A Numerical Method for solving Convection-Reaction-Diffusion Multivalued Equations in Fire Spread Modelling [1]

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Abstract

A numerical method is developed for fire spread simulation modelling. The two dimensional surface model presented takes into account moisture content, radiation, wind and slope effects, which are by far the most important mechanisms in fire spread. We consider the combustion of a porous solid, where the energy conservation equation is applied. The influence of moisture content and eventually heat absorbtion by pyrolysis, can be represented as two free boundaries, and are treated here using a multivalued operator representing the enthalpy. The maximal monotone property of this operator allows the implementation of a numerical algorithm with good convergence properties.

Key words: fire, radiation, moisture, pyrolysis, wind, slope
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1 Introduction

Many existing physical models for fire spread in a porous fuel bed use the principle of energy conservation applied to the preheated fuel. Generally, radiation is considered as the dominant mechanism of the fuel preheating. But in order to obtain reliable rates of fire spread, wind effects and initial vegetation moisture should be taken into account. Physical models from fundamental conservation equations and complex physics have been developed [3,7,9,8]. These valuable

approaches are computationally expensive and too slow to be used in real time mode, even with fast and parallel processing. On the other hand, several works have appeared recently where one or two dimensional physical models are considered in order to simulate fire spread in small computers, with moderate simulation times, see for example [4,5,10]. This paper is a contribution to generally applicable models of fire spread through fuel beds, by means of simple models but taking into account radiation, moisture content, wind and slope effects. Specifically, the influence of moisture content and eventually heat absorption by pyrolysis, can be represented as two free boundaries, and are treated in this paper using a multivalued operator representing the enthalpy. The maximal monotone property of this operator allows the implementation of a numerical algorithm with well-known convergence properties.

2   Governing Equations

2.1   Combustion Model

The non dimensional equations governing the fire spread in a region Ω with boundary Γ are

$$\partial_t e + v.\nabla e - \nabla(\kappa(u)\nabla u) + \alpha u = S(u,y), \tag{1}$$
$$e \in G(u), \tag{2}$$
$$\partial_t y = -g(u)y. \tag{3}$$

The boundary and initial conditions are given by

$$u(x,t) = 0 \quad x \in \Gamma, \ t > 0, \tag{4}$$
$$u(x,0) = u_0(x) \quad x \in \Omega, \tag{5}$$
$$y(x,0) = y_0(x) \quad x \in \Omega. \tag{6}$$

The unknowns of problem (1)-(3) e, u and y are the non-dimensional enthalpy, the non-dimensional temperature and the mass fraction of solid fuel respectively. This model is essentially based on the model presented in [12] where our main contribution is the use of the multivalued operator for the enthalpy. Concerning to the parameters, v is a re-scaled wind velocity taking part into the convective term $v.\nabla e$ representing the energy convected by the gas pyrolisated through the elementary control volume, α in αu represents the energy lost by convection in the vertical direction and $g(u)y$ in equation (3) represents the fuel mass variation by pyrolysis, where $g(u)$ is given by $g(u) = s^+(u,y)\beta$ and

s^+ is

$$s^+(u, y) = \begin{cases} 1 \text{ if } u \geq u_p \text{ and } y > y_e \\ 0 \qquad \text{ in other case} \end{cases}$$

This expression comes from considering a large activation energy in an Arrhenius type law for the loss of fuel, so that it can be assumed that the pyrolysis rate is small for temperatures lower than a certain value [12], and considering a mass fraction lower bound of extinction y_e.

The source term $S(u, y)$ in (1) represents the heat due to combustion and eventually the heat due to non local radiation.

$$S(u, y) = -\partial_t y + R(u, x) = g(u)y + R(u, x) \tag{7}$$

The non local radiation term $R(u, x)$ describes the thermal radiation from the flame above the layer. This term is a convolution operator given by,

$$R(u, x) = \epsilon \int_{\Omega_f(u,y)} f(x - \tilde{x}) \, d\tilde{x},$$

where $\Omega_f(u, y) = \{x \in \Omega; \ u(x) > u_p \ \text{ and } \ y(x) > y_e\}$ is the fire domain on the surface.

When a medium is optically dense, the radiation within it can travel only a short distance before being absorbed and can be approximated by a nonlinear diffusion term in (1), $\kappa(u) = \kappa(1 + u)^3$, see [11].

The non-dimensional enthalpy e is an element of a multivalued operator G, see Fig. 1, given by

$$G(u) = \begin{cases} u & \text{if} & u < u_v, \\ [u_v \, , \, u_v + \lambda_v] & \text{if} & u = u_v, \\ u + \lambda_v & \text{if } u_v < u < u_p, \\ [u_p + \lambda_v \, , \, u_p + \lambda_v + \lambda_p] \text{ if} & u = u_p, \\ u + \lambda_v + \lambda_p & \text{if} & u > u_p, \end{cases}$$

where u_v is the non-dimensional water evaporation temperature and u_p is the non-dimensional solid fuel pyrolysis temperature. The quantities λ_v and λ_p are the non-dimensional evaporation heat and pyrolysis heat, respectively.
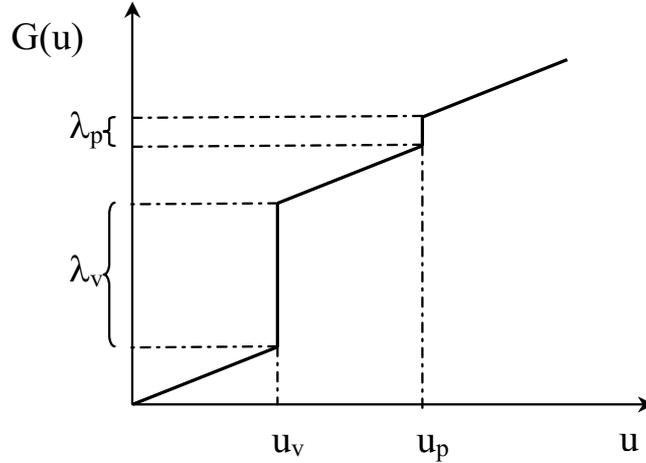
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Fig. 1. Enthalpy operator

It should be noticed that in the burnt zone the multivalued operator does not exactly represent the physical phenomena since water vapor is no more in the porous medium. This drawback can be circumvented setting $\lambda_v = 0$ and $\lambda_p = 0$ in the burnt area.

This model is a variant of models in [12] (chapter one), model I in [10] or model in [13], where we have introduced the influence of moisture content and heat absorption by pyrolysis, by using the enthalpy multivalued operator.

From a numerical point of view, the use of a multivalued operator to treat the enthalpy has been widely used for the Stefan problem in works like [1], among many others, but mainly concerning solid-liquid phase change problems. The works in [2] and the references there in, use it in forest fires simulations but in a complete different approach to the one given here. As far as we know, solving the enthalpy equations by means of a multivalued operator in forest fires simulations had not been developed until now.

2.2 Convection Model

The surface velocity v in equation (1) is computed by means of the model developed in [14]. Denoting $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, for a fixed time t, and any temperature field u defined on Ω, we associate a two dimensional velocity wind field v defined on Ω by

$$v = \frac{\xi}{c}(\nabla^\perp \psi + e\nabla \tilde{u} + v_m^*),$$

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where ψ is the unique solution of

$$-\nabla.(\frac{1}{a}\nabla\psi) = \nabla^{\perp}.\frac{b\nabla\tilde{u} + v_m^*}{a} \quad \text{in } \Omega,$$
$$\psi = 0 \quad \text{on } \Gamma,$$

for $\tilde{u} = u\lambda/(\delta - h)$, where $h(x)$ is the surface height, δ the domain's height and λ is related to the buoyancy forces due to the temperature gradients in the vertical direction. Functions $a(x)$, $b(x)$, $c(x)$ and $e(x)$ take into account the influence of the surface height $h(x)$ and are defined by

$$a(x) = \frac{1}{3}(\delta - h(x))^2(3\xi + \delta - h(x)),$$

$$b(x) = \frac{1}{30}(\delta - h(x))^2(2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(x) - (3\delta + 5\xi)h^2(x) + h^3(x)),$$

$$c(x) = \frac{1}{3}(\delta - h(x))(\delta + 3\xi - h(x))),$$

$$e(x) = \frac{1}{45}(\delta - h(x))^5.$$

The meteorological wind v_m is a data on Γ such that $\int_\Gamma (\delta - h)\, v_m.n\, ds = 0$, and v_m^* is any velocity field defined in Ω such that

$$\nabla.v_m^* = 0, \quad v_m^*.n = (\delta - h)\, v_m.n \text{ on } \Gamma.$$

This gives
$$v = \mathbf{L}(u, v_m, h),$$

where $\mathbf{L}$ is a time independent operator, affine with respect to u and v_m, and non linear with respect to h.

3   Numerical Method

3.1   Time Integration

Let $\Delta t = t^{n+1} - t^n$ be a time step and let y^n, e^n and u^n denote approximations at time step t^n to the exact solution y, e and u, respectively.

We consider an implicit scheme by discretizing the total derivative, see [15],

$$\partial_t e + w.\nabla e \approx \frac{1}{\Delta t}(e^{n+1} - \bar{e}^n),$$

where $\bar{e}^n = e^n \circ X^n$, and $X^n(x) = X(x, t^{n+1} - t^n) \approx x - w\Delta t$ is the position at time t^n of the particle which is at position x at time t^{n+1}. At each time step, we solve,

$$\frac{y^{n+1} - y^n}{\Delta t} = -y^{n+1}g(u^{n+1}), \tag{8}$$

$$\frac{e^{n+1} - \bar{e}^n}{\Delta t} - \kappa(u^{n+1})\Delta u^{n+1} + \alpha u^{n+1} = S^{n+1}, \tag{9}$$

$$e^{n+1} \in G(u^{n+1}). \tag{10}$$

3.2 Iterative Solution at each Time Step

Problem (8), (9), (10) is non linear due to the multivalued operator G. As in Bermúdez and Moreno [16], we consider an exact perturbation of this problem. Let $\omega > 0$ be a given parameter and set,

$$G^\omega = G - \omega I,$$

where I is the identity, then (10) can be written,

$$z^{n+1} = e^{n+1} - \omega u^{n+1} \in G^\omega(u^{n+1}). \tag{11}$$

For λ and ω verifying $\lambda\omega < 1$, the resolvent,

$$J_\lambda^\omega = (I + \lambda G^\omega)^{-1} = ((1 - \lambda\omega)I + \lambda G)^{-1}$$

is a well defined univalued operator and the Yosida approximation of G^ω is given by

$$G_\lambda^\omega = \frac{I - J_\lambda^\omega}{\lambda}.$$

It is easy to check that the inclusion (11) is equivalent to the fixed point equation

$$z^{n+1} = G_\lambda^\omega(u^{n+1} + \lambda z^{n+1}).$$

This suggests the following algorithm for solving (8), (9), (10): For given u^n, y^n and z^n,

(1) Set $u^{n+1,0} = u^n$, $z^{n+1,0} = z^n$.
(2) Compute

$$y^{n+1,i+1} = \frac{y^n}{1 + \Delta tg(u^{n+1,i})}.$$

(3) Compute $u^{n+1,i+1}$ solving

$$(\alpha\Delta t + \omega)u^{n+1,i+1} - \Delta t\kappa(u^{n+1,i})\Delta u^{n+1,i+1} = \bar{e}^n - z^{n+1,i} + \Delta tS^{n+1,i}.$$

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(4) Compute $z^{n+1,i+1} = G_\lambda^\omega(u^{n+1,i+1} + \lambda z^{n+1,i})$.

(5) If $\|z^{n+1,i+1} - z^{n+1,i}\| > Tol$, update $i \leftarrow i+1$ and go to step 2. else end of the loop.

For $\lambda\omega \leq 1/2$ the Yosida approximation G_λ^ω is a Lipschitz operator with constant $1/\lambda$ and the convergence of the algorithm can be proved [16].

3.3 Practical computation of $z^{n+1,i+1}$

In the following we take $\lambda\omega = 1/2$. Set $\bar{u} = u^{n+1,i+1} + \lambda z^{n+1,i}$, then

$$G_\lambda^\omega(\bar{u}) = \frac{1}{\lambda}\bar{u} - \frac{1}{\lambda}J_\lambda^\omega(\bar{u}).$$

It remains to explain how to compute

$$\bar{z} = J_\lambda^\omega(\bar{u}),$$

which is equivalent to solve (for $\lambda\omega = \frac{1}{2}$)

$$(\omega I + G)\bar{z} \ni 2\omega\bar{u}.$$

Then, $\bar{z}$ is given by

$$\bar{z} = \begin{cases} \frac{2\omega\bar{u}}{1+\omega} & \text{if} & 2\omega\bar{u} < (1+\omega)u_v \\ u_v & \text{if} & (1+\omega)u_v < 2\omega\bar{u} < (1+\omega)u_v + \lambda_v \\ \frac{2\omega\bar{u}-\lambda_v}{1+\omega} & \text{if} & (1+\omega)u_v + \lambda_v < 2\omega\bar{u} < (1+\omega)u_p + \lambda_v \\ u_p & \text{if } (1+\omega)u_p + \lambda_v < 2\omega\bar{u} < (1+\omega)u_p + \lambda_v + \lambda_p \\ \frac{2\omega\bar{u}-\lambda_v-\lambda_p}{1+\omega} & \text{if} & (1+\omega)u_p + \lambda_v + \lambda_p < 2\omega\bar{u} \end{cases}$$

4 Numerical Results

4.1 Propagation of a circular fire front with non local radiation

This example corresponds to the numerical approximation of the spread of a circular fire front in a square fuel bed of 3×3 m^2 composed with *Pinus Pinaster* with a fuel load of 1 kg/m^2. We have studied the propagation of the fire front for different moisture contents, taking into account non local radiation but neglecting in a first approach the pyrolysis heat. The fire is ignited at the center of the square. We obtain a circular fire front as it is foreseen.
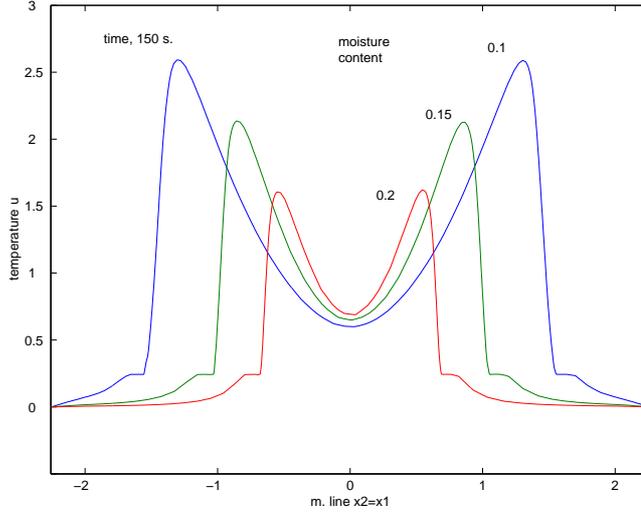
Fig. 2. Temperature at time $150s$ for different moisture contents

We use the following set of parameters: $\kappa(u) = 1.3 \times 10^{-4}$ which is constant since we use non local radiation, $\alpha = 0.02$. We have considered three values for $\lambda_v = 0.6$, 0.9, 1.2 which correspond to a moisture content of 0.1, 0.15, and 0.2 (kg of water/kg of dry fuel).

Fig. 2 shows the temperature profile on the diagonal of the square after 150 seconds from the ignition time for three values of the moisture content, 10%, 15% and 20%. The effects of the moisture can be clearly appreciated in the plate before the fire front where water is still evaporating and the temperature is constant. Furthermore, for higher moisture content we find lower rate of spread and lower maximum temperature in the fire front. In accordance with the experiments. For a given fuel load there is an upper value of fuel moisture above which the fire will not propagate. In this example the critical value is 22% of water. The heat absorbed by pyrolysis is usually much lower than the heat absorbed by water. To see the effect of pyrolysis, we show in Fig. 3 the temperature profiles 50 and 100 seconds after the ignition time, comparing two runs, the first without pyrolysis heat and the second taking a pyrolysis heat equal to $30kJ/kg$.

4.2 Propagation of a linear fire front with wind and slope

We consider the experiments that were carried out in a low speed wind tunnel by Mendes-Lopes in [17]. They were performed in order to observe wind driven effects, slope and moisture content effects in a fire across a bed of pine needles.
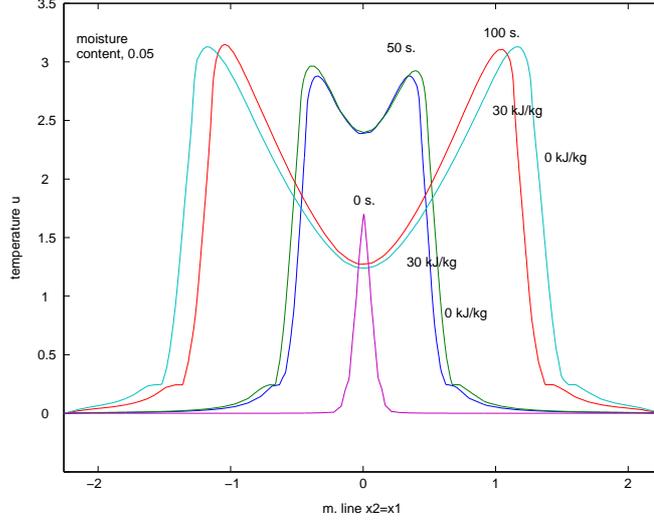
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Fig. 3. Temperature at times $50s.$ and $100s.$ for different pyrolysis heat

The wind speed values cover the range between $-3 \ m/s$ and $3 \ m/s$. The slope can be set at angles from 0^0 up to 15^0 by means of a movable tray. The tray is 2.5 m long and 1.2 m wide. The fuel bed occupies only the central part of the tray (0.70 m wide) and consists of a layer of *Pinus pinaster* needles with a load of approximately 0.5 kg/m^2 on a dry basis. A fuel moisture content equal to 10% is considered. Only local radiation is assumed.

We have used the following set of parameters

$$q = 4., \ \kappa = 5. \times 10^{-5}, \ \alpha = 0.1$$

For a moisture content equal to 10% the corresponding non-dimensional evaporation heat is $\lambda_v = 0.3$. The values of the parameters in the convection model are

$$\xi = 0.1, \ \lambda = 3.$$

Furthermore, in order to take into account the stimulation of combustion by the wind, we consider an empirical law as in Simeoni [18] for the parameter β, more precisely

$$\beta = 0.2 + 0.02v_m^2,$$

where v_m represents the inflow velocity of the wind.

The following configurations have been studied: Wind driven up-slope fires for wind ranging from 0 to 3 m/s, 0% slope and 10% slope, moisture content value equal to 10%.

Fig. 4 shows the influence of wind velocity on the rate of spread for 0^0 and 10^0
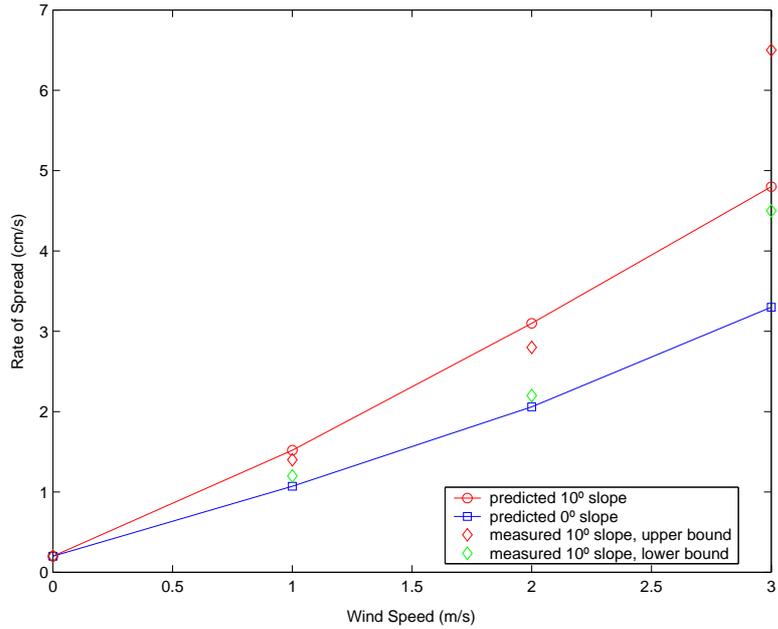
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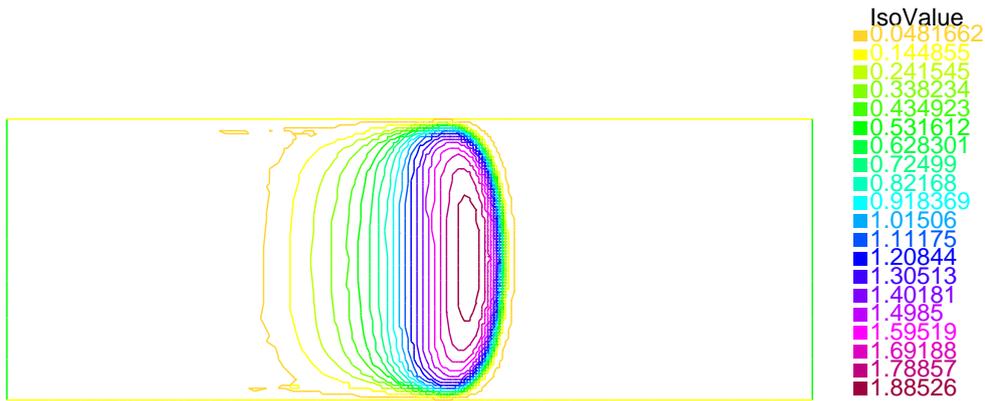Fig. 4. Influence of slope on the rate of spread.

Fig. 5. Temperature contours.

slope. The results are compared with the experimental data given in Mendes-Lopes [17]. A good correlation between computed and experimental results is obtained, up to $2m/s$ wind velocity. The comparison for a wind speed equal to $3m/s$ is less significant because in this case, as indicated in [17], the scattering of data is high due to the turbulent nature of the flow that is enhanced with increasing wind speed. Fig. 5 shows the corresponding typical picture of temperature contours, at time $30s$. The shape of the fire front agrees with the experiments.

The computing software used is the FreeFem++ [6], all the computations have been done using P1-Lagrange finite element approximation and anisotropic adaptivity, the tolerance value of the algorithm presented in this work is 1.e-4. The computational cost for the program taking away the high time con-

suming radiation term is sufficiently low for running real time experiments in hand held computers. This draw back has made us work in efficient radiation models with real-time computations that will be introduced in following works.

The following table shows an example of the time consumed by the cpu differentiating the radiation term and the rest of the program taken from the numerical simulations. The time step used is 0.1 sg which is about the time it takes to solve the problem without the radiation term.

Number of Nodes	Radiation term (cpu)	Rest of the program (cpu)
4000	1.65 sg	0.077 sg
7097	1.77 sg	0.092 sg
10311	2.3 sg	0.1525 sg

5 Conclusions

A simplified fire spread model is presented, taking into account the dominant thermal transfer mechanism in this kind of combustion. The use of multi-valued operators allows to correctly represent the physical phenomena of the problem and it is a good tool to model moisture content from the point of view of mathematics and numerical analysis. The maximal monotone property of this operator allows the implementation of a numerical algorithm with well-known convergence properties. Radiation is a non local phenomena but in certain circumstances can be approximated by a non linear diffusion term. This approach avoids the need of solving a convolution operator representing the radiation from the flame, which is costly from a numerical point of view. The numerical examples show the effect of the vegetation moisture decreasing the velocity of the fire front, as well as, the effect of wind and slope on fire propagation increasing the velocity of the fire front in the corresponding direction.

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