ :

$$M_s(n) \text{ accepts} \Leftrightarrow M_s^*(N_n \otimes 1) \text{ is nilpotent.}$$

These two results can then be combined with the results of the previous section to get the following proposition:

**Proposition 14.**

$$\mathbf{co-NL} \subseteq [P_+]$$

We will now study the encoding of deterministic pointer machines in more detail.

### 5.2. The Encoding of DPMs

We will show that the encoding of deterministic pointer machines satisfies a particular property: their 1-norm is less or equal to 1. We first define what is the 1-norm:

**Definition 15.** We define the family of norms  $\|\cdot\|_1^k$  on  $\mathfrak{M}_k(\mathfrak{M}_6(\mathfrak{A}))$  by:

$$\|(a_{i,j})_{1 \leq i,j \leq k}\|_1^k = \max_{1 \leq j \leq k} \sum_{i=1}^k \|a_{i,j}\|$$

where  $\|a_{i,j}\|$  is the usual norm (the  $C^*$ -algebra norm) on  $\mathfrak{M}_6(\mathfrak{A})$ .

We will simply write  $\|\cdot\|_1$  when the superscript is clear from the context.

A corollary of the following proposition will be used extensively in the proofs.

**Proposition 16.** *Let  $A, B$  be operators such that  $AB^* = B^*A = 0$ . Then:*

$$\|A + B\| = \max\{\|A\|, \|B\|\}$$

*Proof.* Actually, this proposition appears as Theorem 1.7 (b) in a paper of P. J. Maher [23]. This last theorem is stated differently (with conditions on ranges) in the given reference. To obtain the result as we stated it, simply notice that  $AB^* = 0$  implies  $\text{Ran}(B^*) \subseteq \ker(A) = (\text{Ran}(A^*))^\perp$  so  $\text{Ran}(B^*) \perp \text{Ran}(A^*)$ . Similarly,  $B^*A = 0$  implies that  $\text{Ran}(A) \subseteq \ker(B^*) = (\text{Ran}(B))^\perp$  so  $\text{Ran}(A) \perp \text{Ran}(B)$ . Thus, if  $AB^* = B^*A = 0$ , we have  $\text{Ran}(A) \perp \text{Ran}(B)$  and  $\text{Ran}(A^*) \perp \text{Ran}(B^*)$  and one can then apply the theorem of Maher.  $\square$

**Corollary 17.** *The operators  $ff+bf$ ,  $rec$ ,  $bb+fb$ , and  $rr_i^0+rr_i^1+rc_i$  ( $i$  fixed) are of norm 1.*

*Proof.* It is clear that  $bf \times ff^* = 0$  since  $\pi_{\text{out}}\pi_{\text{in}} = 0$ . Similarly,  $ff^* \times bf = 0$ . Thus, using the preceding proposition, we have  $\|ff+bf\| = \max\{\|bf\|, \|ff\|\} = 1$ .

A similar argument shows that  $\|fb+bb\| = 1$ .

Clearly, as a consequence of  $\pi_{ai}\pi_{bi} = 0$  for  $a \neq b$ , we deduce  $\|rec\| = 1$  by Proposition 16.

We are now proving the result for  $rr_i^0+rr_i^1+rc_i$ , so we fix  $1 \leq i \leq p$ . First, notice that  $rc_i \times (rr_i^0)^* = 0$  as a consequence of  $\pi_{\text{mov-back}_i} \times \pi_{\text{back}_i} = 0$ . Conversely,  $(rr_i^0)^* \times rc_i = 0$  as a consequence of<sup>16</sup>  $\pi_{\text{back}_{i+1}} \times \pi_{\text{mov-back}_i} = 0$ . A similar argument shows that  $rc_i \times (rr_i^1)^* = 0$  and  $(rr_i^1)^* \times rc_i = 0$ . Moreover, we know that  $(rr_i^0)^* \times rr_i^1 = 0$  and  $rr_i^1 \times (rr_i^0)^* = 0$  by just looking at the first term of the tensor in their definition. By an iterated use of Proposition 16, we get that  $\|rr_i^0+rr_i^1+rc_i\| = \max\{\|rr_i^0\|, \|textrr_i^1\|, \|rc_i\|\} = 1$ .  $\square$

We are now in possession of all the material needed to characterize the operators coming from the encoding of a deterministic pointer machine.

**Proposition 18.** *Let  $M \in DPM(p)$  be an acyclic machine, and  $M_s^\bullet = \rightarrow \bullet + reject_s$  its encoding as an operator,  $\|M_s^\bullet\|_1 \leq 1$ .*

*Proof.* Since we are working with deterministic machines, the transition relation is functional: for each  $\rho \in C_M$  there is at most one  $t$ , say  $t_\rho$ , such that  $\rho \rightarrow t_\rho$ . Thus:

$$M_s^\bullet = \sum_{\rho \in C_M} \phi_{\rho, t_\rho} + reject_s$$

Since  $M_s^\bullet$  is an element of  $\mathfrak{M}_N(\mathfrak{M}_6(\mathfrak{A}))$ , we will now compute  $\|M_s^\bullet\|_1^N$ . We first show that this matrix has at most one non-null coefficient in each column. Then we will use the fact that these coefficients are of norm at most 1.

To prove this we introduce the set  $P = \{1, \dots, 6\}^p \times Q^1$  of *extended pseudo-configurations*. This set is a basis of the algebra of pseudo-states, and we write  $M_s^\bullet = (a_{\rho, \rho'})_{\rho, \rho' \in P}$ . This way, the 1-norm of  $M_s^\bullet$  is

$$\|M_s^\bullet\|_1 = \max_{\rho' \in P} \left\{ \sum_{\rho \in P} \|A_{\rho, \rho'}\| \right\}$$

<sup>16</sup>When  $i = p$ , we consider that  $\text{back}_{i+1} = \mathbf{q}_0$  to ease the argument.

Let  $\rho = (a_1, \dots, a_n; q)$  be a (non-extended) pseudo-configuration. If  $\rho$  is a pseudo-configuration of the machine (i.e.  $q$  is an element of  $Q$ ), then there is at most one pseudo-configuration  $t_\rho$  such that  $\rho \rightarrow t_\rho$ . This *atomic transition* can be:

- **Accept:** in this case the operator  $\phi_{\rho, t_\rho}$  is equal to 0 and the column is empty.
- **Reject:** in this case the column corresponding to  $\rho$  contains only the operator  $(\mathbf{q} \rightarrow \mathbf{back}_1)$  that encodes the transition  $\rho \rightarrow (a_1, \dots, a_n; \mathbf{back}_1)$ , and the norm of this operator is equal to 1.
- **Move forward a pointer, read a value and change state to  $\mathbf{q}'$ :** then the only extended pseudo-configurations introduced by the encoding is  $\tilde{\rho} = (a_1, \dots, a_n; \mathbf{mov}_j^{\mathbf{q}, \mathbf{q}'})$ . The column corresponding to  $\rho$  contains the operator  $\text{ff} + \text{bf}$ , which is of norm 1 by Corollary 17. The column corresponding to  $\tilde{\rho}$  contains only the operator  $\text{rec}$  whose norm is equal to 1 by Corollary 17.
- **Move backwards:** this case is similar to the previous one.

Now let us take a look at the operator  $\text{reject}_s$ . The extended pseudo-configurations introduced for the encoding of the rejection are  $\tilde{\rho}_i^m = (a_1, \dots, a_n; \mathbf{mov-back}_i)$  and  $\tilde{\rho}_i^b = (a_1, \dots, a_n; \mathbf{back}_i)$  for  $i = 1, \dots, p$ . The column corresponding to  $\rho$  contains only the operator encoding the transition from  $\rho$  to  $\tilde{\rho}_1^b$ , which is a norm 1 operator. Let us now fix  $1 \leq i \leq p$ . The column corresponding to the extended pseudo-state  $\tilde{\rho}_i^b$  contains only the operator  $rm_i$ , which is of norm 1. The column corresponding to  $\tilde{\rho}_i^m$  contains the operator  $rr_i + rc_i$ , which is of norm 1 by Corollary 17.

We just showed that each column of the matrix contains at most one operator different from 0, and that this operator is always of norm 1. Hence, for any fixed extended pseudo-configuration  $\rho$ :

$$\sum_{\rho \in P} \alpha_{\rho, \rho'} \leq 1$$

As a consequence of the definition of the 1-norm,  $\|M^\bullet\|_1 = \max_{\rho' \in P} \{\sum_{\rho \in P} \alpha_{\rho, \rho'}\} \leq 1$ .  $\square$

The last proposition motivates the following definition:

**Definition 19.**

$$P_{+,1} = \{\phi \mid \phi \in P_+ \text{ and } \|\phi\|_1 \leq 1\}$$

Now, by looking carefully at the proof of Proposition 12, one can show that a similar result holds in the particular case of deterministic pointer machines. Indeed, if  $M$  is deterministic, the machine  $M'$  constructed from  $M$  in the proof is also deterministic:

**Proposition 20.** *For all  $M \in DPM(p)$ , there exists  $M' \in DPM(p)$  such that  $M'$  is acyclic and  $\mathcal{L}(M) = \mathcal{L}(M')$ .*

As a consequence, one gets the following result:

**Proposition 21.**

$$\mathbf{L} \subseteq [P_{+,1}]$$

The question is then: is this inclusion strict or did we find a characterization of the class  $\mathbf{L}$ ? To answer this question, we need to look at the proof of the inclusion  $[P_{\geq 0}] \subseteq \mathbf{co-NL}$  and see what happens when we restrict to the operators in  $P_{+,1}$ .

**6. Operators and Logarithmic Space**

In this section, we will show that  $[P_{+,1}] \subseteq \mathbf{L}$ , concluding the proof that  $[P_{+,1}] = \mathbf{L}$ . To prove this inclusion, we will need a technical Lemma proved in our earlier paper:

**Lemma 22** ([2, Lemma 27]). *Let  $N_n$  be a binary representation of an integer  $n$  in  $\mathfrak{M}_6(\mathfrak{N}_0)$  and  $\Phi \in \mathfrak{M}_6(\mathfrak{S})$  be an observation in  $P_{\geq 0}$ . Then there exist an integer  $k$ , an injective morphism  $\psi : \mathfrak{M}_k(\mathbf{C}) \rightarrow \mathfrak{K}$  and two matrices  $M \in \mathfrak{M}_6(\mathfrak{M}_k(\mathbf{C}))$  and  $\bar{\Phi} \in \mathfrak{M}_6(\mathfrak{M}_k(\mathbf{C})) \otimes \mathfrak{E}$  such that  $Id \otimes \psi(M) = N_n$  and  $Id \otimes \psi \otimes Id_{\mathfrak{E}}(\bar{\Phi}) = \Phi$ .*

This lemma is of the utmost importance. Even though the hyperfinite factor  $\mathfrak{R}$  is necessary to have a uniform representation of integers, it is an algebra of operators acting on an infinite-dimensional Hilbert space. It is therefore not obvious that one can check the nilpotency of such an operator with finite resources, and it is of course not possible in general. However, the previous lemma has as a direct consequence that checking the nilpotency of a product  $\Phi(N_n \otimes 1)$  where  $\Phi$  is an observation in  $P_{\geq 0}$  is equivalent to checking the nilpotency of a product of matrices:

**Corollary 23.** *Let  $\Phi, N_n, \bar{\Phi}$  and  $M$  be as in the statement of the previous lemma. Then:*

$\Phi(N_n \otimes 1)$  is nilpotent if and only if  $\bar{\Phi}(M \otimes 1)$  is nilpotent

Notice that moreover the degree of nilpotency of the two products is equal.

**Proposition 24** ([2, Proposition 28]). *If  $\Phi \in P_{\geq 0}$  and  $N_n$  is a representation of  $n \in \mathbf{N}$ , there is a **co-NL** machine that checks if  $\bar{\Phi}(M \otimes 1)$  is nilpotent.*

This proposition, together with Proposition 14 and the fact that  $[P_+] \subseteq [P_{\geq 0}]$  gives us a proof of the following equality:

**Theorem 25** ([2, Theorem 29]).

$$\mathbf{co-NL} = [P_+] = [P_{\geq 0}]$$

We recall how the proof of Proposition 24 works. The algebra we are working with is:

$$\mathfrak{F} = \mathfrak{M}_6(\mathbf{C}) \otimes \underbrace{(\mathfrak{M}_{n+1}(\mathbf{C}) \otimes \cdots \otimes \mathfrak{M}_{n+1}(\mathbf{C}))}_{p \text{ copies}} \rtimes \mathfrak{S}_N \otimes \Omega$$

We let in the following  $K = 6k(\log_2(n)+1)^N N!$  be the dimension of the Hilbert space the algebra  $\mathfrak{F}$  is acting on.

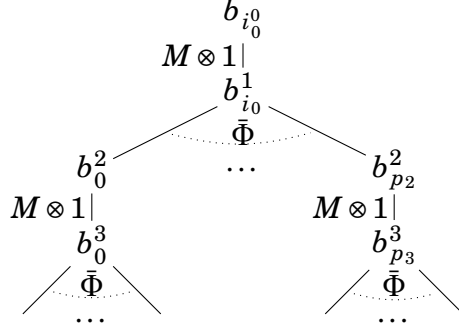
We chose the basis of this algebra defined as the set of elements of the form:

$$(\pi, a_0, a_1, \dots, a_p; \sigma; e)$$

where  $\pi \in \{1, 6\}$  is an element of the basis  $(0o, 0i, 1o, 1i, s, e)$  of  $\mathfrak{M}_6(\mathbf{C})$ ,  $a_i \in \{1, \dots, p\}$  are the elements of the basis chosen to represent the integer  $n$ ,  $\sigma \in \mathfrak{S}_N$  and  $e$  is an element of a basis of  $\Omega$ . When we apply  $M \otimes 1_\Omega$  representing the integer to an element of this basis, we obtain one and only one vector of the basis  $(\pi, a_0, a_1, \dots, a_p; \sigma; e)$ . When we apply to this element the matrix  $\bar{\Phi}$  we obtain a linear positive combination of  $l \in \mathbf{N}$  elements of the basis:

$$\bar{\Phi}(\pi, a_0, a_1, \dots, a_p; \sigma; e) = \sum_{i=0}^l \alpha_i(\rho, a_{\tau_i(0)}, \dots, a_{\tau_i(p)}; \tau_i \sigma; e_i) \quad (1)$$

So we obtain the following picture:



To decide if the product  $\bar{\Phi}(M \otimes 1)$  is nilpotent, it is then sufficient to check, for each possible value of  $b_{i_0}^0$ , that the branches of this tree are finite. Since they are either infinite or of length at most  $K$ , this can be checked by a non-deterministic (to deal with the branchings that appear in the figure above) Turing machine. This machine only uses logarithmic space since it needs only to store at each step the value of  $b_{i_0}^0$ , the current basis element  $b_{i_j}^{2m}$ , and the number of iterations of  $\bar{\Phi}(M \otimes 1)$  that were computed to get from  $b_{i_0}^0$  to  $b_{i_j}^{2m}$ .

Now, let us look at the case when  $\Phi \in P_{+,1}$ . In this particular case, the equation 1 becomes:

$$\bar{\Phi}(\pi, a_0, a_1, \dots, a_p; \sigma; e) = (\rho, a_{\tau_{b_0}(0)}, \dots, a_{\tau_{b_0}(p)}; \tau_{b_0}\sigma; e_{b_0}) \quad (2)$$

Indeed, writing  $\bar{\Phi}$  as a  $\text{Card}(Q) \times \text{Card}(Q)$  matrix (here  $Q$  denotes a basis of  $\Omega$ , which is a matrix algebra from the definition of observations), we use the fact that  $\|\bar{\Phi}\|_1 \leq 1$  to deduce that for all  $q \in Q$ ,  $\sum_{q' \in Q} \|\bar{\Phi}_{q,q'}\| \leq 1$ .

From the equation 2, we have  $\|\bar{\Phi}_{e,e_i}\| = 1$  ( $i = 1, \dots, l$ ), and thus  $\sum_{e' \in Q} \|\bar{\Phi}_{e,e'}\| \geq l$ . Therefore,  $l \leq 1$ .

That is, the previous picture becomes:

$$\begin{array}{c} b_{i_0} \\ M \otimes 1 | \\ b_{i_0}^1 \\ \bar{\Phi} | \\ b_{i_1} \\ M \otimes 1 | \\ b_{i_1}^1 \\ \bar{\Phi} | \\ \vdots \end{array}$$

Notice that the nilpotency degree of  $\bar{\Phi}(M \otimes 1)$  is again at most  $K$ . One can thus easily define a deterministic Turing machine that takes the basis elements one after the other and compute the sequence  $b_{i_0}, b_{i_1}, \dots$ : if the sequence stops before the  $K$ -th step, the machine starts with the next basis element as  $b_{i_0}$ , and if the sequence did not stop at step  $K$  it means the matrix was not nilpotent. This Turing machine needs no more than logarithmic space since it stores only the current basis element (i.e.  $b_{i_0}$ ), the last computed term of the sequence and its index. We just proved:

**Proposition 26.** *If  $\Phi \in P_{+,1}$  and  $N_n$  is a representation of  $n \in \mathbf{N}$ , there is a  $\mathbf{L}$  machine that checks if  $\bar{\Phi}M$  is nilpotent.*

Putting together the results of this section and the last, we can finally state the main theorem of this paper:

**Theorem 27.**

$$\mathbf{L} = [P_{+,1}]$$

Together with Theorem 25, we thus obtained two sets of observations  $P_+$  and  $P_{+,1}$ , satisfying  $P_{+,1} \subsetneq P_+$  and such that  $\mathbf{L} = [P_{+,1}] \subseteq [P_+] = \mathbf{co-NL}$ .

## 7. Conclusion

This work both completes and extends the results obtained in our earlier paper [2] where we showed that the approach recently proposed by Girard [1] for studying complexity classes succeeds in characterizing the complexity class  $\mathbf{co-NL}$ . On one hand, we showed that the non-deterministic pointer machines, which were designed to mimic the computational behavior of operators, are really close to a well-known abstract machine, two-way multi-head finite automata. This result gives a much better insight on the pointer machines, and we hope this will help us in extending the model of pointer machines to study others complexity classes. It moreover strengthens the proof of the fact that the set of operators  $P_+$  characterizes the class  $\mathbf{co-NL}$ . On the other hand, we extended the result by finding a subset of  $P_+$ , defined through a condition on the norm, that corresponds to the encoding of deterministic pointer machines. We showed first that deterministic pointer machines were equivalent to the notion of deterministic two-way multi-head finite automata, hence that  $\mathbf{L}$  is a subset of the language associated to  $P_{+,1}$ . We then showed that the converse inclusion holds, showing that the set of operators  $P_{+,1}$  is indeed a characterization of the complexity class  $\mathbf{L}$ .

The notion of acceptance used in this work (nilpotency) is closely related [24] to the notion of interaction in Girard’s geometry of interaction in the hyperfinite factor (GoI5). This should lead to the possibility of defining types in GoI5 corresponding to complexity classes. For this reason, and the fact that the GoI5 construction allows more classical implicit computational complexity approaches such as defining constrained exponential connectives, the GoI5 framework seems a perfect candidate for a general mathematical framework for the study of complexity. The combined tools offered by geometry of interaction and the numerous tools and invariants of the theory of operators that could be used in this setting offer new perspectives for the obtention of separation results.

We also believe that the approach explored in this paper can be used to obtain characterizations of other complexity classes, and in particular the classes **P** and **co-NP**. These characterizations may be obtained through the use of a more complex group action in the crossed product construction, or by defining a suitable superset of  $P_+$ . An approach for obtaining such a result would be to generalize the notion of pointer machines to get a characterization of **P** or **co-NP**. Since the pointer machines are closely related to the operators and the way the latter interact with the representation of integers, such a result would be a great step towards a characterization of these classes.

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