A fuzzy distance measure for generalized fuzzy numbers

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Abstract
In this paper, first of all the distance measure entitled generalized Hausdorff distance is defined between two generalized fuzzy numbers that has been introduced by Chen [3]. Then using another distance and combining it with the generalized Hausdorff distance, a fuzzy distance measure is introduced with the help of fuzzy distance measure proposed in [1] to generalize fuzzy numbers. The concept is illustrated by solving several testing examples.

Keywords: Fuzzy numbers, Generalized Fuzzy numbers, Fuzzy distance measure.

1 Introduction

The methods of measuring of distance between fuzzy numbers have became important due to the significant applications in diverse fields like remote sensing, data mining, pattern recognition and multivariate data analysis and so on. Several distance measures for precise numbers are well established in the literature. Several researchers focused on computing the distance between fuzzy numbers [2, 4, 7, 11, 12]. Usually the distance methods basically compute crisp distance values for fuzzy numbers. Naturally a logical question occurs to us: if the numbers themselves are not known exactly, how can the distance between them be an exact value? In view of this, Voxman [12] first introduced a fuzzy distance for fuzzy numbers. Therefore a distance measure for fuzzy numbers is that the distance between two uncertain numbers should also be an uncertain number, logically. Abbasbandy and Hajighasemi [1] introduced a new fuzzy distance between two fuzzy numbers. In the most of researches, the authors pointed out to construct the distance measure between normal fuzzy numbers, for instance [1, 2, 12].

Now in this paper, the introduced distance measure between two normal fuzzy numbers in [1] is developed to distance measure between two generalized fuzzy numbers. Such that, the confidence level has a important role in this concept. Now here a fuzzy distance measure between two generalized fuzzy numbers is developed of [1]. Then using this proposed distance, a new concept of fuzzy distance measure between two generalized fuzzy numbers has been introduced in this paper.

The structure of the paper as follows:
In section 2, some basic definitions and results about generalized fuzzy numbers are brought. In section 3, the new distance between generalized fuzzy numbers is introduced then the new concept of fuzzy distance are defined and is compared with proposed method [12]. Finally, conclusions are drawn in Section 4.

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2 Preliminaries

A fuzzy set on a set $X$ is a function $\tilde{A} : X \rightarrow [0, 1]$. The support of $\tilde{A}$, $\text{supp} \tilde{A}$ is the closure of the set $\{ x \in X \mid \tilde{A}(x) > 0 \}$.

**Definition 2.1.** [1] A fuzzy number is a fuzzy set $\tilde{A} : R \rightarrow [0, 1]$ on $R$ satisfying

(i) $\tilde{A}$ is upper semi-continuous;

(ii) $\text{supp} \tilde{A}$ is a closed and bounded interval;

(iii) if $\text{supp} \tilde{A} = [a, b]$, then there exist $c, d$, $a \leq c \leq d \leq b$, such that $\tilde{A}$ is increasing on the interval $[a, c]$, equal to 1 on the interval $[c, d]$ and decreasing on the interval $[d, b]$.

We let $F$ denote the family of all fuzzy numbers. If $\tilde{A} \in F$, then for each $\alpha$, $0 < \alpha \leq 1$, the $\alpha$-cut of $\tilde{A}$, is defined by

$$\tilde{A}_\alpha = \{ x \in X \mid \tilde{A}(x) \geq \alpha \}.$$

The $\alpha$-cut representation of $\tilde{A}$ is the pair of functions, $(\tilde{A}^L(\alpha), \tilde{A}^R(\alpha))$, defined by

$$\tilde{A}^L(\alpha) = \begin{cases} \{ x \mid x \in \tilde{A}_\alpha \} & \text{if } \alpha > 0, \\ \{ x \mid x \in \text{supp} \tilde{A} \} & \text{if } \alpha = 0, \end{cases}$$

and

$$\tilde{A}^R(\alpha) = \begin{cases} \{ x \mid x \in \tilde{A}_\alpha \} & \text{if } \alpha > 0, \\ \{ x \mid x \in \text{supp} \tilde{A} \} & \text{if } \alpha = 0. \end{cases}$$

If $\tilde{A}$ is a fuzzy number then the compliment of $\tilde{A}$, $\tilde{A}_c$, is the fuzzy set defined by $\tilde{A}_c(x) = 1 - \tilde{A}(x)$.

**Definition 2.2.** [3] A generalized trapezoidal fuzzy number $\tilde{A}$ as $\tilde{A} = (a, b; \beta, \gamma; w)$, where $0 < w \leq 1$, and $a$, $b$, $\beta$ and $\gamma$ are non-negative real numbers, $w$ represents the degree of confidence of expert regarding $\tilde{A}$ is a fuzzy subset on the real line $R$, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions:

(i) $\mu_{\tilde{A}}$ is a continuous mapping from $R$ to the closed interval $[0, 1]$;

(ii) $\mu_{\tilde{A}}(x) = 0$, where $-\infty < x \leq a - \beta$;

(iii) $\mu_{\tilde{A}}(x)$, is strictly increasing on $[a - \beta, a]$;

(iv) $\mu_{\tilde{A}}(x) = w$, where $a \leq x \leq b$;

(v) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $[b, b + \gamma]$;

(vi) $\mu_{\tilde{A}}(x) = 0$, where $b + \gamma \leq x < \infty$

If $w = 1$, then the generalized fuzzy number $\tilde{A}$ is called a normal trapezoidal fuzzy number and denoted as $\tilde{A} = (a, b; \beta, \gamma)$. If $a = b$, then $\tilde{A}$ is called a generalized triangular fuzzy number and denoted as $\tilde{A} = (a; \beta, \gamma; w)$. If $\beta = 0$, $\gamma = 0$ and $a = b$ and $w = 1$, then $\tilde{A}$ is called a real number.

The $\alpha$-cut of the $\tilde{A}$ is presented as an interval $[\tilde{A}]_\alpha = [\tilde{A}^L(\alpha), \tilde{A}^R(\alpha)]$ where for a generalized trapezoidal fuzzy number they are $\tilde{A}^L(\alpha) = (a - \beta) + (\frac{\beta}{w}) \alpha$ and $\tilde{A}^R(\alpha) = (b + \gamma) - (\frac{\gamma}{w}) \alpha$ for $0 \leq \alpha \leq w$.

**Definition 2.3.** A fuzzy number $\tilde{A}$ is positive if $\mu_{\tilde{A}}(x) = 0$ for $x < 0$. 
Let $F^g$ be the family of all generalized fuzzy numbers.
If $K$ is the set of compact subsets of $R^2$, and $A$ and $B$ are two subsets of $R^2$ then the Hausdorff metric $H : K \times K \rightarrow [0, \infty)$ is defined by $[12]
\begin{align*}
H(A, B) &= \max \{ \sup_{b \in B} d_\mathcal{F}(b, A), \sup_{a \in A} d_\mathcal{F}(a, B) \},
\end{align*}
where $d_\mathcal{F}$ is the usual Euclidean metric for $R^2$.

In the next definition the Hausdorff metric have been developed to generalized fuzzy numbers, that is useful for the main purpose of this article: introduce a fuzzy number for approach of distance measure and also for similarity measure.

**Definition 2.4.** The generalized Hausdorff metric $d^g$ on $F^g \times F^g$ is defined by
\begin{align*}
d^g(A_1, A_2) &= \sup_{0 \leq \alpha \leq w} \{ H([\tilde{A}_1](\alpha), [\tilde{A}_2](\alpha)) \} + \beta \sup_{w' \leq \alpha \leq w'} | R(\alpha) - L(\alpha) |,
\end{align*}
where
\begin{align*}
\beta &= \begin{cases} 
1 & \text{for } w' \neq w, \\
0 & \text{for } w' = w,
\end{cases}
\end{align*}
and
\begin{align*}
R(\alpha) &= \begin{cases} 
\tilde{A}_1^R(\alpha) & \text{for } w' = w_1 \\
\tilde{A}_2^R(\alpha) & \text{for } w' = w_2,
\end{cases}
\end{align*}
and
\begin{align*}
L(\alpha) &= \begin{cases} 
\tilde{A}_1^L(\alpha) & \text{for } w' = w_1 \\
\tilde{A}_2^L(\alpha) & \text{for } w' = w_2.
\end{cases}
\end{align*}

Where $w = \min\{w_1, w_2\}$ and $w' = \max\{w_1, w_2\}$.

**Definition 2.5.** $[14]$ Addition of two fuzzy numbers $\tilde{A}_1 = (a_1, b_1; \beta_1, \gamma_1)$ and $\tilde{A}_2 = (a_2, b_2; \beta_2, \gamma_2)$ as follows:
\begin{align*}
\tilde{A}_1 + \tilde{A}_2 &= (a_1 + b_2, b_1 + a_2; \beta_1 + \beta_2, \gamma_1 + \gamma_2)
\end{align*}
We suppose that $\tilde{A}_1 + \tilde{A}_2 = (a_1 + b_2, b_1 + a_2; \beta_1 + \beta_2, \gamma_1 + \gamma_2; w)$ for addition of two generalized trapezoidal fuzzy numbers where $w = \min\{w_1, w_2\}$ also $w_1$ and $w_2$ represent the degree of confidence of expert regarding $\tilde{A}_1$ and $\tilde{A}_2$ respectively.

**Definition 2.6.** $[14]$ Subtraction of two fuzzy numbers $\tilde{A}_1 = (a_1, b_1; \beta_1, \gamma_1)$ and $\tilde{A}_2 = (a_2, b_2; \beta_2, \gamma_2)$ as follows:
\begin{align*}
\tilde{A}_1 \ominus \tilde{A}_2 &= (a_1 - b_2, b_1 - a_2; \beta_1 + \beta_2, \gamma_1 + \gamma_2)
\end{align*}
We suppose that $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - b_2, b_1 - a_2; \beta_1 + \beta_2, \gamma_1 + \gamma_2; w)$ for subtraction of two generalized trapezoidal fuzzy numbers where $w = \min\{w_1, w_2\}$ also $w_1$ and $w_2$ represent the degree of confidence of expert regarding $\tilde{A}_1$ and $\tilde{A}_2$ respectively.
2.1 Fuzzy distance given by Voxman

Here, we briefly describe the fuzzy distance measure by Voxman [12]. The fuzzy distance function on \( F, \Delta : F \times F \rightarrow R \), is defined by

\[
\Delta(\tilde{A}_1, \tilde{A}_2)(z) = \sup_{|x-y|=z} \min\{ \tilde{A}_1(x), \tilde{A}_2(y) \}.
\]

For each pair of fuzzy numbers \( \tilde{A}_1 \) and \( \tilde{A}_2 \), let \( \Delta_{\tilde{A}_1, \tilde{A}_2} \) denote the fuzzy number \( \Delta(\tilde{A}_1, \tilde{A}_2) \). If the \( \alpha \)-cut representations of \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are \( (\tilde{A}_1^L(\alpha), \tilde{A}_1^R(\alpha)) \) and \( (\tilde{A}_2^L(\alpha), \tilde{A}_2^R(\alpha)) \), respectively, then the \( \alpha \)-cut representation of \( \Delta_{\tilde{A}_1, \tilde{A}_2} \) is defined by

\[
L_{\alpha} = \begin{cases} 
\max \{ \tilde{A}_2^L(\alpha) - \tilde{A}_1^R(\alpha), 0 \} & \text{if } \frac{1}{2}(\tilde{A}_1^L(1) + \tilde{A}_1^R(1)) \leq \frac{1}{2}(\tilde{A}_2^L(1) + \tilde{A}_2^R(1)), \\
\max \{ \tilde{A}_1^L(\alpha) - \tilde{A}_2^R(\alpha), 0 \} & \text{if } \frac{1}{2}(\tilde{A}_2^L(1) + \tilde{A}_2^R(1)) \leq \frac{1}{2}(\tilde{A}_1^L(1) + \tilde{A}_1^R(1)), 
\end{cases}
\]

and

\[
R_{\alpha} = \max \{ \tilde{A}_1^R(\alpha) - \tilde{A}_2^L(\alpha), \tilde{A}_2^R(\alpha) - \tilde{A}_1^L(\alpha) \}.
\]

3 Distance measure

Let us consider two generalized trapezoidal fuzzy numbers \( \tilde{A}_1 \) and \( \tilde{A}_2 \), denoted by \( \tilde{A}_1 = (a_1, b_1; b_2, c_1) \) and \( \tilde{A}_2 = (a_2, b_2; b_2, c_2) \). The \( \alpha \)-cut of \( \tilde{A}_1 \) and \( \tilde{A}_2 \) represented using two intervals \( \tilde{A}_1^\alpha = [A_1^1(\alpha), A_1^2(\alpha)] \) and \( \tilde{A}_2^\alpha = [A_2^1(\alpha), A_2^2(\alpha)] \) for all \( \alpha \in [0, 1] \), respectively.

Here we extend the distance measure proposed in [1] for generalized fuzzy numbers, then by using it we introduce a new similarity measure for generalized fuzzy numbers.

3.1 a new distance measure for generalized fuzzy numbers

First, suppose that the \( R(\alpha) \) and \( L(\alpha) \) be defined in definition 2.4. Now the distance between two generalized fuzzy numbers \( \tilde{A}_1 \) and \( \tilde{A}_2 \) can be defined by

\[
d(\tilde{A}_1, \tilde{A}_2) = \left| \int_0^1 (1 - \alpha)(\tilde{A}_2^R(\alpha) - \tilde{A}_2^L(\alpha)) + \alpha(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha))d\alpha \right|
\]

\[
+ \left| \int_0^w \alpha(R(\alpha) - L(\alpha))d\alpha \right| \quad (3.1)
\]

for \( w \geq \frac{1}{2} \) and if \( w < \frac{1}{2} \) can be defined by

\[
d(\tilde{A}_1, \tilde{A}_2) = \left| \int_0^1 (1 - \alpha)(\tilde{A}_1^R(\alpha) - \tilde{A}_2^L(\alpha)) + \alpha(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha))d\alpha \right|
\]

\[
+ \left| \int_0^w \alpha(R(\alpha) - L(\alpha))d\alpha \right| \quad (3.2)
\]

Remark 3.1. For \( w = 1 \) the distance between \( \tilde{A}_1 \) and \( \tilde{A}_2 \) is distance be defined by Abdashandy and Hajighasemi in [1].

See Table 1 the results of calculation \( d \) and \( d_{\infty} \) for some normal triangular fuzzy numbers in [1].
Table 1: The results of calculation $d$ and $d_{\infty}$

<table>
<thead>
<tr>
<th>$\tilde{A}_1$</th>
<th>$\tilde{A}_2$</th>
<th>$d(\tilde{A}_1, \tilde{A}_2)$</th>
<th>$d_{\infty}(\tilde{A}_1, \tilde{A}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4,3,1)</td>
<td>(0,1,2)</td>
<td>$\frac{22}{7}$</td>
<td>4</td>
</tr>
<tr>
<td>(3,2,2)</td>
<td>(4,3,1)</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(4,1,1)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(4,1,1)</td>
<td>(6,2,2)</td>
<td>2.25</td>
<td>3</td>
</tr>
<tr>
<td>(2,1,4)</td>
<td>(3,2,2)</td>
<td>0.125</td>
<td>1</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(6,1,1)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(3,2,2)</td>
<td>(3,1,1)</td>
<td>0.25</td>
<td>1</td>
</tr>
</tbody>
</table>

**Theorem 3.1.** For generalized fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$, we have

(i) $d(\tilde{A}_1, \tilde{A}_2) \geq 0$ and $d(\tilde{A}_1, \tilde{A}_1) = 0$;

(ii) $d(\tilde{A}_1, \tilde{A}_2) = d(\tilde{A}_2, \tilde{A}_1)$;

(iii) $d(\tilde{A}_1, \tilde{A}_2) \leq d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2)$.

**Proof.** We consider only (iii). Suppose $\tilde{A}_1$ and $\tilde{A}_2$ have $\alpha$ cut representations as before, and $\tilde{A}_3$ has $\alpha$ cut representation

$$[\tilde{A}_3]_\alpha = [\tilde{A}_3^L(\alpha), \tilde{A}_3^R(\alpha)]$$

By definition for $w \geq \frac{1}{2}$, we have

$$d(\tilde{A}_1, \tilde{A}_2) = \int_{0}^{1} \left[ (1 - \alpha)(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha) + \tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha)) + \alpha(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha) + \tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha)) \right] d\alpha$$

$$+ \int_{\frac{1}{2}}^{1} \frac{1}{\alpha}(\tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha) + (1 - \alpha)(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha))) d\alpha$$

Since $\int_{0}^{\frac{1}{2}} \alpha(R(\alpha) - L(\alpha)) d\alpha$ is a positive number then we can add it to right side from the unequal, so we obtain

$$d(\tilde{A}_1, \tilde{A}_2) \leq \int_{0}^{\frac{1}{2}} [(1 - \alpha)(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha)) + \alpha(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha))] d\alpha$$

$$+ \int_{\frac{1}{2}}^{1} \frac{1}{\alpha}(\tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha) + (1 - \alpha)(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha))) d\alpha$$

$$+ \int_{0}^{1} \alpha(R(\alpha) - L(\alpha)) d\alpha$$

$$+ \int_{\frac{1}{2}}^{1} [(1 - \alpha)(\tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha)) + \alpha(\tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha))] d\alpha$$

$$+ \int_{\frac{1}{2}}^{1} \frac{1}{\alpha}(\tilde{A}_1^L(\alpha) - \tilde{A}_2^L(\alpha) + (1 - \alpha)(\tilde{A}_1^R(\alpha) - \tilde{A}_2^R(\alpha))) d\alpha$$

$$+ \int_{\frac{1}{2}}^{1} \alpha(R(\alpha) - L(\alpha)) d\alpha = d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2)$$
Now for \( w < \frac{1}{2} \) we have the similar proof.

### 3.1.1 Properties

Since we introduce this distance by dominance, similarity Hausdorff distance we can be proved these properties:

(i) \( d(\tilde{A}_1 + \tilde{A}_2, \tilde{A}_2 + \tilde{A}_3) = d(\tilde{A}_1, \tilde{A}_3) \),

(ii) \( d(\tilde{A}_1 + \tilde{A}_2, \tilde{0}) \leq d(\tilde{A}_1, \tilde{0}) + d(\tilde{A}_2, \tilde{0}) \),

(iii) \( d(\lambda \tilde{A}_1, \lambda \tilde{A}_2) = \lambda d(\tilde{A}_1, \tilde{A}_2) \).

where \( \tilde{A}_1, \tilde{A}_2, \) and \( \tilde{A}_3 \) are three generalized trapezoidal fuzzy numbers and \( \lambda \) is a real number.

Since in the next section we need the unequal \( d(\tilde{A}_1, \tilde{A}_2) \leq d^\infty(\tilde{A}_1, \tilde{A}_2) \), then here we say this important relation between \( d \) and \( d^\infty \) in the form:

**Theorem 3.2.** For two generalized fuzzy numbers \( \tilde{A}_1 \) and \( \tilde{A}_2 \), we have

\[
0 \leq d(\tilde{A}_1, \tilde{A}_2) \leq d^\infty(\tilde{A}_1, \tilde{A}_2).
\]

**Proof.** By definition \( d(\ldots, \ldots) \), we have,

\[
d(\tilde{A}_1, \tilde{A}_2) = \frac{1}{2} \int_0^1 (1 - \alpha) (A^0_1(\alpha) - A^0_2(\alpha)) d\alpha + \frac{1}{2} \int_0^1 \alpha (A^1_1(\alpha) - A^1_2(\alpha)) d\alpha
\]

\[
\int_{\frac{1}{2}}^w \alpha (A^0_1(\alpha) - A^0_2(\alpha)) d\alpha + \int_{\frac{1}{2}}^w (1 - \alpha) (A^1_1(\alpha) - A^1_2(\alpha)) d\alpha + \int_w^w \alpha (R(\alpha) - L(\alpha)) d\alpha
\]

By assumption \( \sup_{0 \leq \alpha \leq w} \{H(\tilde{A}_1, \tilde{A}_2, \alpha)\} = M \) and \( \sup_{w < \alpha \leq w'} \{R(\alpha) - L(\alpha)\} = M' \), we have \( |A^0_1(\alpha) - A^0_2(\alpha)| \leq M \), \( |A^1_1(\alpha) - A^1_2(\alpha)| \leq M \), \( |R(\alpha) - L(\alpha)| \leq M' \), and mean value theorem for integrals, we obtain

\[
d(\tilde{A}_1, \tilde{A}_2) \leq M \int_{\frac{1}{2}}^w (1 - \alpha) d\alpha + M \int_{\frac{1}{2}}^w \alpha d\alpha
\]

\[
+ M \int_{\frac{1}{2}}^w \alpha d\alpha + M \int_{\frac{1}{2}}^w (1 - \alpha) d\alpha + M' \int_w^w \alpha d\alpha
\]

since \( 0 \leq \alpha \leq 1 \)

\[
\leq M \int_0^1 (1 - \alpha) d\alpha + M \int_0^1 \alpha d\alpha + M' \int_w^w d\alpha = M + (w' - w)M' \leq M + M'
\]

Therefore

\[
d(\tilde{A}_1, \tilde{A}_2) \leq d^\infty(\tilde{A}_1, \tilde{A}_2).
\]

### 3.2 Fuzzy distance measure

As mentioned at the beginning, in the literature often the exiting distance measure give crisp value for two fuzzy numbers. This is our motivation to develop a new distance measure which will give a fuzzy number as the distance between two generalized fuzzy numbers. Let two generalized trapezoidal fuzzy numbers \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are given. Here we introduce a new concept of fuzzy distance between two generalized fuzzy numbers.

By \( d(\ldots, \ldots) \) and \( d^\infty(\ldots, \ldots) \), we can introduce the fuzzy distance by a symmetric generalized triangular fuzzy number as follows:

\[
d(\tilde{A}_1, \tilde{A}_2) = \left( x, \theta, \theta; w \right)
\]

where \( w = \min\{w_1, w_2\} \), \( x = \frac{d(\tilde{A}_1, \tilde{A}_2) + d^\infty(\tilde{A}_1, \tilde{A}_2)}{2} \), and \( \theta = \frac{d^\infty(\tilde{A}_1, \tilde{A}_2) - d(\tilde{A}_1, \tilde{A}_2)}{2} \), with \( \alpha \) cut representation \( \lambda_{\alpha}(\tilde{A}_1, \tilde{A}_2), \rho_{\alpha}(\tilde{A}_1, \tilde{A}_2) \) where \( \lambda_{\alpha}(\tilde{A}_1, \tilde{A}_2) = x + (1 - \alpha)\theta \) and \( \rho_{\alpha}(\tilde{A}_1, \tilde{A}_2) = x - (1 - \alpha)\theta \).

The proposed fuzzy distance (3.3) satisfies fuzzy distance properties followed in Kaleva and Seikkala [8].
Theorem 3.3. For fuzzy numbers $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$, we have

(i) $d(\tilde{A}_1, \tilde{A}_2) = 0$ if only if $\tilde{A}_1 = \tilde{A}_2$;

(ii) $d(\tilde{A}_1, \tilde{A}_2) = d(\tilde{A}_2, \tilde{A}_1)$;

(iii) $\lambda_\alpha(\tilde{A}_1, \tilde{A}_2) \leq \lambda_\alpha(\tilde{A}_1, \tilde{A}_3) + \lambda_\alpha(\tilde{A}_2, \tilde{A}_3)$ and $\rho_\alpha(\tilde{A}_1, \tilde{A}_2) \leq \rho_\alpha(\tilde{A}_1, \tilde{A}_3) + \rho_\alpha(\tilde{A}_2, \tilde{A}_3)$;

(iv) $d(\tilde{A}_1, \tilde{A}_2)$ is positive.

Proof.

(i) By definition of fuzzy zero, $\tilde{0}(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0 \end{cases}$, from assumption $d(\tilde{A}_1, \tilde{A}_2) = 0$, we obtain $d(\tilde{A}_1, \tilde{A}_2) + d_\infty(\tilde{A}_1, \tilde{A}_2) = 0$. Since $d(\tilde{A}_1, \tilde{A}_2)$ and $d_\infty(\tilde{A}_1, \tilde{A}_2)$ are positive numbers, we have $d(\tilde{A}_1, \tilde{A}_2) = d_\infty(\tilde{A}_1, \tilde{A}_2) = 0$ and hence $\tilde{A}_1 = \tilde{A}_2$. Also, converse is obvious.

(ii) By properties of $d(\ldots)$ and $d_\infty(\ldots)$, it is obvious.

(iii) By definition of $\lambda_\alpha(\tilde{A}_1, \tilde{A}_2)$, we have

$$\lambda_\alpha(\tilde{A}_1, \tilde{A}_2) = d(\tilde{A}_1, \tilde{A}_2) + \alpha \left( \frac{d_\infty(\tilde{A}_1, \tilde{A}_2) - d(\tilde{A}_1, \tilde{A}_2)}{2} \right) = \left( 1 - \frac{\alpha}{2} \right) d(\tilde{A}_1, \tilde{A}_2) + \frac{\alpha}{2} d_\infty(\tilde{A}_1, \tilde{A}_2)$$

$$\leq \left( 1 - \frac{\alpha}{2} \right) (d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2)) + \frac{\alpha}{2} (d_\infty(\tilde{A}_1, \tilde{A}_3) + d_\infty(\tilde{A}_3, \tilde{A}_2)) = \lambda_\alpha(\tilde{A}_1, \tilde{A}_3) + \lambda_\alpha(\tilde{A}_3, \tilde{A}_2),$$

because $(1 - \frac{\alpha}{2}) > 0$. For $\rho(\tilde{A}_1, \tilde{A}_2)$, we have the similar proof.

(iv) is clear from (3.3).
3.2 Comparing by using Ambiguity and fuzziness

Now, it is obvious that, for the case of fuzzy distance measure some vagueness is always presented in the distance measure. but certainly, in real life decision making situation, the distance measure with less vagueness is more suitable and acceptable to us especially from the stability point of view. Suppose we have tow fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ with the same central values but with different spreads. Then $\tilde{A}_1$ is expected to be better than $\tilde{A}_2$ in the sense of stability or preciseness if

$$
Amb(\tilde{A}_1) < Amb(\tilde{A}_2) \quad \text{and} \quad Fuzz(\tilde{A}_1) < Fuzz(\tilde{A}_2)
$$

Considering this, our interest is now to compare between the proposed fuzzy distance measure and the fuzzy distance given by voxman withe help of the following definitions:

Delgado et al. [5, 6] have extensively studied two attributes of fuzzy numbers, ambiguity and fuzziness. Ambiguity may be seen as a ‘global spread’ of the membership function, whereas the fuzziness involve a comparison between the fuzzy set and its complement. These concepts are defined as follow:

$$
Amb(\tilde{A}) = \int_0^1 S(\alpha)[R(\alpha) - L(\alpha)]d\alpha,
$$

$$
Fuzz(\tilde{A}) = \int_0^1 S(\alpha)|q - p|d\alpha - \left\{ \int_0^1 S(\alpha)|L_\alpha - L(\alpha)|d\alpha + \int_0^1 S(\alpha)|R(\alpha) - R(\alpha)|d\alpha + \int_\frac{1}{2}^1 S(\alpha)|L(\alpha) - R(\alpha)|d\alpha \right\},
$$

Table 2: Calculation the fuzzy distance measure for generalized fuzzy numbers

<table>
<thead>
<tr>
<th>set</th>
<th>$\tilde{A}_1$</th>
<th>$\tilde{A}_2$</th>
<th>$d(\tilde{A}_1, \tilde{A}_2)$</th>
<th>$d_F(\tilde{A}_1, \tilde{A}_2)$</th>
<th>$d(\tilde{A}_1, \tilde{A}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.25:0.15,0.15;1)</td>
<td>0.0125</td>
<td>0.05</td>
<td>(0.03:0.01,0.01;1)</td>
</tr>
<tr>
<td>2</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.2:0.3:0.1,0.1;1)</td>
<td>0.0215</td>
<td>0.14</td>
<td>(0.08:0.05,0.05;1)</td>
</tr>
<tr>
<td>3</td>
<td>(0.3;0.0:0.0,0.1)</td>
<td>(0.3;0.0:0.0,0.1)</td>
<td>0.1</td>
<td>0.1</td>
<td>(0.1:0.0,0.1)</td>
</tr>
<tr>
<td>4</td>
<td>(0.2;0.0:0.0,0.1)</td>
<td>(0.3;0.0:0.0,0.1)</td>
<td>0.1</td>
<td>0.1</td>
<td>(0.1:0.0,0.1)</td>
</tr>
<tr>
<td>5</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.6:0.1,0.1;1)</td>
<td>0.0783</td>
<td>0.8571</td>
<td>(0.46:0.38,0.38;1)</td>
</tr>
<tr>
<td>6</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.55;0.31,0.31;1)</td>
<td>0.2359</td>
<td>0.8648</td>
<td>(0.55:0.31,0.31;1)</td>
</tr>
<tr>
<td>7</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.2:0.3:0.1,0.1;1)</td>
<td>0.3</td>
<td>0.3</td>
<td>(0.3:0.0,0.1)</td>
</tr>
<tr>
<td>8</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.55:0.15,0.15;1)</td>
<td>0.2875</td>
<td>0.35</td>
<td>(0.31:0.03,0.03;1)</td>
</tr>
<tr>
<td>9</td>
<td>(0.2;0.3:0.1,0.1;1)</td>
<td>(0.5:0.6:0.1,0.1;1)</td>
<td>0.3</td>
<td>0.3</td>
<td>(0.3:0.0,0.1)</td>
</tr>
<tr>
<td>10</td>
<td>(0.2:0.1,0.1;1)</td>
<td>(0.3;0.0,0.1)</td>
<td>0.075</td>
<td>0.2</td>
<td>(0.13:0.06,0.06;1)</td>
</tr>
<tr>
<td>11</td>
<td>(0.4:0.3,0.3;1)</td>
<td>(0.4:0.1,0.1;1)</td>
<td>0.05</td>
<td>0.2</td>
<td>(0.125:0.07,0.07;1)</td>
</tr>
<tr>
<td>12</td>
<td>(0.3;0.5,0.1,0.1;1)</td>
<td>(0.4:0.1,0.1;1)</td>
<td>0.05</td>
<td>0.1</td>
<td>(0.75:0.025,0.025;1)</td>
</tr>
<tr>
<td>13</td>
<td>(0.4;0.0,0.4;1)</td>
<td>(0.4:0.1,0.1;1)</td>
<td>0.125</td>
<td>0.3</td>
<td>(0.21:0.08,0.08;1)</td>
</tr>
<tr>
<td>14</td>
<td>(0.3;0.4,0.0;1)</td>
<td>(0.6;0.7,0;0.1)</td>
<td>0.3</td>
<td>0.3</td>
<td>(0.3:0.0,0.1)</td>
</tr>
<tr>
<td>15</td>
<td>(0.5;0.6,0.1,0.1;1)</td>
<td>(0.2:0.3:0.1,0.1;1)</td>
<td>0.1714</td>
<td>0.3</td>
<td>(0.23:0.06,0.06;1)</td>
</tr>
</tbody>
</table>

3.2.1 Comparing by using Ambiguity and fuzziness

Now, it is obvious that, for the case of fuzzy distance measure some vagueness is always presented in the distance measure. but certainly, in real life decision making situation, the distance measure with less vagueness is more suitable and acceptable to us especially from the stability point of view. Suppose we have tow fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ with the same central values but with different spreads. Then $\tilde{A}_1$ is expected to be better than $\tilde{A}_2$ in the sense of stability or preciseness if

$$
Amb(\tilde{A}_1) < Amb(\tilde{A}_2) \quad \text{and} \quad Fuzz(\tilde{A}_1) < Fuzz(\tilde{A}_2)
$$

Considering this, our interest is now to compare between the proposed fuzzy distance measure and the fuzzy distance given by voxman withe help of the following definitions:
where supp $\tilde{A} = [p, q]$ and $(L(\alpha), R(\alpha))$ be the $\alpha$-cut representations of $\tilde{A}$. Also $\tilde{A}_c$ be the complement of $\tilde{A}$ with $\alpha$-cut representations $(L_c(\alpha), R_c(\alpha))$. The function $S : [0, 1] \rightarrow [0, 1]$ is an increasing function and $S(0) = 0$ and $S(1) = 1$, [12]. We say that $S$ is a regular reducing function if $\int_0^1 S(\alpha) d\alpha = \frac{1}{2}$. A routine calculation shows for $S(\alpha) = \alpha$, we have

$$Fuzz(\tilde{A}) = \int_0^{\frac{1}{2}} [R(\alpha) - L(\alpha)] d\alpha + \int_{\frac{1}{2}}^1 [L(\alpha) - R(\alpha)] d\alpha.$$  

Here we develop ambiguity and fuzziness to generalized fuzzy numbers, that is useful for our purpose i.e. comparing proposed fuzzy distance measure with fuzzy distance given by voxman in [12].

Let us consider the generalized trapezoidal fuzzy numbers $\tilde{A} = (a, b, \beta, \gamma, w)$, the $\alpha$-cut of it denoted by intervals $[\tilde{A}]_\alpha = [L(\alpha), R(\alpha)]$, we can define the generalized ambiguity and fuzziness for $\tilde{A}$ with $S(\alpha) = \alpha$ by

$$Amb(\tilde{A}) = \int_0^{w} \alpha [R(\alpha) - L(\alpha)] d\alpha,$$

$$Fuzz(\tilde{A}) = \int_0^{w} [R(\alpha) - L(\alpha)] d\alpha + \int_{w}^1 [L(\alpha) - R(\alpha)] d\alpha.$$  

where $w > \frac{1}{2}$ and also

$$Fuzz(\tilde{A}) = \int_0^{\frac{w}{2}} [R(\alpha) - L(\alpha)] d\alpha + \int_{\frac{w}{2}}^1 [L(\alpha) - R(\alpha)] d\alpha.$$  

where $w \leq \frac{1}{2}$.

Table 3 shows that the ambiguity and the fuzziness of $\tilde{d}(\tilde{A}_1, \tilde{A}_2)$ are less than of the ambiguity and fuzziness of $\Delta(\tilde{A}_1, \tilde{A}_2)$, which is defined by Voxman [12], for some examples. We can see that, when the support of $\tilde{A}_1$ and $\tilde{A}_2$ are disjoint, then $d(\tilde{A}_1, \tilde{A}_2) = d_\infty(\tilde{A}_1, \tilde{A}_2)$ and in this case, $A(d(\tilde{A}_1, \tilde{A}_2)) = F(d(\tilde{A}_1, \tilde{A}_2)) = 0$. 

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Table 3: Comparison of ambiguity and fuzziness

<table>
<thead>
<tr>
<th>$\tilde{A}_1$</th>
<th>$\tilde{A}_2$</th>
<th>$A(\tilde{d}(\tilde{A}_1, \tilde{A}_2))$</th>
<th>$F(\tilde{d}(\tilde{A}_1, \tilde{A}_2))$</th>
<th>$A(\Delta(\tilde{A}_1, \tilde{A}_2))$</th>
<th>$F(\Delta(\tilde{A}_1, \tilde{A}_2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3,2,2)</td>
<td>(4,3,1)</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{68}{75}$</td>
<td>$\frac{17}{20}$</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(4,1,1)</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>(4,1,1)</td>
<td>(6,2,2)</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{16}$</td>
<td>$\frac{53}{64}$</td>
<td>$\frac{4}{3}$</td>
</tr>
<tr>
<td>(2,1,4)</td>
<td>(3,2,2)</td>
<td>$\frac{7}{38}$</td>
<td>$\frac{7}{32}$</td>
<td>$\frac{203}{216}$</td>
<td>1</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>(6,1,1)</td>
<td>0</td>
<td>0</td>
<td>$\frac{2}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>(3,2,2)</td>
<td>(3,1,1)</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{16}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>

4 Conclusion

Though several distance measures [2, 4, 7, 11, 12] have already been introduced for computing crisp distance measure for fuzzy numbers, common sense reasoning leads to exploration of the existence of fuzziness. We believe that the distance measure between two uncertain numbers should also be an uncertain number: If the uncertainty in the form of fuzziness is inherent within the fuzzy numbers, the distance measure value should be fuzzy. Voxman [12] first introduced the concept of fuzzy distance measure for normal fuzzy numbers. In this paper a new distance measure and the generalized Hausdorff distance is introduced between generalized fuzzy numbers. Also here, a fuzzy distance measure between two generalized fuzzy numbers is developed from [1]. Finally the method proposed in this paper computed a fuzzy distance value with less fuzziness and ambiguity in comparison of Voxman’s method.

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