Euler–Rodrigues frames on spatial Pythagorean-hodograph curves

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Abstract

We investigate the properties of a special kind of frame, which we call the Euler–Rodrigues frame (ERF), defined on the spatial Pythagorean-hodograph (PH) curves. It is a frame that can be naturally constructed from the PH condition. It turns out that this ERF enjoys some nice properties. In particular, a close examination of its angular velocity against a rotation-minimizing frame yields a characterization of PH curves whose ERF achieves rotation-minimizing property. This computation leads into a new fact that this ERF is equivalent to the Frenet frame on cubic PH curves. Furthermore, we prove that the minimum degree of non-planar PH curves whose ERF is an rotation-minimizing frame is seven, and provide a parameterization of the coefficients of those curves.

Keywords: Euler–Rodrigues frame; Pythagorean-hodograph curve; Rotation-minimizing frame; Quaternion

1. Introduction

A sweep surface (Jüttler and Mäurer, 1999; Pottmann and Wagner, 1998; Wang and Joe, 1997) is a preferred technique to model surfaces in the CAD systems. It is created by extruding a planar profile (or cross-section) curve along a spatial spine (or axial) curve. During the extrusion, the profile curve is always on the normal plane of the spine curve, which requires a local coordinate system, i.e., an ordered orthonormal basis on the normal plane. By selecting such ordered orthonormal basis on each normal plane, we build two vector fields on the spine curve. Together with the unit tangent vector field, those two vector fields constitute a frame (field) on the spine curve.
In this article, we investigate the properties of a frame that is defined on the spatial Pythagorean-hodograph (PH) curves. This frame comes out naturally from the quaternion formulation (Farouki et al., 2002a, 2002b) of PH curves. As a special instance of “PH representation map” proposed by Choi et al. (2002), this formulation entails a rational parameterization of special orthogonal group SO(3) of \( \mathbb{R}^3 \), known as the Euler–Rodrigues parameterization (Altmann, 1986; Bottema and Roth, 1990) in kinematics. The three column vectors of SO(3) comprise a frame on the PH curve with the first one as the unit tangent vector of the curve. Hence, we propose to name this frame Euler–Rodrigues frame (ERF) of PH curves.

The biggest practical advantage of having the ERF over the usual frames such as Frenet frames (Gray, 1993; O'Neill, 1966) in the geometric modeling systems is that it is rational. For example, in order to construct a rational sweep surface or generalized cylinders (Kim et al., 1994), frames should also be represented as a rational function of the spine curve parameter, which cannot be expected of Frenet frames for general curves. (In this regard, Wagner and Ravani (1997) studied a special type of rational curves whose Frenet frame is rational.) Moreover, ERF is defined “everywhere” on PH curves in contrast to Frenet frames that cannot be defined on curves’ inflection points (i.e., where the curvature vanishes), which incurs an abrupt flip of their orientation near those points.

Another issue in deciding which frame to adopt is their variation of orientation. Here we only consider frames whose, say, first vector coincides with the tangent vector of the curves. Then the remaining two vectors, to keep orthogonal to the first one, are forced to change their directions accordingly. Without breaking the orthogonality, however, the two vectors have one degree of freedom to rotate on the normal plane of the curves. A frame that does not execute such extra rotation on the normal plane is called a rotation-minimizing frame (RMF) (Bishop, 1975; Guggenheimer, 1989; Klok, 1986; Mäurer and Jüttler, 1999; Wang and Joe, 1997), and it serves as a reference to other frames. Put in the language of differential geometry, this amounts to saying that each of the vector fields that constitute the frame is a parallel section of the normal bundle of the curve. Rotation-minimizing property is regarded as a desirable attribute of frames in the CAD systems since a rotation-minimizing frame prevents an unexpected distortion of the sweep surface that might occur when an arbitrary frame is used instead. Once an RMF is established on a given curve, its controlled rotation can be readily introduced by a curve on SO(2).

The ERFs are not in general RMFs. In fact, since the RMF condition is expressed by an ordinary differential equation, it is difficult to obtain its closed-form expression for general curves. However, we will show that the rate of angular deviation of an ERF from an RMF can be expressed by rational functions, and in particular if the PH spine curves are cubic or quintic, those rational functions admit, without any numerical approximation, an exact integration. (In this regard, Farouki (2002) has shown that the Frenet frame also enjoys the equivalent property.) Hence, we can have, though not rational, exact formula for RMF on PH cubic and quintic curves, and if necessary, we can employ any rational approximation (such as the one introduced in (Mäurer and Jüttler, 1999)) of the angular deviation to get rational approximation of the RMF. Moreover, we will characterize the class of PH curves whose ERFs satisfy rotation-minimizing property.

The existence of ERF was first indicated by Jüttler (1998a). In fact, the origin of quaternion representation of PH curves can also be traced back to the work of Dietz, Hoschek, and Jüttler (1993) to characterize rational curves on the unit sphere in \( \mathbb{R}^3 \).

We develop our results as follows. In Section 2 we present a review of basic properties of quaternion algebra that we need to explain the quaternion representation of PH curves in Section 3. After the explanation of RMF and the related issues in Section 4, we define the ERF and compute its angular velocity in Section 5. In this section, we investigate the features of ERF on cubic PH curves and
then prove that the minimum degree of non-planar PH curves whose ERF satisfies rotation-minimizing property is seven. Finally, some concluding remarks follow in Section 6.

2. Quaternion algebra

As we have mentioned, our results are based on observations on the quaternions that link PH curves and rotations in 3-dimensional space together. Thus it is advantageous to review basic properties of quaternion algebra, particularly its description of 3-dimensional rotations. Readers interested in this subject should refer more comprehensive textbooks such as (Altmann, 1986) or lately published (Kuipers, 1999). Also, its direct applications in the motion planning can be found in (Kim and Nam, 1995).

Consider a 4-dimensional real vector space \( \mathbb{R}^4 \), and let \( \{1, i, j, k\} \) be its standard basis

\[
1 = (1, 0, 0, 0), \quad i = (0, 1, 0, 0), \quad j = (0, 0, 1, 0), \quad k = (0, 0, 0, 1).
\]

This vector space is qualified as the algebra \( \mathbb{H} \) of (real) quaternions after furnished with multiplication induced by the rules

\[
i^2 = j^2 = k^2 = ijk = -1
\]

and \( 1 \) acting as identity element. It follows from (1) that

\[
i j = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]

Thus the quaternion multiplication is non-commutative, i.e., \( AB \neq BA \), for general quaternions. In this article, quaternion numbers will always be denoted by calligraphic letters, such as \( A = a1 + axi + ayj + azk \). (We usually omit \( 1 \) in \( a1 \) and denote it by the scalar notation \( a \).)

Then the product of two quaternions is explicitly given by

\[
AB = (ab - axbx - ayby - azbz) + (abx + axb + aybz - azby)i + (aby - axbz + ayb + azbx)j + (abz + axby - aybx + azb)k.
\]

The 3-dimensional subspace \( \mathbb{R}^3 \) is naturally embedded in \( \mathbb{H} \) by identifying each vector \( a = (ax, ay, az) \) with the quaternion \( 0 + axi + ayj + azk \). This identification proves to be useful when we come to 3-dimensional rotations. In fact, for any quaternion \( A = a + a_i i + a_j j + a_k k \), we call \( a_i i + a_j j + a_k k \) the “vector” part, while \( a \) the “scalar” part.

The product of \( A \) with its \textit{conjugate} \( A^* \), defined as \( a - bi - cj - dk \), is simply a positive real number

\[
AA^* = a^2 + a_i^2 + a_j^2 + a_k^2,
\]

which turns the algebra \( \mathbb{H} \) into a \textit{field} with the inverse \( A^{-1} = A^*/(AA^*) \) for \( A \neq 0 \). Here the positive square root of \( AA^* \), denoted by \( |A| \), is called the \textit{magnitude} of \( A \). Also note that \( (AB)^* = B^* A^* \).

The unit quaternions, i.e., quaternions with unit magnitude, compactly describe 3-dimensional rotations. A 3-dimensional rotation is characterized by its axis, designated by a unit vector \( n \), and angle \( \theta \). Then the corresponding unit quaternion is (employing the decomposition of quaternion into scalar and vector part)

\[
U = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n.
\]
Now, for any given vector $v$ in $\mathbb{R}^3$, its transform under the specified rotation is expressed by quaternion product $UvU^*$. (Here, $v$ is naturally regarded as quaternion.) Note that the correspondence between unit quaternions and 3-dimensional rotations is two-to-one; both $U$ and $-U$ describe exactly the same rotation. (This is mathematically equivalent to saying that SU(2) double-covers SO(3).)

3. Quaternion representation of PH curves

The PH curves considered in this article are polynomial curves in the 3-dimensional space given by $r(t) = (x(t), y(t), z(t))$ satisfying

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$$

for some real polynomial $\sigma(t)$. The first attempt to characterize their components is made in the concluding remarks of (Farouki and Sakkalis, 1990) as follows:

$$x'(t) = h(t)[u(t)^2 - v(t)^2 - w(t)^2],$$
$$y'(t) = 2h(t)u(t)v(t),$$
$$z'(t) = 2h(t)u(t)w(t),$$

where $h(t)$, $u(t)$, $v(t)$, and $w(t)$ are real polynomials. This is, however, only a sufficient condition, as Farouki et al. admitted. Although they later completely analyzed cubic PH curves (Farouki and Sakkalis, 1994) in terms of control polygons, the correct conditions were still unknown.

But a necessary and sufficient condition had been already discovered in other context. In their search of rational expressions of curves and surfaces on the unit sphere, Dietz et al. (1993) proved that the relatively prime polynomials $x'(t)$, $y'(t)$, $z'(t)$, and $\sigma(t)$ of (2) had the form

$$x'(t) = a(t)^2 + b(t)^2 - c(t)^2 - d(t)^2,$$
$$y'(t) = 2[a(t)d(t) + b(t)c(t)],$$
$$z'(t) = 2[-a(t)c(t) + b(t)d(t)],$$
$$\sigma(t) = \pm[a(t)^2 + b(t)^2 + c(t)^2 + d(t)^2],$$

with real polynomials $a(t)$, $b(t)$, $c(t)$, and $d(t)$. Those $x'(t)$, $y'(t)$, $z'(t)$, and $\sigma(t)$ with common factor $h(t)$ also admit similar representation. By denoting $x'(t) = h(t)\tilde{x}'(t)$, $y'(t) = h(t)\tilde{y}'(t)$, $z'(t) = h(t)\tilde{z}'(t)$, and $\sigma(t) = h(t)\tilde{\sigma}(t)$, it is easy to see that $\tilde{x}'(t)$, $\tilde{y}'(t)$, $\tilde{z}'(t)$, and $\tilde{\sigma}(t)$ are then relatively prime Pythagorean quadruples and thus

$$\tilde{x}'(t) = \tilde{a}(t)^2 + \tilde{b}(t)^2 - \tilde{c}(t)^2 - \tilde{d}(t)^2,$$
$$\tilde{y}'(t) = 2[\tilde{a}(t)\tilde{d}(t) + \tilde{b}(t)\tilde{c}(t)],$$
$$\tilde{z}'(t) = 2[-\tilde{a}(t)\tilde{c}(t) + \tilde{b}(t)\tilde{d}(t)],$$
$$\tilde{\sigma}(t) = \pm[\tilde{a}(t)^2 + \tilde{b}(t)^2 + \tilde{c}(t)^2 + \tilde{d}(t)^2]$$

for some polynomials $\tilde{a}(t)$, $\tilde{b}(t)$, $\tilde{c}(t)$, and $\tilde{d}(t)$.
By the way, those expressions in (3) are familiar in the province of motion designs (Bottema and Roth, 1990; Jüttler, 1995, 1998b; Jüttler and Wagner, 1999); after divided by $\Delta = a^2 + b^2 + c^2 + d^2$, they constitute the first column of the SO(3) matrix $U$.

$$U = \frac{1}{\Delta} \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(-ad + bc) & 2(ac + bd) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(-ab + cd) \\ 2(-ac + bd) & 2(ab + cd) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}$$

(5)

where $a$, $b$, $c$, and $d$ are called the Euler–Rodrigues parameters (Altmann, 1986). This observation entails an elegant formulation of PH curves (Choi et al., 2002). Since the matrix $U$ represents a rotational transformation in $\mathbb{R}^3$, its first column, $U (1, 0, 0)^T$, is the image of the vector $i = (1, 0, 0)$ under the rotation. Now, as we mentioned in the previous section, such rotation can also be represented by $U i U^*$ with some unit quaternion $U$. (This representation was exploited by Miura (2000) for designing fair curves.) Inspired by this correspondence, we can readily verify that

$$A i \tilde{A}^* = (a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(-ac + bd)k$$

(6)

for $A = a + bi + cj + dk$. Hence we have an equivalent formulation of PH curves

$$r'(t) = A(t) i \tilde{A}^*(t)$$

(7)

for (3), and

$$r'(t) = h(t), \tilde{A}(t) i \tilde{A}^*(t)$$

(8)

for (4), by quaternion polynomials $A(t) = a(t) + b(t)i + c(t)j + d(t)k$ and $\tilde{A}(t) = \tilde{a}(t) + \tilde{b}(t)i + \tilde{c}(t)j + \tilde{d}(t)k$.

4. Rotation minimizing frames

An ordered set $E = \{e_1(t), e_2(t), e_3(t)\}$ of mutually orthogonal unit vector fields on a curve $r(t)$ is called a frame (field) on the curve. In this article we will only consider positively-oriented frame fields, i.e., $e_1(t) \cdot (e_2(t) \times e_3(t)) \equiv 1$. We call $E$ adapted if $e_1(t) = r'(t)/\|r'(t)\|$. Then, the plane spanned by $\{e_2(t), e_3(t)\}$ and passing $r(t)$ is called normal plane to the curve. Adapted frames are useful to create surfaces; given a planar curve $p(u) = (p_1(u), p_2(u))$ and a spatial curve $r(t)$, the sweep surface $S(t, u)$ is defined by

$$S(t, u) = r(t) + p_1(u) e_2(t) + p_2(u) e_3(t).$$

In this context, we call $p(u)$ the profile curve and $r(t)$ the spine curve (Pottmann and Wagner, 1998). Note that the profile curve always lies on the normal plane.

An important example of adapted frames is the Frenet frame $F = \{t, n, b\}$ (O’Neill, 1966) (hereafter, the time parameter $t$ will be sometimes omitted for brevity’s sake):

$$t = \frac{r'}{|r'|}, \quad n = \frac{(r' \times r'') \times r'}{|r' \times r''||r'|}, \quad b = \frac{r' \times r''}{|r' \times r''|}.$$  

(9)

Although the Frenet frame is an indispensable tool for the theory of curves in 3-dimensional Euclidean space, it has several drawbacks in applications: first, it cannot be defined on the inflection points; second, it is generally non-rational function of $t$; and finally it has undesirable rotation about $t$. 

But “there is more than one way to frame a curve” (Bishop, 1975). Knowing that the undesirable rotation of the Frenet frame may incur the distortion of the corresponding sweep surface, we need to construct an adapted frame whose normal plane stays as still as possible. Let us elaborate on this idea. Given a curve \( r(t) \), we set \( e_1(t) = r'(t)/|r'(t)| \). Selecting an arbitrary unit vector \( e_2(t) \) on the normal plane of \( r(t) \) for each \( t \) and then defining \( e_2(t) = e_1(t) \times e_2(t) \), which is also on the normal plane, determines an adapted frame \( E = \{ e_1(t), e_2(t), e_3(t) \} \). We want to pick each unit vector in such a way that the resultant vector field \( e_2(t) \) varies as small as possible. The variation can be decomposed into two components; since \( |e_2(t)| \) is constant, we have \( e_2'(t) \perp e_2(t) \), i.e., \( e_2'(t) = ae_1(t) + be_3(t) \). While the coefficient \( a \) makes for the inevitable change of \( e_2(t) \) to keep orthogonal to the varying \( e_1(t) \), \( b \) is responsible for the rotation of \( e_2(t) \) on the normal plane. Hence, to achieve “minimum” variation of \( e_2(t) \), we must have \( b = 0 \). Klok (1986) computed the coefficient \( a \) and showed that \( e_2(t) \) should satisfy the following ordinary differential equation

\[
e'(t) = -\frac{r''(t) \cdot e(t)}{|r'(t)|^2} r'(t) .
\] (10)

We call an adapted frame \( E = \{ e_1(t), e_2(t), e_3(t) \} \) a rotation-minimizing frame (RMF) if \( e_2(t) \) and \( e_3(t) \) are the solutions of (10). This is equivalent to saying that the normal connections of both \( e_2(t) \) and \( e_3(t) \) on \( r(t) \) are zero, i.e., both the projections of \( e_2'(t) \) and \( e_3'(t) \) to the normal plane are zero;

\[
e_2'(t) \cdot e_3(t) = e_3'(t) \cdot e_2(t) = 0. \] (11)

Consequently, \( e_2'(t) \) and \( e_3'(t) \) are parallel to \( e_1(t) \). Note that a curve can have an infinite number of RMFs; distinct sets of initial positions of \( e_2(t) \) and \( e_3(t) \) produce different frames.

Now any other adapted frame \( F = \{ f_1(t), f_2(t), f_3(t) \} \) can be represented by an angular deviation \( \theta(t) \) from an RMF:

\[
f_1(t) = e_1(t), \quad \begin{pmatrix} f_2(t) \\ f_3(t) \end{pmatrix} = \begin{pmatrix} \cos \theta(t) & \sin \theta(t) \\ -\sin \theta(t) & \cos \theta(t) \end{pmatrix} \begin{pmatrix} e_2(t) \\ e_3(t) \end{pmatrix} .
\]

Since frames are usually obtained from mutually orthogonal vector fields by dividing their magnitudes, it is convenient to have a formula for computing angular deviation that is still valid for non-unit \( f_2(t) \) and \( f_3(t) \). Now, note that we have in general (here, we are not assuming that \( |f_i(t)| = 1 \))

\[
f_2(t) = |f_2(t)| \left[ e_2(t) \cos \theta(t) + e_3(t) \sin \theta(t) \right],
\]

\[
f_3(t) = |f_3(t)| \left[ -e_2(t) \sin \theta(t) + e_3(t) \cos \theta(t) \right].
\]

Using (11) (and \( e_k' \cdot e_k = 0 \) since \( |e_k| \) is constant), we can easily verify that the angular velocity \( \omega(t):= \theta'(t) \) satisfies

\[
f_2'(t) \cdot f_2(t) = |f_2(t)||f_2(t)|\omega(t),
\]

\[
f_3'(t) \cdot f_3(t) = -|f_2(t)||f_3(t)|\omega(t).
\]

Thus we have

\[
\omega(t) = \frac{f_2'(t) \cdot f_3(t)}{|f_2(t)||f_3(t)|} = \frac{-f_2(t) \cdot f_3'(t)}{|f_2(t)||f_3(t)|} .
\] (12)

The angular velocity is positive when the frame is rotating counterclockwise (with respect to an RMF) about \( f_1(t) \) axis.
Remark 1. The angular velocity $\omega$ used here does not coincide with that of the frames in the kinematics, where the angular velocity matrix $\Omega$ is defined by $\Omega = F^T F'$ ($F$ is considered as a matrix such that its columns correspond to $f_i$'s), and the angular velocity is the vector representation of $\Omega$ (Bottema and Roth, 1990). In fact, our version of angular velocity is just the $(2, 3)$-element of the Cartan matrix $(F^T)'F$ of the frame (Bishop, 1975; O’Neill, 1966). RMF can be defined by using either the angular velocity matrix (Jüttler and Wagner, 1999) or the Cartan matrix (Bishop, 1975). In this article, the angular velocity $\omega$ always refers to the time-derivative of the deviation angle $\theta$ as presented above.

Example 2. The angular velocity $\omega_F(t)$ of Frenet frame is
\[
\omega_F = \frac{\det(r', r'', r''')}{|r' \times r''|^2} |r'| = \tau |r'|, \tag{13}
\]
where $\tau(t)$ is the torsion of the curve (Gray, 1993). (Here, it is convenient to use $(r' \times r'') \times r'$ and $r' \times r''$ for $n$ and $b$.) The sign convention of torsion here is given by $b' = -\tau n$. That is, the curve has a positive torsion when the normal plane spanned by $\{n, b\}$ rotates counterclockwise.

5. Euler–Rodrigues frames

In this section, we introduce a rational frame naturally defined on PH curves. First, suppose a PH curve $r(t)$ satisfies (7). Then, the unit quaternion $U(t) = A(t)/|A(t)|$ defines a rotation in $\mathbb{R}^3$, and thus three vector fields
\[
\begin{align*}
    f_1(t) &= U(t)iU^*(t), \\
    f_2(t) &= U(t)jU^*(t), \\
    f_3(t) &= U(t)kU^*(t)
\end{align*}
\]
constitute an adapted rational frame $F = \{f_1(t), f_2(t), f_3(t)\}$. Note that $f_1(t)$, $f_2(t)$, and $f_3(t)$ correspond exactly to the column vectors of $U$ of (5).

Definition 3. Let $A(t) = a(t) + b(t)i + c(t)j + d(t)k$ be a quaternion polynomial and $r(t)$ a PH curve defined by $r'(t) = A(t)iA^*(t)$. The Euler–Rodrigues frame (ERF) $F = \{f_1(t), f_2(t), f_3(t)\}$ of $r(t)$ is an adapted frame defined by (14).

Now suppose $r(t)$ satisfies (8). Similarly we can define the unit quaternion $\tilde{U}(t) = \tilde{A}(t)/|\tilde{A}(t)|$ and a rational frame $\tilde{F} = \{\tilde{f}_1(t), \tilde{f}_2(t), \tilde{f}_3(t)\}$,
\[
\begin{align*}
    \tilde{f}_1(t) &= \tilde{U}(t)i\tilde{U}^*(t), \\
    \tilde{f}_2(t) &= \tilde{U}(t)j\tilde{U}^*(t), \\
    \tilde{f}_3(t) &= \tilde{U}(t)k\tilde{U}^*(t).
\end{align*}
\]
Now on the interval of $t$ such that $h(t) > 0$, since $\tilde{f}_1(t)$ points to the direction of $r'(t)$ in that interval, $\tilde{F}$ is an adapted frame of $r(t)$. In this case, we define $\tilde{F}$ as an ERF of $r(t)$. (We need to reverse the direction of $\tilde{f}_1(t)$, and one of the other $\tilde{f}_i(t)$'s in the region of $h(t) < 0$.)

Remark 4. Our subsequent analysis on ERF is focused on the PH curves of type (7). However, we found that the properties of ERF on the PH curves of type (8) are not much different. We will come back to this issue in Section 5.4.
We compute the angular deviation of an ERF against the RMF using (12). Here we apply \( f_2(t) = A(t)jA^*(t) \) and \( f_3(t) = A(t)kA^*(t) \).

\[
\omega = 2 \frac{-a'b + ab' + c'd - cd'}{a^2 + b^2 + c^2 + d^2}.
\] (15)

Real polynomials \( a(t), b(t), c(t), \) and \( d(t) \) of degree \( n \) construct a PH curve of odd degree \( 2n + 1 \). Since the leading terms of the numerator cancel each other, the angular velocity is a rational function whose numerator and denominator are of degree \( 2n - 2 \) and \( 2n \). We can obtain the angular deviation by integrating (15) using a partial fraction decomposition of the rational function. For cubic and quintic PH curves, this can be carried out exactly since the denominators are of degree 2 and 4, thereby admitting the exact factorizations. For higher degree curves, a numerical method is required to factorize the denominators whose degree is greater than 5.

In the following subsections, we discuss the ERF on cubic, quintic, and PH curves of degree 7 represented by \( r'(t) = A(t)iA^*(t) \) in detail. The corresponding quaternion polynomials \( A(t) \) are written in Bernstein form:

\[
A(t) = \sum_{k=0}^{n} \binom{n}{k} A_k (1-t)^{n-k} t^k,
\] (16)

where \( A_k \) are quaternion constants. When we write \( A(t) = a(t) + b(t)i + c(t)j + d(t)k \), the Bernstein form of the real polynomials \( a(t), b(t), c(t), \) and \( d(t) \) are

\[
a(t) = \sum_{k=0}^{n} \binom{n}{k} a_k (1-t)^{n-k} t^k,
b(t) = \sum_{k=0}^{n} \binom{n}{k} b_k (1-t)^{n-k} t^k,
c(t) = \sum_{k=0}^{n} \binom{n}{k} c_k (1-t)^{n-k} t^k,
d(t) = \sum_{k=0}^{n} \binom{n}{k} d_k (1-t)^{n-k} t^k.
\] (17)

Note that we have \( A_k = a_k + b_ki + c_kj + d_kk \).

5.1. ERF on cubic PH curves

We denote the linear polynomials \( a(t), b(t), c(t), \) and \( d(t) \) of \( A(t) \) in Bernstein form:

\[
a(t) = a_0(1-t) + a_1t,
b(t) = b_0(1-t) + b_1t,
c(t) = c_0(1-t) + c_1t,
d(t) = d_0(1-t) + d_1t.
\] (18)

By substituting (18) in (15), we have

\[
\omega = 2 \frac{a_0b_1 - a_1b_0 - c_0d_1 + c_1d_0}{a^2 + b^2 + c^2 + d^2}.
\] (19)
Since the numerator is constant and the denominator is always positive, the angular velocity does not change its sign, i.e., the frame rotates only in one direction. Note that the denominator is the speed $|r'(t)|$ of the curve, so the product of the frame’s angular velocity and the curve’s speed is constant.

In view of (19) an ERF of a cubic PH curve becomes an RMF if and only if

$$a_0b_1 - a_1b_0 - c_0d_1 + c_1d_0 = 0.$$  \hspace{1cm} (20)

We know that PH condition (2) imposes a constraint on the geometry of the cubic curves. Namely, PH cubic curves are generic helices, spatial curves whose curvature and torsion are in constant proportion (Farouki and Sakkalis, 1994). RMF constraint (20) limits the curves’ degree of freedom once more—only planar PH cubic curves satisfy the constraint. One may prove this by a direct computation, but we have found that it is an easy corollary of the following unexpected fact.

**Theorem 5.** An ERF maintains a constant angle against the Frenet frame on cubic PH curves.

For the proof, it suffices to show that the rate of angular deviation $\omega_F$ of Frenet frame is equal to (19), which can be verified straightforwardly by employing (13) with (7).

The constant angle $\Delta \theta$ between $\mathbf{f}_3(t)$ of ERF and $\mathbf{b}(t)$ of Frenet frame satisfies the following equation:

$$\cos \Delta \theta = \frac{a_0d_1 - a_1d_0 - b_0c_1 + b_1c_0}{\sqrt{D}},$$

where

$$D = (a_0a_1 - c_0c_1)^2 - (a_0a_1 + d_0d_1)^2 + (a_0c_1 - b_1d_0)^2$$
$$+ (a_0d_1 + b_1c_0)^2 + (a_1c_0 - b_0d_1)^2 + (a_1d_0 + b_0c_1)^2$$
$$- (b_0b_1 + c_0c_1)^2 + (b_0b_1 - d_0d_1)^2.$$

Hence, given a PH cubic curve, its ERF is an RMF if and only if so is its Frenet frame. But the Frenet frame is an RMF if and only if its torsion is constantly zero (13), i.e., if and only if the curve is planar.

**Corollary 6.** The ERF of a cubic PH curve is an RMF if and only if the curve is planar.

So, cubic PH curves whose ERF possesses rotation-minimizing property are planar PH curves, known as “Tschirnhausen cubics” (Farouki and Sakkalis, 1990). Rotation-minimizing condition is too severe for cubic PH curves to retain their identity as workable spatial curves. But they are nevertheless useful as spine curves for rational sweep surfaces. Their angular velocity (19) admits exact integration in terms of trigonometric functions. A rational approximation of this deviation angle, such as the one in (Mäurer and Jüttler, 1999), can be used to adjust the ERF to an RMF. Hence, we can obtain rational approximations of RMF on arbitrary cubic PH curves.

5.2. ERF on quintic PH curves

We denote quadratic polynomials $a(t), b(t), c(t),$ and $d(t)$ of $A(t)$ in Bernstein form:

$$a(t) = a_0(1 - t)^2 + 2a_1(1 - t)t + a_2t^2,$$
$$b(t) = b_0(1 - t)^2 + 2b_1(1 - t)t + b_2t^2,$$
$$c(t) = c_0(1 - t)^2 + 2c_1(1 - t)t + c_2t^2,$$
$$d(t) = d_0(1 - t)^2 + 2d_1(1 - t)t + d_2t^2.$$ \hspace{1cm} (21)
By substituting (21) in (15), we have
\[
\omega = 4 \frac{A(1-t)^3 + (A-C)(1-t)^2t + (B-C)(1-t)t^2 + Bt^3}{a^2 + b^2 + c^2 + d^2}.
\] (22)

where
\[
A = a_0 b_1 - a_1 b_0 - c_0 d_1 + c_1 d_0, \\
B = a_1 b_2 - a_2 b_1 - c_1 d_2 + c_2 d_1, \\
C = a_2 b_0 - a_0 b_2 - c_2 d_0 + c_0 d_2.
\] (23)

Note that the numerator of (22) is in fact quadratic although it is of cubic Bernstein form.

The ERF of a quintic PH curve is an RMF if and only if the system of homogeneous quadratic equations \( A = 0, B = 0, C = 0 \) is satisfied. Since this is a system of three equations for twelve real unknowns, we expect these twelve unknowns to be expressed in terms of nine independent parameters.

**Theorem 7.** The ERF of a quintic PH curve is an RMF if and only if the twelve coefficients \( a_i, b_i, c_i, \) and \( d_i \) for \( i = 0, 1, 2 \) are expressed by nine independent variables \( r, s, u, v, w, x, y, \theta, \phi, \) and \( \phi \) as follows.

If \( a_0^2 + b_0^2 > 0 \) and \( c_0^2 + d_0^2 > 0, \)
\[
\begin{align*}
a_0 & = r \cos \theta, \quad b_0 = r \sin \theta, \quad c_0 = s \cos \phi, \quad d_0 = s \sin \phi, \\
a_1 & = u \cos \theta - v \sin \theta, \quad b_1 = u \sin \theta + v \cos \theta, \\
c_1 & = w \cos \phi - \frac{r \cos \phi}{s} \sin \phi, \quad d_1 = w \sin \phi + \frac{r \sin \phi}{s} \cos \phi, \\
a_2 & = x \cos \theta - y \sin \theta, \quad b_2 = x \sin \theta + y \cos \theta, \\
c_2 & = \frac{1}{r \cos \phi} (r w y - s u y + s v x) \cos \phi - \frac{r \cos \phi}{s} \sin \phi, \\
d_2 & = \frac{1}{r \cos \phi} (r w y - s u y + s v x) \sin \phi + \frac{r \cos \phi}{s} \cos \phi,
\end{align*}
\] (24)

where \( r > 0 \) and \( s > 0. \)

If \( a_0 = b_0 = 0 \) then we may write \( c_0 = s_0 \cos \phi \) and \( d_0 = s_0 \sin \phi, \) or if \( c_0 = d_0 = 0 \) then we may write \( a_0 = r_0 \cos \theta \) and \( b_0 = r_0 \sin \theta. \) In either case, the remaining coefficients are expressed as follows:
\[
\begin{align*}
a_1 & = r_1 \cos \theta, \quad b_1 = r_1 \sin \theta, \quad c_1 = s_1 \cos \phi, \quad d_1 = s_1 \sin \phi, \\
a_2 & = r_2 \cos \theta, \quad b_2 = r_2 \sin \theta, \quad c_2 = s_2 \cos \phi, \quad d_2 = s_2 \sin \phi.
\end{align*}
\] (25)

**Proof.** In view of (23), the equations \( A = 0, B = 0, C = 0 \) are equivalent to the following ones
\[
\frac{1}{2} \det \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} = \frac{1}{2} \det \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix},
\] (26)
\[
\frac{1}{2} \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \frac{1}{2} \det \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix},
\] (27)
\[
\frac{1}{2} \det \begin{pmatrix} a_2 & a_0 \\ b_2 & b_0 \end{pmatrix} = \frac{1}{2} \det \begin{pmatrix} c_2 & c_0 \\ d_2 & d_0 \end{pmatrix}.
\] (28)
Let $P_i = (a_i, b_i)$ and $Q_i = (c_i, d_i)$ for $i = 0, 1, 2$. Then the above equations geometrically mean that the (signed) area of $\triangle OP_iP_{i+1}$ is equal to that of $\triangle OQ_iQ_{i+1}$.

Suppose $P_0$ and $Q_0$ lie on the positive half of $x$-axis, i.e., $a_0 = r$, $b_0 = 0$, $c_0 = s$, and $d_0 = 0$ with $r > 0$ and $s > 0$ (see Fig. 1). Set $a_1 = u$ and $b_1 = v$. Then the LHS of (26), the area of $\triangle OP_0P_1$ is $rv/2$.

This compels $d_1$, the (signed) height of $\triangle OQ_0Q_1$ to be $rv/s$, while $c_1$ is arbitrary and to be denoted by $w$.

The same argument goes with (28); we set $a_2 = x$ and $b_2 = y$, and then we must have $d_2 = ry/s$. But $c_2$ is no longer arbitrary. A direct expansion of (27) shows that we must have $c_2 = \frac{1}{r}(rwy - suy + svx)$.

We restore $P_0$ and $Q_0$ to their original positions by rotations about the origin: $P_0 = (r \cos \theta, r \sin \theta)$ and $Q_0 = (s \cos \phi, s \sin \phi)$. And accordingly, $P_1$, $Q_1$, $P_2$, and $Q_2$ will have expression (24).

Now suppose $a_0 = b_0 = 0$ or $c_0 = d_0 = 0$. Then all determinants in equation (26) and (28) are zero, and thus so are those in (27). This is equivalent to saying that the vectors $(a_0, b_0)$, $(a_1, b_1)$, and $(a_2, b_2)$ are parallel; and so are the vectors $(c_0, d_0)$, $(c_1, d_1)$, and $(c_2, d_2)$. These vectors can be compactly represented by two argument angles $\theta$ and $\phi$ as in (25). Note that in this case we should allow non-positive $r_i$ and $s_i$, for $i = 0, 1, 2$. $\square$

It looks like that quintic PH curves have sufficient degrees of freedom for their ERF to be an RMF. But the following somewhat disappointing result shows that they are not much better than cubic PH curves.

**Theorem 8.** The ERF of a quintic PH curve is an RMF if and only if the curve is planar.

**Proof.** Let $r(t)$ be such a quintic curve. With a suitable rotation, we may assume that its initial tangent vector $r'(0)$ is in the direction of the positive $x$-axis. Since $r'(0) = A_0 i A_0^*$, we must have $c_0 = d_0 = 0$ (Farouki et al., 2002a). Then the coefficients of $A(t)$ takes the form of (25), and thus we can write

\[
\begin{align*}
a(t) &= r(t) \cos \theta, & c(t) &= s(t) \cos \phi, \\
b(t) &= r(t) \sin \theta, & d(t) &= s(t) \sin \phi,
\end{align*}
\]

where

\[
r(t) = r_0(1-t)^2 + 2r_1(1-t)t + r_2t^2 \quad \text{and} \quad s(t) = 2s_1(1-t)t + s_2t^2.
\]

This leads to the following expression of $r'(t)$:
\[
\mathbf{r}'(t) = \left(r(t)^2 - s(t)^2, 2r(t)s(t)\sin(\theta + \phi), -2r(t)s(t)\cos(\theta + \phi)\right)
\]
\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \xi & -\sin \xi \\
0 & \sin \xi & \cos \xi
\end{pmatrix}
\begin{pmatrix}
r(t)^2 - s(t)^2 \\
2r(t)s(t)
\end{pmatrix}
\]

where \(\xi = \theta + \phi - \pi/2\). Hence, \(\mathbf{r}(t)\) is on a plane in which the \(x\)-axis lies entirely. \(\square\)

We found that the minimum degree of PH curves that are not planar and their ERF satisfies rotation-minimizing condition is seven.

5.3. ERF on PH curves of degree 7

We denote cubic polynomials \(a(t), b(t), c(t),\) and \(d(t)\) of \(\mathcal{A}(t)\) in Bernstein form:
\[
\begin{align*}
a(t) &= a_0(1-t)^3 + 3a_1(1-t)^2t + 3a_2(1-t)t^2 + a_3t^3, \\
b(t) &= b_0(1-t)^3 + 3b_1(1-t)^2t + 3b_2(1-t)t^2 + b_3t^3, \\
c(t) &= c_0(1-t)^3 + 3c_1(1-t)^2t + 3c_2(1-t)t^2 + c_3t^3, \\
d(t) &= d_0(1-t)^3 + 3d_1(1-t)^2t + 3d_2(1-t)t^2 + d_3t^3.
\end{align*}
\]

The angular velocity of the PH curve of degree 7 is computed to be
\[
\omega = \frac{6}{a^2 + b^2 + c^2 + d^2} \left[ A(1-t)^5 + (A + 2B)(1-t)^4t + (2B + 3C + D)(1-t)^3t^2 \\
+ (3C + D + 2E)(1-t)^2t^3 + (2E + F)(1-t)t^4 + Ft^5 \right],
\]

where
\[
\begin{align*}
A &= a_0b_1 - a_1b_0 - c_0d_1 + c_1d_0, \\
B &= a_0b_2 - a_2b_0 - c_0d_2 + c_2d_0, \\
C &= a_1b_2 - a_2b_1 - c_1d_2 + c_2d_1, \\
D &= a_0b_3 - a_3b_0 - c_0d_3 + c_3d_0, \\
E &= a_1b_3 - a_3b_1 - c_1d_3 + c_3d_1, \\
F &= a_2b_3 - a_3b_2 - c_2d_3 + c_3d_2.
\end{align*}
\]

Note that the numerator of (31) is in fact quartic although it is of quintic Bernstein form.

The ERF of a PH curve of degree 7 is an RMF if and only if the system of homogeneous quadratic equations
\[
\begin{align*}
A &= B = 3C + D = E = F = 0
\end{align*}
\]

are satisfied. We found that it is advantageous to write the coefficients of \(\mathcal{A}(t)\) in polar coordinates:
\[
\begin{align*}
(a_i, b_i) &= (r_i \cos \theta_i, r_i \sin \theta_i), \\
(c_i, d_i) &= (s_i \cos \phi_i, s_i \sin \phi_i)
\end{align*}
\]

for \(i = 0, \ldots, 3\). The characterization of solutions of (33) is easy if the initial tangent vector \(\mathbf{r}'(0)\) points to the positive \(x\)-axis.
Lemma 9. Suppose that $c_0 = d_0 = 0$. Then any solution of (33) should satisfy
\[ \theta_0 = \theta_1 = \theta_2 \pmod{\pi}, \]
and the representations of remaining coefficients are divided into following two cases:

Case 1 ($\theta_3 = \theta_0 \pmod{\pi}$). This is a degenerate case. We have $\phi_1 = \phi_2 = \phi_3 \pmod{\pi}$, while $r_i$ and $s_j$ are arbitrary nonnegative numbers for $i = 0, \ldots, 3$ and $j = 1, 2, 3$.

Case 2 ($\theta_3 \neq \theta_0 \pmod{\pi}$). Then we have arbitrary $\phi_1$, $\phi_2$, and $\phi_3$ under the condition $\phi_i \neq \phi_j \pmod{\pi}$ for $i \neq j$, and
\[
\begin{align*}
    r_0 &= 3 \frac{s_1 s_2}{r_3} \frac{\sin(\phi_2 - \phi_1)}{\sin(\theta_3 - \theta_0)}, \\
    r_1 &= \frac{s_1 s_3}{r_3} \frac{\sin(\phi_3 - \phi_1)}{\sin(\theta_3 - \theta_1)}, \\
    r_2 &= \frac{s_2 s_3}{r_3} \frac{\sin(\phi_3 - \phi_2)}{\sin(\theta_3 - \theta_2)},
\end{align*}
\]
while $s_1$, $s_2$, $s_3$, and $r_3$ are arbitrary positive numbers.

Proof. We write (33) in matrix form:
\[
\begin{align*}
    \det \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} &= \det \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix}, \quad (36) \\
    \det \begin{pmatrix} a_0 & a_2 \\ b_0 & b_2 \end{pmatrix} &= \det \begin{pmatrix} c_0 & c_2 \\ d_0 & d_2 \end{pmatrix}, \quad (37) \\
    3 \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} + \det \begin{pmatrix} a_0 & a_3 \\ b_0 & b_3 \end{pmatrix} &= 3 \det \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} + \det \begin{pmatrix} c_0 & c_3 \\ d_0 & d_3 \end{pmatrix}, \quad (38) \\
    \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} &= \det \begin{pmatrix} c_1 & c_3 \\ d_1 & d_3 \end{pmatrix}, \quad (39) \\
    \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} &= \det \begin{pmatrix} c_2 & c_3 \\ d_2 & d_3 \end{pmatrix} \quad (40)
\end{align*}
\]
Since $c_0 = d_0 = 0$, all determinants of (36) and (37) are zero, which means that the vectors $(a_0, b_0), (a_1, b_1)$, and $(a_2, b_2)$ are parallel. This leads to the condition (35). Then, (38) is reduced to
\[
\det \begin{pmatrix} a_0 & a_3 \\ b_0 & b_3 \end{pmatrix} = 3 \det \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}. \quad (41)
\]
Suppose first that $\theta_3 = \theta_0 \pmod{\pi}$. Then the vector $(a_3, b_3)$ is also parallel to $(a_0, b_0)$ and thus to $(a_1, b_1)$ and $(a_2, b_2)$. It follows that all determinants of (36)–(40) are zero. Thus the vectors $(c_i, d_i)$ for $i = 1, 2, 3$ are parallel and this can be represented by the condition $\phi_1 = \phi_2 = \phi_3 \pmod{\pi}$.

Now suppose that $\theta_3 \neq \theta_0 \pmod{\pi}$. Then the vector $(a_3, b_3)$ is parallel to none of $(a_i, b_i)$ for $i = 0, 1, 2$. So no pair of $(c_i, d_i)$ for $i = 1, 2, 3$ is parallel, which leads to the condition $\phi_i \neq \phi_j \pmod{\pi}$ for $i \neq j$. Now take the Eq. (41). In polar coordinates, this is equivalent to
\[
r_0 r_3 \sin(\theta_3 - \theta_0) = 3 s_1 s_2 \sin(\phi_2 - \phi_1), \quad (42)
\]
and thus
\[ r_0 = \frac{s_1 s_2 \sin(\phi_2 - \phi_1)}{s_3 \sin(\theta_3 - \theta_0)}. \]  \hfill (43)

Similarly, we can obtain the expressions of \( r_1 \) and \( r_2 \) from (39) and (40). \( \square \)

The characterization of general PH curves of degree 7 whose ERF satisfies rotation-minimizing condition is now an easy corollary.

**Theorem 10.** The ERF of a PH curve of degree 7 is an RMF if and only if the coefficients of the PH curve is represented as follows:
\[ a_0 = r_0 \cos \theta_0 \cos \frac{\alpha}{2}, \]
\[ b_0 = r_0 \sin \theta_0 \cos \frac{\alpha}{2}, \]
\[ c_0 = r_0 \sin(\theta_0 - \beta) \sin \frac{\alpha}{2}, \]
\[ d_0 = r_0 \cos(\theta_0 - \beta) \sin \frac{\alpha}{2}, \]
and for \( i = 1, 2, 3, \)
\[ a_i = r_i \cos \theta_i \cos \frac{\alpha}{2} - s_i \sin(\phi_i - \beta) \sin \frac{\alpha}{2}, \]
\[ b_i = r_i \sin \theta_i \cos \frac{\alpha}{2} - s_i \cos(\phi_i - \beta) \sin \frac{\alpha}{2}, \]
\[ c_i = r_i \sin(\theta_i - \beta) \sin \frac{\alpha}{2} + s_i \cos \phi_i \cos \frac{\alpha}{2}, \]
\[ d_i = r_i \cos(\theta_i - \beta) \sin \frac{\alpha}{2} + s_i \sin \phi_i \cos \frac{\alpha}{2}, \]
where the variables \( r_j, s_j, \theta_j, \) and \( \phi_j \) satisfy the conditions of Lemma 9, and \( 0 \leq \alpha < \pi \) and \( 0 \leq \beta < 2\pi \) are arbitrary real numbers.

**Proof.** Let us denote the direction of \( \mathbf{r}'(0) \) by spherical coordinates:
\[ \mathbf{r}'(0) = |\mathbf{r}'(0)|(\cos \alpha \mathbf{i} + \sin \alpha \cos \beta \mathbf{j} + \sin \alpha \sin \beta \mathbf{k}). \]  \hfill (44)

Note that here we take \( x \)-axis as the polar axis, \( \alpha \) the latitudinal angle, and \( \beta \) the longitudinal angle of the spherical coordinate system. Now consider rotations that moves \( \mathbf{i} \) to \( \mathbf{r}'(0)/|\mathbf{r}'(0)|. \) Among the one-parameter family of such rotations (Farouki et al., 2002a), we choose the one with minimum rotational angle—\( \alpha. \) The corresponding rotational axis is in the direction of
\[ \mathbf{i} \times \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = -\sin \alpha \sin \beta \mathbf{j} + \sin \alpha \cos \beta \mathbf{k}. \]  \hfill (45)

Thus, its unit vector \( \mathbf{n} = -\sin \beta \mathbf{j} + \cos \beta \mathbf{k} \) and a half angle \( \alpha/2 \) comprise the quaternion expression of the rotation:
\[ \mathbf{U} = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \mathbf{n} = \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \sin \beta \mathbf{j} + \sin \frac{\alpha}{2} \cos \beta \mathbf{k}. \]  \hfill (46)
Now $\tilde{r}(t) = U^* r(t) U$ is a curve whose initial tangent vector points to the positive $x$-axis, and its parameterization is given by Lemma 9. Since $r(t) = U \tilde{r}(t) U^*$ and thus $A_i = U \tilde{A}_i$, the presented expressions of $A_i$ are results of a direct computation using (46) and Lemma 9. $\blacksquare$

**Remark 11.** One can prove that the degenerate case of Lemma 9, as the quintic case does, produces planar PH curves of degree 7. Planar PH curves of degree 7 have been scrutinized by Jüttler (2000) for the $G^2$ Hermite interpolation problems.

**Example 12.** Fig. 2 shows a PH curve of degree 7 as well as its Frenet frame and ERF. For clarity, we only depicted the binormal vector $b$ of the Frenet frame and $f_3$ of ERF. Observe how the angular deviation of the Frenet frame changes along the curve. Fig. 3 shows the graph of the curve’s torsion, confirming that this curve is really spatial. It is constructed using Lemma 9, and the following table reports the parameter values of the curve starting at the origin.

<table>
<thead>
<tr>
<th>$r_3$</th>
<th>$\theta_0, \theta_1, \theta_2$</th>
<th>$\theta_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.60</td>
<td>1.38$\pi$</td>
<td>0.0687$\pi$</td>
<td>1.07</td>
<td>1.63</td>
<td>0.301</td>
<td>1.26$\pi$</td>
<td>1.37$\pi$</td>
<td>0.0421$\pi$</td>
</tr>
</tbody>
</table>
Knowing that the PH curves of degree 7 can afford rotation-minimizing Euler–Rodrigues frames, a natural question must be about the flexibility of such class of curves, that is, whether they can be designed to meet the prescribed shape requirements. Such requirements are generally expressed by the Hermite interpolation problem.

The Hermite interpolation problem for PH curves has been scrutinized by many authors. For planar PH curves, Farouki and Neff (1995) showed that the minimal degree to interpolate $C^1$ data is 5. Later, Moon et al. (2001) discussed the new means to differentiate among the quintic interpolants with better shape. Furthermore, Jüttler (2000) treated PH curves of degree 7 to interpolate $G^2$ data.

For spatial PH curves, Jüttler and Mäurer (1999) gave a detailed analysis on $G^1$ Hermite interpolation of cubic PH curves. Recently, Farouki et al. (2002a) solved $C^1$ problem with quintics, and showed that the interpolants admit a closed-form expression as a two-parameter family.

General spatial PH curves of degree 7 have 4 additional free parameters compared to the quintics. So, it seems that 5 constraints of (33) can be satisfied by suitable choices of 6 free parameters. However, we couldn’t come up with any closed-form solutions, and believe this problem deserves its own separate discussion. Instead, we executed the following experiments to investigate the flexibility of those PH curves of degree 7. Fig. 4 shows the randomly computed endpoints $\mathbf{r}(1)$ of PH curves of degree 7 satisfying (33) with its starting point $\mathbf{r}(0)$ at the origin, its initial tangent $\mathbf{r}'(0) = (1, 0, 0)$, and its final tangent $\mathbf{r}'(1) = (1/2, \sqrt{3}/2, 0)$. This figure and other results suggest that it is highly likely that the $C^1$ Hermite interpolation problem is solvable for most (if not all) practical cases.

Fig. 4. A plot of points $\mathbf{r}(1)$ of PH curves of degree 7 whose ERF is an RMF with $\mathbf{r}(0)$ at the origin, $\mathbf{r}'(0) = (1, 0, 0)$, and $\mathbf{r}'(1) = (1/2, \sqrt{3}/2, 0)$. 
5.4. **ERF on PH curves with common factors**

Now suppose a PH curve \( r(t) \) satisfying (8) and let \( \tilde{r}(t) \) be a curve defined by

\[
\tilde{r}'(t) = \tilde{A}(t) i \tilde{A}^*(t).
\]

Then, obviously \( \tilde{r}(t) \) is a PH curve. And if \( h(t) > 0 \), \( \tilde{r}(t) \) has the same ERF as \( r(t) \). (The case \( h(t) < 0 \) can be treated similarly.) The important observation is that the angular velocity of their ERF is the same.

Now consider a quartic PH curve \( r(t) \) with linear \( h(t) \) and \( \tilde{A}(t) \), and suppose its ERF is an RMF. Then, by Corollary 6, we must have planar \( \tilde{r}(t) \), that is, its tangent vector always lies on a plane. But the tangent vector of \( r(t) \) is parallel to that of \( \tilde{r}(t) \), so \( r(t) \) must also be a planar curve. Thus we can conclude that there is no non-planar quartic PH curve whose ERF is an RMF.

The above argument can be readily generalized up to sextic PH curves whether they are of the form (7) or (8). Thus, we have justified our claim that the minimum degree of a non-planar PH curve whose ERF is an RMF is seven and it must be of the form (7).

**6. Concluding remarks**

The Euler–Rodrigues frame is a rational frame that is naturally induced from the quaternion representation of PH curves. By computing its angular velocity against an RMF, we characterized cubic, quintic, and PH curves of degree 7 whose ERF fulfills rotation-minimizing property. Several topics that could not be addressed here deserve further investigation. In particular regarding PH curves of degree 7, these include rational approximation schemes of RMF using their ERF and the flexibility assessment of their subclass whose ERF are RMF.

**References**


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