



Adaptive finite time coordinated consensus for high-order multi-agent systems: Adjustable fraction power feedback approach



Yujuan Wang^{a,b,1}, Yongduan Song^{a,b,1,*}, Miroslav Krstic^{c,2}, Changyun Wen^{d,2}

^a Key Laboratory of Dependable Service Computing in Cyber Physical Society, Ministry of Education, Chongqing University, China

^b School of Automation, Chongqing University, Chongqing, 400044, China

^c Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

^d School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, 639798, Singapore

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ABSTRACT

Finite-time consensus for multi-agent systems (MAS) with high-order nonlinear uncertain dynamics under directed communication topology is *still* an open problem due to the involvement of high-order nonlinear uncertain dynamics; one-way directed communication constraints; and unknown time-varying control effectiveness gain. In this paper, based upon the locally defined consensus error and fractionally composed virtual error, a number of useful intermediate results are derived, with which, a finite time consensus solution is established for networked MAS with high-order uncertain dynamics under single-way directed communication topology. By including fraction power integration as part of the Lyapunov function candidate and by using inductive analysis, it is shown that the proposed distributed solution is able to achieve consensus in finite time with sufficient accuracy. The benefits and effectiveness of the developed algorithm are also confirmed by numerical simulations.

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1. Introduction

Finite-time convergence behavior is of special importance in cooperative control of MAS, but the vast majority research on finite time control of MAS has been focused on linear single or double integrator ([12,21,26,28], to just name a few). Although effort on finite time consensus of MAS with first- or second-order feedback linearizable nonlinear dynamics has been made by [3,5,10,13] by using homogeneous (local) approximation, results on finite time distributed control of MAS with general high-order nonlinear dynamics are scarce, and extending the finite time control methods for single or double integrator to higher-order case encounters significant technical challenge. The main hinderance stems from the fact that in the presence of high-order dynamics, the commonly used filtering technique [20] cannot be used with Babarlat lemma or UUB (uniformly ultimately bounded) theory to derive finite time convergence because the convergence of the filtered error to zero or to a small value cannot ensure the finite time convergence of the original error.

* Corresponding author. Fax: +86 23 65112173.

E-mail addresses: iamwyj123456789@163.com (Y. Wang), ydsong@cqu.edu.cn (Y. Song), krstic@ucsd.edu (M. Krstic), ecywen@ntu.edu.sg (C. Wen).

¹ Senior Member, IEEE.

² Fellow, IEEE.

The underlying problem becomes even more challenging when non-parametric uncertainties and unknown time-varying control effectiveness gains (or non-affine dynamics) are involved. In fact, the existence of non-parametric uncertainties literally makes it impossible to use continuous control to achieve zero steady-state error in finite time even for non-networked (single) system. Although zero error convergence is no longer feasible for such situation, convergence with sufficient steady state accuracy is acceptable for most practical applications. Motivated by the work of [2], where it is shown that the finite-time convergence systems with nonlinearities possess many good features including faster convergence rate, better disturbance rejection and robustness against uncertainties, it is thus practically meaningful and plausible to pursue achieving finite time convergence with sufficient accuracy. However, new issues and challenges arise along with this effort. One is that if the control effectiveness gain is unknown and time-varying, literally any designed protocol will be polluted or disguised, making the control impact on the agent uncertain and unclear such that the widely used negative feedback control method is inapplicable for direct cancelation. Furthermore, to guarantee global finite-time stable control with sufficient accuracy for high-order nonlinear MAS, the derivative of the Lyapunov function $V(x, t)$ must satisfy the relation of $\dot{V}(x, t) \leq -cV(x, t)^\alpha + \zeta$ for suitable constants $c > 0$, $0 < \alpha < 1$ and bounded constant $\zeta > 0$, which brings about another technical difficulty to adaptive control design because it is non-trivial to synthesize the adaptive compensation to fulfill such relation. The problem under consideration is made further complicated by the one-way directed communication topology, which imposes significant challenge to extend existing finite time adaptive consensus methods derived from undirected graph to directed graph. As such, the problem of distributed finite time control of MAS with high-order nonlinear dynamics has not been well addressed, this is particular true when there involve non-parametric uncertainties and disturbances in the model and the communication is one-way directed.

In this work we present a solution to this problem by focusing on achieving consensus of MAS in finite time with sufficient accuracy. To circumvent the aforementioned technical obstacles, we first introduce the locally defined neighborhood error and the fractionally composed virtual error, which are embedded into the control scheme; and then we design the fraction power feedback law and fraction power adaptive updated law for the parameter estimation, which renders the crucial relation $\dot{V}(x, t) \leq -cV(x, t)^\alpha + \zeta$ to be satisfied. To carry out rigorous proof of the stability by inductive analysis, we add the fraction power integration term to the Lyapunov function candidate. In addition, to cope with the directed communication topology we use the recent result of [23] on the newly constructed Laplacian deduced from eigenvalue theory, such that the distributed consensus via finite time control is realized.

Notation. Throughout this paper, argument in a variable or function sometimes is dropped if no confusion is likely to occur and the initial time t_0 is set as $t_0 = 0$ without loss of generality; $|\cdot|$ is the absolute value of a real number; for a vector $X = [x_1, \dots, x_N]^T$, $X^h = [x_1^h, \dots, x_N^h]^T$ with $h \in R$, $\|X\|$ denotes the Euclidean norm of the vector X , and $\text{diag}\{X\} = \text{diag}\{x_1, \dots, x_N\}$; $J = \{1, 2, \dots, N\}$ denotes the set of node indexes.

2. Problem formulation and preliminaries

Consider a group of networked systems consisting of N subsystems with high-order nonlinear dynamics, modeled by

$$\begin{aligned} \dot{x}_{i,m} &= x_{i,m+1}, \quad m = 1, \dots, n-1 \\ \dot{x}_{i,n} &= f_i(\tilde{x}_i, u_i) + f_{di}(\tilde{x}_i, t), \\ y_i &= x_{i,1}, \quad i = 1, \dots, N, \end{aligned} \tag{1}$$

where $\tilde{x}_i = [x_{i,1}, \dots, x_{i,n}]^T \in R^n$, $x_{i,m} \in R$, $u_i \in R$ and $y_i \in R$ are system state, control input, and control output, respectively, $f_i(\cdot): R^n \times R \rightarrow R$ is a smooth but unknown non-affine function, $f_{di}(\cdot)$ denotes all the uncertainties and disturbances acting on the i th subsystem.

Remark 1. It should be mentioned that the general high-order systems as described by (1) include the first-order and second-order systems as specific cases. So the proposed control schemes are naturally applicable to lower-order systems.

Let the communication network among the N subsystems be represented by a directed graph $\mathcal{G} = (\iota, \varepsilon)$ and each subsystem can be seen as a node in such graph [16], where $\iota = \{\iota_1, \iota_2, \dots, \iota_N\}$ denotes the set of nodes, $\varepsilon \subseteq \iota \times \iota$ is the set of edges between two distinct nodes. The set of neighbors of node ι_i is denoted by $\mathcal{N}_i = \{\iota_j \in \iota | \varepsilon_{ji} \in \varepsilon\}$, in which $\varepsilon_{ji} = (\iota_j, \iota_i) \in \varepsilon$ indicates that node i can obtain information from node j . The graph topology is represented by the weighted adjacency matrix $\mathcal{A} = [a_{ij}]$, where $a_{ij} > 0 \Leftrightarrow \varepsilon_{ji} \in \varepsilon$, and $a_{ij} = 0$, otherwise. Moreover, it is assumed that $a_{ii} = 0$ for all $i = 1, \dots, N$. The in-degree matrix $\mathcal{B} = \text{diag}\{B_1, \dots, B_N\} \in R^{N \times N}$ is defined as $B_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ such that the Laplacian matrix is defined by $L = [l_{ij}] = \mathcal{B} - \mathcal{A} \in R^{N \times N}$.

The objective is to establish fully distributed consensus protocols for general high-order nonlinear MAS in the presence of non-affine dynamics and non-parametric uncertainties as described by (1) via finite time control method such that not only the stable cooperative output consensus is achieved, but also the finite time convergence is ensured with sufficient accuracy, i.e., there exists a finite time T^* such that for all $t \geq T^*$ and all $i = 1, \dots, N$, $|e_i(t)| < \epsilon$, where $e_i(t)$ denotes the local neighborhood error, $\epsilon > 0$ is a constant which can be made sufficiently small, and meanwhile, all the internal signals in the closed-loop systems are bounded.

Note that by non-affine dynamics, the control action enters into the system through an implicit and uncertain way, making the controller design much more challenging. To proceed, the non-affine dynamic model (1) is first transformed

into an affine form by using the mean value theorem [15], that is, there exists a constant $\xi_i \in (0, u_i)$ ($i \in J$) such that

$$f_i(\bar{x}_i, u_i) = f_i(\bar{x}_i, 0) + \frac{\partial f_i(\bar{x}_i, \xi_i)}{\partial u_i} u_i. \tag{2}$$

Denote $g_i(\bar{x}_i, \xi_i) = \partial f_i(\bar{x}_i, \xi_i) / \partial u_i$. By substituting (2) into (1), one gets

$$\dot{x}_{i,n} = g_i(\bar{x}_i, \xi_i) u_i + L_{f_i}(\bar{x}_i, t), \tag{3}$$

where $L_{f_i}(\bar{x}_i, t) = f_i(\bar{x}_i, 0) + f_{di}(\bar{x}_i, t)$ denotes the lumped uncertainties. Before moving on, we impose the following fundamental conditions.

Assumption 1. The directed communication network \mathcal{G} is strongly connected.

Assumption 2. $g_i(\cdot)$ is unknown and time-varying but bounded away from zero, that is, there exist unknown constants g_{li} and g_{ui} such that $0 < g_{li} \leq |g_i(\cdot)| \leq g_{ui} < \infty$ and $g_i(\cdot)$ is sign-definite (we assume $\text{sgn}(g_i) = +1$ in this paper without loss of generality).

Assumption 3. For the lumped uncertainties $L_{f_i}(\cdot)$, there exists an unknown constant $\alpha_i \geq 0$ and a known scalar function $\psi_i(\bar{x}_i) \geq 0$ such that $|L_{f_i}(\cdot)| \leq \alpha_i \psi_i(\cdot)$. In addition, if \bar{x}_i is bounded, so is $\psi_i(\bar{x}_i)$.

Remark 2. In Assumption 2, although g_{li} is unknown, we can always estimate its lower bound, i.e., $\underline{g} < g_{li}$ for some constant $\underline{g} > 0$. In addition, Assumption 2 is commonly imposed in most existing works in addressing the tracking control problem, for instance [6,14,19,22,24,25]. Assumption 3 is related to the extraction of the core information from the nonlinearities of the system, which can be readily done for any practical system with only crude model information.

3. Main results

The focus in this section is on deriving the finite time output consensus control for MAS with dynamics (1). We first introduce the neighborhood error and local virtual error, and then derive some useful properties, which, as can be seen in the sequel, play a crucial role in developing the adaptive finite time control algorithm. At last, the inductive control design and stability analysis are conducted.

To develop the distributed finite-time control for high-order MAS under directed topology, we define the following local position error $e_{i,1}$ and local virtual errors $e_{i,m}$ as

$$\begin{aligned} e_{i,1} &= \sum_{j \in \mathcal{N}_i} a_{ij} (y_i - \varpi_i - y_j + \varpi_j), \quad i, j \in J \\ e_{i,m} &= x_{i,m}^{1/q_m} - x_{i,m}^{*1/q_m}, \quad i \in J, m = 2, \dots, n \end{aligned} \tag{4}$$

where ϖ_i are constants denoting the final consensus configuration such that $y_i - y_j = \varpi_i - \varpi_j$ ($i, j \in J$); $q_m = (4ln + 3 - 2m) / (4ln + 1) > 0$ ($l \in \mathbb{Z}_+$); $x_{i,m}^*$ ($m = 2, \dots, n$) is the virtual control given by $x_{i,m}^* = -\beta_{i,m-1} e_{i,m-1}^{q_m}$ with $\beta_{i,m-1}$ being positive constant to be designed. Denote $y = [y_1, \dots, y_N]^T$, $\varpi = [\varpi_1, \dots, \varpi_N]^T$, and $E = [e_{1,1}, \dots, e_{N,1}]^T$, then it holds that $E = L(y - \varpi)$.

Remark 3. It is noted that unlike the existing method [9] in defining the fraction power q_m , we have introduced the integer l into q_m here, such treatment leads to the fraction power feedback control capable of adjusting the control precision and finite convergence time, as shown in sequel.

The following properties are crucial for the development of finite time control for MAS under consideration. See Appendix for the proof.

Property 1. Let $\Xi = 1/2(PL + L^T P)$, where $P = \text{diag}\{p\}$ and $p = [p_1, \dots, p_N]^T$ is the left eigenvector of L associated with the zero eigenvalue. Under Assumption 1, $\forall E^s \neq 0_N$, there exists a constant $k_s > 0$ such that

$$(E^s)^T \Xi E^s \geq k_s (E^s)^T E^s, \tag{5}$$

where $s = (4ln - 1) / (4ln + 1)$, $l \in \mathbb{Z}_+$.

Property 2. Suppose $V : \mathcal{D} \rightarrow \mathbb{R}$ is a continuous function satisfying:

- (i) V is positive definite.
- (ii) There exist real numbers $\tilde{c} > 0$, $\alpha \in (0, 1)$ and $0 < \zeta < \infty$, and an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of the origin such that

$$\dot{V}(x, t) \leq -\tilde{c}V(x, t)^\alpha + \zeta, \quad x \in \mathcal{U} \setminus \{0\}. \tag{6}$$

Then there exists a finite time T^* (called the settling-time),

$$T^* \leq \frac{V(x, 0)^{1-\alpha}}{(1-\mu_1)\tilde{c}(1-\alpha)}, \quad 0 < \mu_1 < 1 \tag{7}$$

and for $\forall t \geq T^*$, we have $V(x, t)$ is bounded by

$$V(x, t) < (\zeta / (\mu_1 \tilde{c}))^{\frac{1}{\alpha}}. \tag{8}$$

Property 3. Consider a continuous function of the form,

$$\dot{\vartheta}(t) = -k_{\vartheta} \vartheta(t)^h + q\zeta(t), \tag{9}$$

where $k_{\vartheta} > 0$, $0 < h = h_1/h_2 \leq 1$ ($h_1, h_2 > 0$ are positive odd integers), $q > 0$ is a constant, and $\zeta(t) \geq 0$ is a scalar function. Then, for any given bounded initial condition $\vartheta(0) \geq 0$, we have $\vartheta(t) \geq 0, \forall t \geq 0$.

Property 4. Let $\tilde{\zeta} = \zeta - \hat{\zeta}$. It then holds that

$$\tilde{\zeta} \hat{\zeta}^{\frac{3s-1}{1+s}} \leq \frac{1+s}{4s} \left[\left(2^{\frac{8s(s-1)}{(1+s)^2}} - 2^{\frac{2(s-1)}{1+s}} \right) \tilde{\zeta}^{\frac{4s}{1+s}} + \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{-\frac{16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta^{\frac{4s}{1+s}} \right], \tag{10}$$

where s is given the same as in Property 1.

With these intermediate results and the lemmas as listed in Appendix, we are ready to address the finite time consensus control design for the high-order nonlinear MAS (1).

The design procedure consists of n steps. At each of the first m ($m = 1, 2, \dots, n-1$) step, a virtual control $x_{i,m+1}^*$ is designed, and finally, the true control law u_i is derived at step n . In the following, we denote by $x_k = [x_{1,k}, \dots, x_{N,k}]$ for $k = 1, \dots, n$.

Step 1: Construct the first part of Lyapunov function candidate as

$$V_1(x_1) = \frac{1}{(1+s)k_s} (E^{\frac{1+s}{2}})^T P E^{\frac{1+s}{2}}, \tag{11}$$

where s and k_s are defined as in Property 1. With simple computation, it is obtained that,

$$\begin{aligned} \dot{V}_1(x_1) &= \frac{1}{k_s} (E^s)^T P \dot{E} = \frac{1}{k_s} (E^s)^T P L x_2 \\ &= \frac{1}{k_s} (E^s)^T P L x_2^* + \frac{1}{k_s} (E^s)^T P L (x_2 - x_2^*), \end{aligned} \tag{12}$$

where $x_2 = [x_{1,2}, \dots, x_{N,2}]^T$. By introducing the virtual control $x_2^* = -\beta_1 E^s$ ($\beta_1 > 0$ is design constant) into (12), we have

$$\begin{aligned} \dot{V}_1(x_1) &= -\frac{\beta_1}{k_s} (E^s)^T P L E^s + \frac{1}{k_s} (E^s)^T P L (x_2 - x_2^*) \\ &= -\frac{\beta_1}{k_s} (E^s)^T \Xi E^s + \frac{1}{k_s} \sum_{i=1}^N (x_{i,2} - x_{i,2}^*) \sum_{j=1}^N \ell_{ji} (e_{j,1})^s. \end{aligned} \tag{13}$$

where ℓ_{ji} is the (j, i) th element of (PL) . Let $\ell_{\max} = \max_{j,i \in \mathcal{J}} |\ell_{ji}|$, a straightforward derivation by using Lemmas 2 and 3 in Appendix,

$$\begin{aligned} \frac{1}{k_s} \sum_{i=1}^N (x_{i,2} - x_{i,2}^*) \sum_{j=1}^N \ell_{ji} (e_{j,1})^s &\leq \frac{1}{k_s} \sum_{i=1}^N 2^{1-s} |e_{i,2}|^s \sum_{j=1}^N |\ell_{ji}| |e_{j,1}|^s \leq 2^{1-s} \ell_{\max} \frac{1}{k_s} \sum_{i=1}^N |e_{i,2}|^s \sum_{j=1}^N |e_{j,1}|^s \\ &\leq \frac{1}{2N} \left(\sum_{i=1}^N |e_{i,1}|^s \right)^2 + \tilde{C}_2 \left(\sum_{i=1}^N |e_{i,2}|^s \right)^2 \leq \frac{1}{2} \sum_{i=1}^N (e_{i,1})^{2s} + C_2 \sum_{i=1}^N (e_{i,2})^{2s} \end{aligned} \tag{14}$$

in which the fact $(\sum_{i=1}^N x_i)^2 \leq N \sum_{i=1}^N x_i^2$ has been used, and $C_2 = N\tilde{C}_2$ is a fixed constant.

In light of Property 1 and (14), it is deduced from (13) that

$$\dot{V}_1(x_1) \leq -\beta_1 \sum_{i=1}^N (e_{i,1})^{2s} + \frac{1}{2} \sum_{i=1}^N (e_{i,1})^{2s} + C_2 \sum_{i=1}^N (e_{i,2})^{2s}. \tag{15}$$

Step 2: Consider the second part of Lyapunov function candidate

$$V_2(x_1, x_2) = V_1(x_1) + W_2(x_1, x_2), \tag{16}$$

with

$$W_2(x_1, x_2) = \sum_{i=1}^N \int_{x_{i,2}^*}^{x_{i,2}} \left(\tau^{\frac{1}{q_2}} - x_{i,2}^{*\frac{1}{q_2}} \right)^{1+s-q_2} d\tau. \tag{17}$$

It is obvious that $W_2(\dots)$ is C^1 , then $V_2(\cdot)$ is C^1 , and moreover, $V_2(\cdot)$ is positive definite and $V_2(\cdot) \leq \bar{k}_b \sum_{i=1}^N (e_{i,1}^{1+s} + e_{i,2}^{1+s})$, where $\bar{k}_b = \max\{\frac{1}{(1+s)k_s}, 2\}$. Note that

$$\begin{aligned} \frac{\partial W_2}{\partial x_{i,2}} &= e_{i,2}^{1+s-q_2} = e_{i,2}, \\ \frac{\partial W_2}{\partial x_{j,1}} &= -(1+s-q_2) \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \int_{x_{i,2}^*}^{x_{i,2}} \left(\tau^{\frac{1}{q_2}} - x_{i,2}^{*\frac{1}{q_2}} \right)^{s-q_2} d\tau \\ &= -\frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} (x_{i,2} - x_{i,2}^*), \quad j \in \bar{\mathcal{N}}_i = \mathcal{N}_i \cup i. \end{aligned} \tag{18}$$

The derivative of $W_2(t)$ is thus derived as

$$\begin{aligned} \dot{W}_2(\cdot) &= \sum_{i=1}^N \left[e_{i,2} \dot{x}_{i,2} - (x_{i,2} - x_{i,2}^*) \sum_{j \in \bar{\mathcal{N}}_i} \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \dot{x}_{j,1} \right] \\ &\leq \sum_{i=1}^N \left[e_{i,2} (x_{i,3} - x_{i,3}^*) + e_{i,2} x_{i,3}^* + 2^{1-q_2} \beta_1^{1/q_2} \cdot \left| e_{i,2} |^{q_2} \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2}) \right| \right]. \end{aligned} \tag{19}$$

According to Lemma 4 in Appendix, we have

$$|x_{i,2}| \leq |e_{i,2} + x_{i,2}^{*1/q_2}|^{q_2} \leq |e_{i,2}|^{q_2} + \beta_1 |e_{i,1}|^{q_2}, \tag{20}$$

which implies that

$$\begin{aligned} 2^{1-q_2} \beta_1^{1/q_2} \sum_{i=1}^N |e_{i,2}|^{q_2} \left| \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2}) \right| &\leq 2^{1-q_2} \beta_1^{1/q_2} \sum_{i=1}^N |e_{i,2}|^{q_2} \left(r_1 |x_{i,2}| + r_2 \sum_{j \in \mathcal{N}_i} |x_{j,2}| \right) \\ &\leq 2^{1-q_2} \beta_1^{1/q_2} \sum_{i=1}^N \left[r_1 |e_{i,2}|^{2q_2} + r_1 \beta_1 |e_{i,2}|^{q_2} |e_{i,1}|^{q_2} + r_2 |e_{i,2}|^{q_2} \sum_{j \in \mathcal{N}_i} (|e_{j,2}|^{q_2} + \beta_1 |e_{j,1}|^{q_2}) \right], \end{aligned} \tag{21}$$

with $r_1 = \max_{i \in \mathcal{I}} \{ \sum_{j \in \mathcal{N}_i} a_{ij} \}$ and $r_2 = \max_{i \in \mathcal{I}, j \in \mathcal{I}} \{ a_{ij} \}$. Note that

$$\begin{aligned} 2^{1-q_2} \beta_1^{1+1/q_2} r_1 |e_{i,2}|^{q_2} |e_{i,1}|^{q_2} &\leq 2^{2(1-q_2)} \beta_1^{2(1+1/q_2)} r_1^2 |e_{i,2}|^{2q_2} + \frac{1}{4} |e_{i,1}|^{2q_2}, \\ 2^{1-q_2} \beta_1^{1/q_2} r_2 |e_{i,2}|^{q_2} \sum_{j \in \mathcal{N}_i} |e_{j,2}|^{q_2} &\leq 2^{-q_2} \beta_1^{1/q_2} r_2 \sum_{j \in \mathcal{N}_i} (|e_{i,2}|^{2q_2} + |e_{j,2}|^{2q_2}), \\ 2^{1-q_2} \beta_1^{1+1/q_2} r_2 |e_{i,2}|^{q_2} \sum_{j \in \mathcal{N}_i} |e_{j,1}|^{q_2} &\leq \sum_{j \in \mathcal{N}_i} \left(2^{2(1-q_2)} \beta_1^{2(1+1/q_2)} r_2^2 c_{\mathcal{N}} |e_{i,2}|^{2q_2} + \frac{1}{4c_{\mathcal{N}}} |e_{j,1}|^{2q_2} \right), \end{aligned} \tag{22}$$

in which $c_{\mathcal{N}} = \max_{i \in \mathcal{I}} \{ \dim\{\mathcal{N}_i\} \}$, and $\dim\{\mathcal{N}_i\}$ denotes the number of the elements in \mathcal{N}_i .

Substituting (22) into (21) yields that

$$2^{1-q_2} \beta_1^{1/q_2} \sum_{i=1}^N |e_{i,2}|^{q_2} \left| \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2}) \right| \leq \frac{1}{2} \sum_{i=1}^N e_{i,1}^{2q_2} + \bar{C}_2 \sum_{i=1}^N e_{i,2}^{2q_2} \tag{23}$$

where $\bar{C}_2 = 2^{1-q_2} \beta_1^{1/q_2} r_1 + 2^{2(1-q_2)} \beta_1^{2(1+1/q_2)} r_1^2 + 2^{1-q_2} \beta_1^{1/q_2} \cdot r_2 c_{\mathcal{N}} + 2^{2(1-q_2)} \beta_1^{2(1+1/q_2)} r_2^2 c_{\mathcal{N}}^2$.

Combining (16), (15), (19) and (23) yields that

$$\dot{V}_2 \leq -(\beta_1 - 1) \sum_{i=1}^N (e_{i,1})^{2s} + (C_2 + \bar{C}_2) \sum_{i=1}^N (e_{i,2})^{2s} + \sum_{i=1}^N [e_{i,2} (x_{i,3} - x_{i,3}^*) + e_{i,2} x_{i,3}^*]. \tag{24}$$

Note that $q_3 = q_2 - 2/(4ln + 1)$, and then $1 + q_3 = 2q_2 = 2s$. Thus by choosing $\beta_1 > n - 1 + k_c$ with $k_c > 0$ being a design parameter and the virtual control $x_{i,3}^* = -\beta_{i,2} e_{i,2}^{q_3}$ with $\beta_{i,2} \geq n - 2 + k_c + C_2 + \bar{C}_2$, we get that

$$\dot{V}_2 \leq -(n - 2 + k_c) \sum_{i=1}^N \sum_{m=1}^2 (e_{i,m})^{2s} + \sum_{i=1}^N e_{i,2} (x_{i,3} - x_{i,3}^*). \tag{25}$$

Step k ($3 \leq k \leq n - 1$): To proceed, we conduct the proof by using an inductive argument. Suppose at step $k - 1$, there exists a C^1 Lyapunov function $V_{k-1}(x_1, \dots, x_{k-1})$, which is positive definite and satisfies

$$V_{k-1}(\cdot) \leq \bar{k}_b \sum_{i=1}^N (e_{i,1}^{1+s} + \dots + e_{i,k-1}^{1+s}) \tag{26}$$

such that

$$\dot{V}_{k-1}(\cdot) \leq -(n - k + 1 + k_c) \sum_{i=1}^N \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^N e_{i,k-1}^{1+s-q_{k-1}} (x_{i,k} - x_{i,k}^*). \tag{27}$$

We claim that (26) and (27) also hold at step k . To prove this, consider

$$V_k(x_1, \dots, x_k) = V_{k-1}(\cdot) + W_k(x_1, \dots, x_k), \tag{28}$$

with

$$W_k(x_1, \dots, x_k) = \sum_{i=1}^N \int_{x_{i,k}^*}^{x_{i,k}} (\tau^{\frac{1}{q_k}} - x_{i,k}^{*\frac{1}{q_k}})^{1+s-q_k} d\tau. \tag{29}$$

It is not hard to check that $W_k(\cdot)$ is C^1 , and moreover,

$$\begin{aligned} \frac{\partial W_k}{\partial x_{i,k}} &= e_{i,k}^{1+s-q_k}, \text{ and for } m = 1, \dots, k-1, \\ \frac{\partial W_k}{\partial x_{i,m}} &= -(1+s-q_k) \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \int_{x_{i,k}^*}^{x_{i,k}} (\tau^{\frac{1}{q_k}} - x_{i,k}^{*\frac{1}{q_k}})^{s-q_k} d\tau. \end{aligned} \tag{30}$$

Therefore, $V_k(\cdot)$ is C^1 , positive definite and satisfies $V_k(\cdot) \leq \bar{k}_b \sum_{i=1}^N (e_{i,1}^{1+s} + \dots + e_{i,k}^{1+s})$.

The derivative of $V_k(\cdot)$ is derived, by combining (27), as

$$\begin{aligned} \dot{V}_k(\cdot) &\leq -(n - k + 1 + k_c) \sum_{i=1}^N \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^N e_{i,k-1}^{1+s-q_{k-1}} (x_{i,k} - x_{i,k}^*) + \sum_{i=1}^N e_{i,k}^{1+s-q_k} x_{i,k+1} \\ &\quad + \sum_{i=1}^N \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \mathcal{N}_i} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right). \end{aligned} \tag{31}$$

To continue, we examine the second term and the fourth term on the right hand side of (31). First, according to Lemma 3 in Appendix, it holds that

$$|e_{i,k-1}^{1+s-q_{k-1}} (x_{i,k} - x_{i,k}^*)| \leq 2^{1-q_k} |e_{i,k-1}|^{1+s-q_{k-1}} |e_{i,k}|^{q_k} \leq \frac{e_{i,k-1}^{2s}}{2} + C_{i,k} e_{i,k}^{2s}, \tag{32}$$

with $C_{i,k}$ a constant. As for the fourth term, we have

$$\left| \sum_{i=1}^N \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \mathcal{N}_i} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \leq \sum_{i=1}^N (1+s-q_k) 2^{1-q_k} |e_{i,k}|^s \cdot \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \mathcal{N}_i} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right|. \tag{33}$$

Now we show that

$$\sum_{i=1}^N \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \mathcal{N}_i} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \leq \sum_{i=1}^N \left(\sum_{m=1}^k \varrho_{ki,m} |e_{i,m}|^s \right), \tag{34}$$

for a constant $\varrho_{ki,m} \geq 0$. This is done by inductive argument. First of all, it is deduced from (18)–(20) that

$$\left| \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \frac{\partial x_{i,2}^{*\frac{1}{q_2}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \leq \sum_{i=1}^N \beta_1^{1/q_2} \left| \sum_{j \in \mathcal{N}_i} a_{ij} (x_{i,2} - x_{j,2}) \right| \leq \sum_{i=1}^N \left(\sum_{m=1}^2 \varrho_{2i,m} |e_{i,m}|^s \right), \tag{35}$$

where $\varrho_{2i,m} \geq 0$ is a constant. Assume that for $k-1$, it holds,

$$\sum_{i=1}^N \left| \left(\sum_{m=2}^{k-2} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \mathcal{N}_i} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \leq \sum_{i=1}^N \left(\sum_{m=1}^{k-1} \varrho_{(k-1)i,m} |e_{i,m}|^s \right) \tag{36}$$

with $\varrho_{(k-1)i,m} \geq 0$ ($m = 1, \dots, k-1$) being constants. Recall that $e_{i,m} = x_{i,m}^{1/q_m} - x_{i,m}^{*1/q_m}$ and $x_{i,m}^* = -\beta_{i,m-1} e_{i,m-1}^{q_m}$, thus $x_{i,m}^{*1/q_m} = -\beta_{i,m-1}^{1/q_m} e_{i,m-1}$ and

$$|x_{i,m}| \leq |e_{i,m} + x_{i,m}^{*1/q_m}|^{q_m} \leq |e_{i,m}|^{q_m} + \beta_{i,m-1} |e_{i,m-1}|^{q_m}, \tag{37}$$

which, together with (36), yields

$$\begin{aligned}
 \sum_{i=1}^N \left| \sum_{m=2}^{k-1} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \tilde{\mathcal{N}}_i} \frac{\partial x_{i,k}^{*\frac{1}{q_k}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| &= \sum_{i=1}^N \left| -\beta_{i,k-1}^{1/q_k} \left(\sum_{m=2}^{k-1} \frac{\partial e_{i,k-1}}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \tilde{\mathcal{N}}_i} \frac{\partial e_{i,k-1}}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \\
 &\leq \sum_{i=1}^N \beta_{i,k-1}^{1/q_k} \left| \frac{x_{i,k-1}^{\frac{1}{q_{k-1}}-1}}{q_{k-1}} \dot{x}_{i,k-1} - \sum_{m=2}^{k-2} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{i,m}} \dot{x}_{i,m} - \sum_{j \in \tilde{\mathcal{N}}_i} \frac{\partial x_{i,k-1}^{*\frac{1}{q_{k-1}}}}{\partial x_{j,1}} \dot{x}_{j,1} \right| \\
 &\leq \sum_{i=1}^N \beta_{i,k-1}^{1/q_k} \left[\frac{1}{q_{k-1}} \cdot \left(e_{i,k-1}^{1-q_{k-1}} + \beta_{i,k-2}^{\frac{1}{q_{k-1}}-1} e_{i,k-2}^{1-q_{k-1}} \right) \cdot (|e_{i,k}|^{q_k} + |e_{i,k-1}|^{q_k} \beta_{i,k-1}) + \sum_{m=1}^{k-1} \varrho_{(k-1)i,m} |e_{i,m}|^s \right] \\
 &\leq \sum_{i=1}^N \sum_{m=1}^k \varrho_{ki,m} |e_{i,m}|^s.
 \end{aligned} \tag{38}$$

Therefore,

$$\left| \sum_{i=1}^N \left(\sum_{m=2}^{k-1} \frac{\partial W_k}{\partial x_{i,m}} \dot{x}_{i,m} + \sum_{j \in \tilde{\mathcal{N}}_i} \frac{\partial W_k}{\partial x_{j,1}} \dot{x}_{j,1} \right) \right| \leq \sum_{i=1}^N (1+s-q_k) 2^{1-q_k} |e_{i,k}|^s \left(\sum_{m=1}^k \varrho_{ki,m} |e_{i,m}|^s \right) \leq \frac{1}{2} \sum_{i=1}^N \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^N \bar{C}_{i,k} e_{i,k}^{2s} \tag{39}$$

with $\bar{C}_{i,k} > 0$ being a constant.

Substituting (32) and (39) into (31) leads to

$$\dot{V}_k(\cdot) \leq -(n-k+k_c) \sum_{i=1}^N \sum_{m=1}^{k-1} e_{i,m}^{2s} + \sum_{i=1}^N (C_{i,k} + \bar{C}_{i,k}) e_{i,k}^{2s} + \sum_{i=1}^N e_{i,k}^{1+s-q_k} (x_{i,k+1} - x_{i,k+1}^*) + \sum_{i=1}^N e_{i,k}^{1+s-q_k} x_{i,k+1}^*. \tag{40}$$

By introducing the virtual control $x_{i,k+1}^* = -\beta_{i,k} e_{i,k}^{q_{k+1}}$, with $\beta_{i,k} \geq n-k+k_c + C_{i,k} + \bar{C}_{i,k} > 0$, we get from (40) that

$$\dot{V}_k(\cdot) \leq -(n-k+k_c) \sum_{i=1}^N \sum_{m=1}^k e_{i,m}^{2s} + \sum_{i=1}^N e_{i,k}^{1+s-q_k} (x_{i,k+1} - x_{i,k+1}^*). \tag{41}$$

Step n: Consider the n th part of Lyapunov function candidate,

$$V_n(x_1, \dots, x_n) = V_{n-1}(\cdot) + W_n(x_1, \dots, x_n), \tag{42}$$

where

$$W_n(x_1, \dots, x_n) = \sum_{i=1}^N \int_{x_{i,n}^*}^{x_{i,n}} (\tau^{\frac{1}{q_n}} - x_{i,n}^{*\frac{1}{q_n}})^{1+s-q_n} d\tau. \tag{43}$$

Clearly, $V_n(\cdot)$ so defined is C^1 , positive definite and satisfies $V_n(\cdot) \leq \bar{k}_b \sum_{i=1}^N (e_{i,1}^{1+s} + \dots + e_{i,n}^{1+s})$.

From the above inductive argument, one can conclude that

$$\dot{V}_n(\cdot) \leq -k_c \sum_{i=1}^N \sum_{m=1}^{n-1} e_{i,m}^{2s} + \sum_{i=1}^N e_{i,n}^{1+s-q_n} \dot{x}_{i,n} + \sum_{i=1}^N (C_{i,n} + \bar{C}_{i,n}) e_{i,n}^{2s}. \tag{44}$$

where $C_{i,n} > 0$ and $\bar{C}_{i,n} > 0$ are constants.

The actual control u_i is designed as

$$u_i = -\beta_{i,n} e_{i,n}^{q_{n+1}} - \frac{\hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n}^{1+s-q_n} \tag{45}$$

with the updated law

$$\dot{\hat{\zeta}}_i = -\sigma_i \gamma_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} + \frac{\sigma_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)}, \tag{46}$$

where $\hat{\zeta}_i$ is the estimation of ζ_i (here ζ_i is a virtual parameter to be defined later), $\psi_i(\cdot)$ is a scalar and readily computable function, $\beta_{i,n}$, σ_i , γ_i and η_i are positive design parameters chosen arbitrarily by the designer. Now we are ready to present the following result.

Theorem 1. Consider the networked high-order nonlinear MAS (1) under Assumptions 1–3. If the distributed adaptive control scheme (45)–(46) with the virtual control $x_{i,m}^*$ ($m = 1, \dots, n$) is applied, then for any initial conditions satisfying $\hat{\zeta}_i(0) \geq 0$ and $V(0) \leq \mu$ ($\mu > 0$ is a bounded constant), finite time stable consensus is achieved in that

(1) the neighborhood error E converges to a small residual set Θ_0 , namely

$$\Theta_0 = \left\{ \|E\| \leq \left(\frac{[(1+s)k_s k_b]^{2s}}{\mu_1 \mu_2 k_a} \varsigma \right)^{\frac{1}{2s}} \right\}, \tag{47}$$

and the parameter estimate error $\tilde{\zeta}$ converges to a small region Θ_1 , given by

$$\Theta_1 = \left\{ \|\tilde{\zeta}\| \leq \left[\left(\frac{2\sigma_{\max} k_b}{\underline{g}} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}} \right\} \tag{48}$$

in a finite time T^* satisfying

$$T^* \leq \frac{V(0)^{1-\frac{2s}{1+s}}}{(1-\mu_1)\tilde{c}(1-\frac{2s}{1+s})}, \tag{49}$$

where $k_a = \min\{k_c, \frac{1+s}{4s}\underline{g}\gamma_i(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}})\}$, $k_b = \max\{\bar{k}_b, \frac{g}{2\sigma_i}\}$, $\varsigma = \sum_{i=1}^N \frac{\eta_i^2}{4} + \sum_{i=1}^N \frac{1+s}{4s}\underline{g}\gamma_i(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{-\frac{16s(s-1)^2}{(1+s)^3}}}{4s})\zeta_i^{\frac{4s}{1+s}}$, $0 < \mu_1 < 1$, $0 < \mu_2 \leq 1$, $\tilde{\zeta} = [\tilde{\zeta}_1, \dots, \tilde{\zeta}_N]^T$, and $\sigma_{\max} = \max\{\sigma_i\}$;

(2) all of the signals in the controlled system are semi-globally uniformly ultimately bounded.

Proof. Choose the Lyapunov function candidate as

$$V = V_n + \sum_{i=1}^N \frac{g}{2\sigma_i} \tilde{\zeta}_i^2. \tag{50}$$

Taking the derivative of V along (3) yields that

$$\begin{aligned} \dot{V}(\cdot) &\leq -k_c \sum_{i=1}^N \sum_{m=1}^{n-1} e_{i,m}^{2s} + \sum_{i=1}^N e_{i,n}^{1+s-q_n} (g_i u_i + L_{fi}) \\ &\quad + \sum_{i=1}^N (C_{i,n} + \bar{C}_{i,n}) e_{i,n}^{2s} + \sum_{i=1}^N \tilde{\zeta}_i \left(-\frac{g}{\sigma_i} \dot{\hat{\zeta}}_i \right). \end{aligned} \tag{51}$$

By Assumption 3 and using Young's inequality, we have, for any $\eta_i > 0$, that

$$e_{i,n}^{1+s-q_n} L_{fi} \leq \alpha_i \psi_i |e_{i,n}|^{1+s-q_n} \leq \frac{\alpha_i^2}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} + \frac{\eta_i^2}{4}. \tag{52}$$

Let $\zeta_i = g^{-1} \alpha_i^2$. By using (45) and (52), it is deduced from (51) that

$$\begin{aligned} \dot{V}(\cdot) &\leq -k_c \sum_{i=1}^N \sum_{m=1}^{n-1} e_{i,m}^{2s} - \sum_{i=1}^N (g_i \beta_{i,n} - C_{i,n} - \bar{C}_{i,n}) e_{i,n}^{2s} \\ &\quad + \sum_{i=1}^N \left(-\frac{g_i \hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} + \frac{g \zeta_i}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} \right) \\ &\quad + \sum_{i=1}^N \frac{\eta_i^2}{4} + \sum_{i=1}^N \tilde{\zeta}_i \left(-\frac{g}{\sigma_i} \dot{\hat{\zeta}}_i \right). \end{aligned} \tag{53}$$

According to Property 3 and (46), it holds that $\hat{\zeta}_i(t) \geq 0$ for any given initial estimate $\hat{\zeta}_i(0) \geq 0$. If $\beta_{i,n}$ is chosen such that $\beta_{i,n} g - C_{i,n} - \bar{C}_{i,n} > k_c$, it thus follows from (53) that

$$\dot{V}(\cdot) \leq -k_c \sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{2s} + \sum_{i=1}^N \frac{g}{\eta_i^2} \psi_i^2 e_{i,n}^{2(1+s-q_n)} (\zeta_i - \hat{\zeta}_i) + \sum_{i=1}^N \frac{\eta_i^2}{4} + \sum_{i=1}^N \tilde{\zeta}_i \left(-\frac{g}{\sigma_i} \dot{\hat{\zeta}}_i \right). \tag{54}$$

By applying the adaptive law for $\hat{\zeta}_i$ given in (46), one has

$$\dot{V}(\cdot) \leq -k_c \sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{2s} + \sum_{i=1}^N g \gamma_i \tilde{\zeta}_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} + \sum_{i=1}^N \frac{\eta_i^2}{4}. \tag{55}$$

It is deduced from Property 4 that

$$\tilde{\zeta}_i \hat{\zeta}_i^{\frac{3s-1}{1+s}} \leq -\frac{1+s}{4s} \left(2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} \right) \tilde{\zeta}_i^{\frac{4s}{1+s}} + \frac{1+s}{4s} \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{-\frac{16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta_i^{\frac{4s}{1+s}}, \tag{56}$$

where $2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}} > 0$. It then holds that

$$\dot{V}(\cdot) \leq -k_c \sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{2s} - \sum_{i=1}^N \frac{1+s}{4s} g \gamma_i (2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}}) \tilde{\zeta}_i^{\frac{4s}{1+s}} + \varsigma \tag{57}$$

with

$$\varsigma = \sum_{i=1}^N \frac{\eta_i^2}{4} + \sum_{i=1}^N \frac{1+s}{4s} g \gamma_i \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{\frac{-16s(s-1)^2}{(1+s)^3}}}{4s} \right) \zeta_i^{\frac{4s}{1+s}}. \tag{58}$$

Let $k_a = \min\{k_c, \frac{1+s}{4s} g \gamma_i (2^{\frac{2(s-1)}{1+s}} - 2^{\frac{8s(s-1)}{(1+s)^2}})\}$. Then (57) can be expressed as

$$\dot{V}(\cdot) \leq -k_a \left(\sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{2s} + \sum_{i=1}^N (\tilde{\zeta}_i^{\frac{2}{1+s}})^{2s} \right) + \varsigma. \tag{59}$$

In the sequel, we prove that there exists a finite time $T^* > 0$ and a bounded constant c_V such that $V(t) < c_V$ when $t \geq T^*$.

Let $k_b = \max\{k_b, \frac{g}{2\sigma_i}\}$, and thus

$$V(\cdot) \leq k_b \left(\sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{1+s} + \sum_{i=1}^N (\tilde{\zeta}_i^{\frac{2}{1+s}})^{1+s} \right) \tag{60}$$

which further implies, by using Lemma 4 in Appendix, that

$$V^{\frac{2s}{1+s}}(\cdot) \leq k_b^{\frac{2s}{1+s}} \left(\sum_{i=1}^N \sum_{m=1}^n e_{i,m}^{2s} + \sum_{i=1}^N \tilde{\zeta}_i^{\frac{4s}{1+s}} \right). \tag{61}$$

Let $\tilde{c} = \frac{\mu_2 k_a}{k_b^{\frac{2s}{1+s}}}$ ($0 < \mu_2 \leq 1$). It is straightforward from (59) and (61) that

$$\dot{V}(\cdot) \leq -\tilde{c} V^{\frac{2s}{1+s}}(\cdot) + \varsigma. \tag{62}$$

Upon using Property 2, it is seen that there exists a finite time T^* satisfying

$$T^* \leq \frac{V(0)^{1-\frac{2s}{1+s}}}{(1-\mu_1)\tilde{c}(1-\frac{2s}{1+s})}, \tag{63}$$

such that for $\forall t \geq T^*$,

$$V(\cdot) < \left(\frac{\varsigma}{\mu_1 \tilde{c}} \right)^{\frac{1+s}{2s}} = k_b \left(\frac{\varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1+s}{2s}} = c_V. \tag{64}$$

Next, we examine the steady-state errors of all agents. Note that

$$\|E\| \leq [(1+s)k_s V_1(\cdot)]^{\frac{1}{1+s}} \leq [(1+s)k_s V(\cdot)]^{\frac{1}{1+s}} \leq \left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}}. \tag{65}$$

Let $\tilde{\zeta} = [\tilde{\zeta}_1, \dots, \tilde{\zeta}_N]^T$, and $\sigma_{\max} = \max\{\sigma_i\}$. Then we have from (50) that

$$\|\tilde{\zeta}\| \leq \sqrt{\frac{2\sigma_{\max}}{g} V(\cdot)} \leq \left[\left(\frac{2\sigma_{\max} k_b}{g} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}}. \tag{66}$$

Thus it has been established that under the proposed finite time control scheme, the neighborhood error converges to a small region Θ_0 given as

$$\Theta_0 = \left\{ \|E\| \leq \left(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}} \right\}, \tag{67}$$

and the parameter estimate errors converge to the region Θ_1 ,

$$\Theta_1 = \left\{ \|\tilde{\zeta}\| \leq \left[\left(\frac{2\sigma_{\max} k_b}{g} \right)^{\frac{2s}{1+s}} \frac{\varsigma}{\mu_1 \mu_2 k_a} \right]^{\frac{1+s}{4s}} \right\} \tag{68}$$

in the finite time T^* as given in (49). \square

4. Comparison with regular linear feedback consensus method

Note that if we set the fraction power $s = 1$, $q_1 = q_2 = \dots = q_n = 1$, the proposed finite time control scheme reduces to the regular non-finite time control and the convergence result is in asymptotical sense. In this case, the local virtual error $e_{i,m}$ ($i \in J, m = 2, \dots, n$) becomes $e_{i,m} = x_{i,m} - x_{i,m}^*$, with the virtual control $x_{i,m}^*$ given as

$$x_{i,m}^* = -\beta_{i,m-1}e_{i,m-1}, \tag{69}$$

and the actual control u_i becomes

$$u_i = -\beta_{i,n}e_{i,n} - \frac{\hat{\zeta}_i}{\eta_i^2} \psi_i^2 e_{i,n} \tag{70}$$

with the updated law

$$\dot{\hat{\zeta}}_i = -\sigma_i \gamma_i \hat{\zeta}_i + \frac{\sigma_i}{\eta_i^2} \psi_i^2 e_{i,n}, \tag{71}$$

then the ultimately uniformly bounded consensus is achieved. More specifically, with this protocol, the neighborhood error converges to

$$\Theta_2 = \{ \|E\| \leq (\frac{2k_b \zeta}{\mu_1 \mu_2 k_a})^{\frac{1}{2}} \}, \tag{72}$$

and the parameter estimate errors converges to

$$\Theta_3 = \{ \|\tilde{\zeta}\| \leq (\frac{2\sigma_{\max} k_b \zeta}{g \mu_1 \mu_2 k_a})^{\frac{1}{2}} \} \tag{73}$$

where the variables $\tilde{\zeta}$ and ζ correspond to the values with $s = 1$ and $q_1 = \dots = q_n = 1$.

Remark 4. It is seen that in these two control schemes, the steady-state errors are controlled by the convergence regions, i.e., $\Theta_0, \Theta_1, \Theta_2$ and Θ_3 , respectively, all of which can be reduced as small as desired by choosing the control parameters. Nevertheless, due to control saturation constraint, the control parameters are not allowed to be set too large. While in the proposed method one can choose the fractional power $s = (4ln - 1)/(4ln + 1)$ properly to enhance the disturbance rejection performance and increase the control precision without the need for excessively large control parameters. For instance, suppose $\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \zeta}{\mu_1 \mu_2 k_a} < 1$ is satisfied (this is allowed because all the parameters in k_a and k_b , i.e., k_c, γ_i and σ_i are free

design parameters), note that $1/s$ is much larger than 1, which makes $(\frac{[(1+s)k_s k_b]^{\frac{2s}{1+s}} \zeta}{\mu_1 \mu_2 k_a})^{\frac{1}{2s}} \ll (\frac{2k_b k_s \zeta}{\mu_1 \mu_2 k_a})^{\frac{1}{2}}$, thus leading to better control precision as compared with linear feedback based non-finite time control method.

Remark 5. It is worth noting that the proposed finite time control scheme is essentially different from the regular UUB control, here the feedback error and the adaptive compensation are not used directly in building the control scheme and the updated law, instead, the fractional power of the feedback error of the form $e_{i,m}^{q_m}$ and the fractional power of adaptive term $\hat{\zeta}_i^{\frac{3s-1}{1+s}}$ are embedded into the algorithm, it is such special treatment that makes it possible to achieve sufficient control precision in finite time by the adaptive approach. In other words, for the proposed control, the convergence precision is adjustable and the convergence time is easily adjustable without the need for high gain to reach the given precision, whereas for regular UUB control, although it can reach a bound in a finite time, both the bound and the finite time cannot be prescribed and one can only enlarge the control gain to get a better control precision.

Remark 6. Note that if letting the number of the agents N be equal to 1 and the neighborhood error $e_{i,1} = \sum_{j \in \mathcal{N}_i} a_{ij}(x_{i,1} - x_{j,1})$ be replaced by $x_{i,1}$, and accordingly $r_1 = \max_{i \in J} \{ \sum_{j \in \mathcal{N}_i} a_{ij} \} = 1$ and $r_2 = \max_{i,j \in J} \{ a_{ij} \} = 0$, then the proposed control scheme is immediately applicable to single (non-networked) systems with high-order nonlinear dynamics. Compared with the existing works on single nonlinear systems by [8,9], and [11], the control scheme presented herein is able to deal with much less restrictive uncertainties, i.e., non-parametric uncertainties and external disturbances, gracefully. In addition, in defining the fraction power q_m , we introduce the integer l into q_m here, such treatment makes the fraction power feedback control capable of adjusting the control precision and finite convergence time, leading to a more favorable solution for finite time control of high-order systems.

Remark 7. Compared with the recent work [23], where the system is second-order dynamic, this work extends the result to high-order case, which poses significant technical challenge. In fact, the main hinderance stems from the fact that in the presence of high-order dynamics, the commonly used filtering technique [20] cannot be used with Babarlat lemma or UUB theory to derive finite time convergence because the convergence of the filtered error to zero or to a small value cannot ensure the finite time convergence of the original error. Compared with the other only two works, [27] and [10], on adaptive finite time consensus for nonlinear MAS, where the systems are second-order; the communication is undirected;

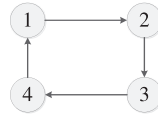


Fig. 1. The directed communications among the four agents.

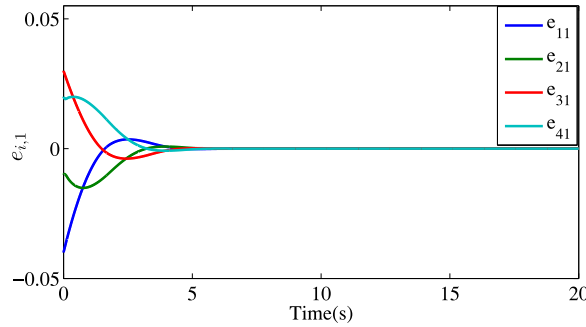


Fig. 2. Position neighborhood errors under the finite time controller.

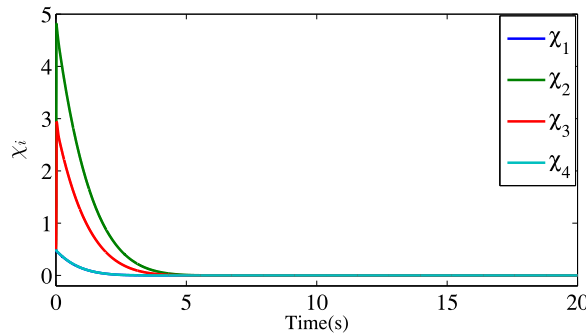


Fig. 3. The parameter estimation under the finite time controller.

and the uncertainties are assumed to exhibit the linear parametric property, this work proposes an adaptive distributed control solution to the problem of the cooperative consensus of nonlinear MAS with general high-order dynamics involving non-parametric uncertainties and disturbances under directed communication, by focusing on achieving consensus in finite time with sufficient accuracy.

5. Numerical simulations

In order to demonstrate the feasibility of the proposed finite-time control, two numerical examples are given.

Example 1. We simulate on a group of third-order MAS with non-affine dynamics in the form of (1) with

$$\dot{x}_{i,3} = (0.6 + 0.1 \exp(-x_{i,2}^2 - x_{i,3}^2))u_i + 0.1 \sin(u_i) + f_{di}(\cdot), \tag{74}$$

for $i = 1, 2, 3, 4$, where the system uncertainties f_{di} is taken as $f_{di}(\cdot) = d_i + 0.5(-1)^i x_{i,1}^2 + 0.5(-1)^i (x_{i,2}^2) + 0.5x_{i,3}^2$, and d_i denotes disturbances, taken as random and bounded by $|d_i| \leq 0.1$. The communication topology is directed and connected as shown in Fig. 1. Each edge weight is taken as 0.1. The left eigenvector of L associated with eigenvalue 0 is $[1, 1, 1, 1]^T$.

The simulation runs for 20s. The initial condition are $x_1(0) = [-0.2, -0.3, 0, 0.2]^T$, $x_2(0) = [0.2, 0.3, 0, -0.2]^T$, and $x_3(0) = \mathbf{0}_4$. The simulation is conducted by applying the control laws given in (45)–(46), where the control parameters are taken as: $l = 1$, $\beta_1 = 4$, $\beta_{i,2} = 12$, $\beta_{i,3} = 40$, $\sigma_i = 1$, $\gamma_i = 1$ and $\eta_i = 0.8$. The scalar function $\psi_i(\bar{x}_i) = 0.2 + x_{i,1}^2 + x_{i,2}^2 + x_{i,3}^2$. In addition, the initial values of the estimates are chosen as $\hat{\zeta}_i(0) = 0.5$ for $i = 1, 2, 3, 4$. The error convergence result of the four agents are depicted in Fig. 2. Fig. 3 represents the parameter estimation $\hat{\zeta}_i$ ($i = 1, 2, 3, 4$).

To show that better performance is achieved with our proposed scheme, we also tested the convergence property between the proposed finite-time control scheme (45)–(46) and the typical non-finite-time based adaptive scheme (69)–(71). Both the two control laws are applied to system (74) by using the same design parameters. It is observed from Fig. 4 that the convergence rate is faster and the error precision is better with the finite time controller compared with the non-finite time controller.

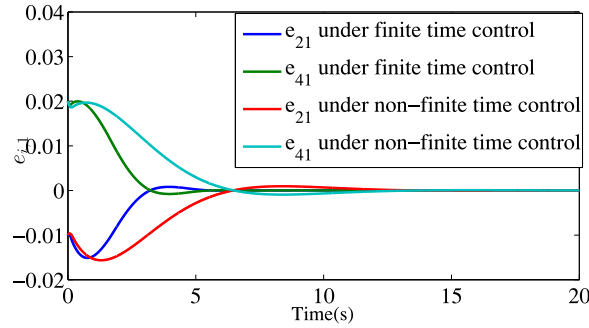


Fig. 4. Position neighborhood errors under the two different controllers.

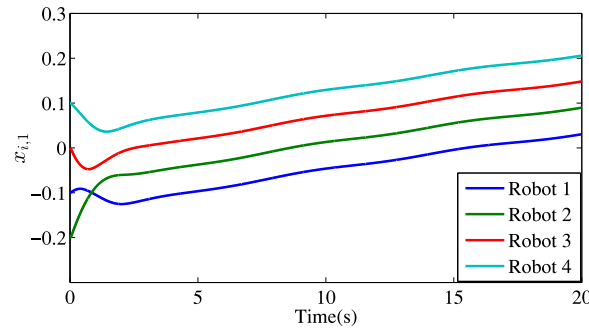


Fig. 5. Position trajectories of the four agents under the finite time controller.

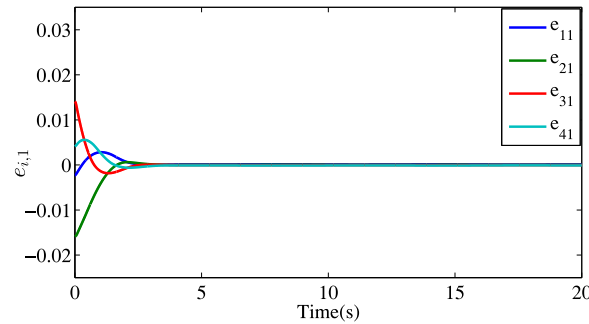


Fig. 6. Position neighborhood errors under the finite time controller.

Example 2. We consider a one-link manipulator [4] with nonaffine actuation system,

$$D_i \ddot{q}_i + B_i \dot{q}_i + N_i \sin(q_i) = f(u_i), \tag{75}$$

for $i = 1, 2, 3, 4$, where q_i , \dot{q}_i and \ddot{q}_i denote the link angular position, velocity and acceleration of the i th robot, respectively; $f(u_i) = u_i + 0.5\cos(u_i)$ is the driving torque in which u_i is the motor current, $G_g(t) = -0.1 \sin(t)$ is the external disturbance. For convenience, we consider the same communication condition as in Example 1 with each edge weight being 0.1. We set the final consensus configuration ϖ_i as $\varpi = 0.15[\pi/2, 5\pi/8, 3\pi/4, 7\pi/8]^T$. The initial condition are $x_1(0) = [-0.1, -0.2, 0, 0.1]^T$, $x_2(0) = \mathbf{0}_4$, and $\zeta(0) = 0.5\mathbf{1}_4$.

Fig. 5 is the position trajectories of the four robots, and Fig. 6 is the neighborhood errors. For comparison, we also compared the convergence property of the proposed control with the typical non-finite-time based control (69)–(71). Fig. 7 demonstrates that the convergence property including the convergence rate and the error precision is better with the finite time controller as compared with the non-finite time method, as theoretically predicted.

6. Conclusions

The majority of the research on finite time consensus control has been focused on MAS described by single or double integrator under undirected topology. This work explicitly addressed the problem of finite-time consensus for MAS with high-order nonlinear uncertain dynamics under directed communication constraints. By integrating an adjustable fractional

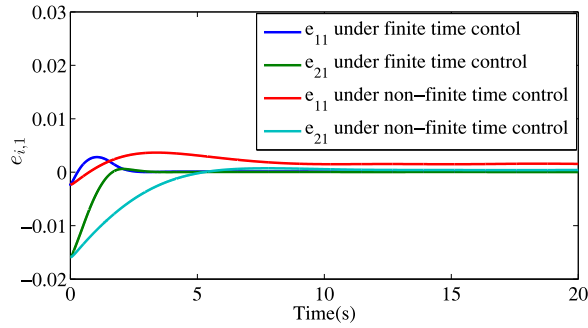


Fig. 7. Position neighborhood errors under the two different controllers.

power feedback control method with adaptive control, not only the inherent uncertainties with unknown and time-varying control gains can be effectively compensated but also the cooperative consensus can be achieved in finite time with sufficient accuracy.

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Appendix

Part I. Preliminaries

Lemma 1 [1]. Suppose there exists a continuously differentiable function $V(x, t) : U_0 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ ($U_0 \subset \mathbb{R}^n$ is an open neighborhood of the origin), a real number $c > 0$ and $0 < \alpha < 1$, such that $V(x, t)$ is positive definite and $\dot{V}(x, t) + cV(x, t)^\alpha \leq 0$ on U ($U \subset U_0$), then $V(x, t)$ is locally in finite-time convergent with a finite settling time satisfying $T_0^* \leq \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$, such that for any given initial state $x(t_0) \in U \setminus \{0\}$, $\lim_{t \rightarrow T_0^*} V(x, t) = 0$ and $V(x, t) = 0$ for $t \geq T_0^*$.

Lemma 2 [17]. Let c, d be positive real numbers and $\gamma(x, y) > 0$ a real-valued function. Then, $|x|^c|y|^d \leq \frac{c\gamma(x, y)|x|^{c+d}}{c+d} + \frac{d\gamma(x, y)^{-c/d}|y|^{c+d}}{c+d}$.

Lemma 3 [17]. If $0 < h = h_1/h_2 \leq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x^h - y^h| \leq 2^{1-h}|x - y|^h$.

Lemma 4 [7]. For $x_i \in \mathbb{R}, i = 1, 2, \dots, N, 0 < h \leq 1$, then $(\sum_{i=1}^N |x_i|)^h \leq \sum_{i=1}^N |x_i|^h \leq N^{1-h}(\sum_{i=1}^N |x_i|)^h$.

Part II. Proof of Property 1–Property 4

A. Proof of Property 1

Proof. Before moving on, we first give an important property on Ξ , which has been proved in our previous work [23]: $\forall X \neq 0_N, X^T \Xi X = 0$ if and only if $X = c1_N$ with c being a nonzero constant, and moreover, $0 < \min_{X \neq c1_N, X \neq 0_N} \frac{X^T \Xi X}{X^T X} \leq \lambda_2(\Xi)$ (we denote by $\lambda_i(\Xi)$ the eigenvalue of Ξ). In the following, we prove that indeed $E^s \neq c1_N$, and then $\min_{E^h \neq 0_N} \frac{(E^h)^T \Xi E^h}{(E^h)^T E^h} > 0$. Let $k_s = \min_{E^h \neq 0_N} \frac{(E^h)^T \Xi E^h}{(E^h)^T E^h}$, then it holds that $(E^h)^T \Xi E^h \geq k_s (E^h)^T E^h$. Note that $p^T E = p^T L(y - \varpi) = 0$, i.e., $p_1 e_{1,1} + \dots + p_N e_{N,1} = 0$, thus for $E \neq 0_N$, it is impossible that $\text{sgn}(e_{i,1}) = 1$ (or $\text{sgn}(e_{i,1}) = -1$) for all $i = 1, 2, \dots, N$ due to the fact that $p_i > 0$ ([18]). Hence, $E \neq c1_N$ with c being a nonzero constant. Note that $\text{sgn}(e_{i,1}) = \text{sgn}(e_{i,1}^h)$ for $i = 1, \dots, N$, it is readily established that $E^h \neq c1_N$. \square

B. Proof of Property 2

Proof. Let $\Delta = \{x : V(x, t) < (\frac{\zeta}{\mu_1 \tilde{c}})^{\frac{1}{\alpha}}, 0 < \mu_1 < 1\}$. According to Theorem 5.2 in [2] for any $x \notin \Delta$ for all $t \in [0, t_x]$, it holds that $V(x, t) \geq (\zeta/\mu_1 \tilde{c})^{\frac{1}{\alpha}}$ i.e., $\zeta \leq \mu_1 \tilde{c} V(x, t)^\alpha$, for all $t \in [0, t_x]$. This fact, together with (6), implies that

$$\dot{V}(x, t) \leq -\tilde{c}V(x, t)^\alpha + \mu_1 \tilde{c}V(x, t)^\alpha = -(1 - \mu_1) \tilde{c}V(x, t)^\alpha \tag{76}$$

for all $t \in [0, t_x]$. Note that $V(x, t) \geq (\zeta/\eta_1 \tilde{c})^\alpha > 0$ for $t \in [0, t_x]$. According to Lemma 1 and (75), we know $t_x < \frac{V(x,0)^{1-\alpha}}{(1-\mu_1)\tilde{c}(1-\alpha)} = T^*$. Thus (8) holds for $\forall t \geq T^*$. \square

C. Proof of Property 3

Proof. The assertion is vacuously true if $\zeta(t) \equiv 0$. In fact, in this case (9) becomes

$$\dot{\vartheta}(t) = -k_\vartheta \vartheta(t)^h, \tag{77}$$

and by direct integration we obtain the solution to (76) as

$$\vartheta(t) = [|\vartheta(0)|^{1-h} - k_\vartheta(1-h)t]^{-\frac{1}{1-h}}, t < \bar{T}, \vartheta(0) \neq 0, = 0, \quad \text{otherwise} \tag{78}$$

with $\bar{T} = \frac{|\vartheta(0)|^{1-h}}{k_\vartheta(1-h)}$, which implies that $\vartheta(t) \geq 0 \forall t \geq 0$ for any given bounded initial condition $\vartheta(0) \geq 0$.

We now consider the case that $\zeta(t)$ is not always equal to 0. If there exists some time constant $t_1 > 0$ such that $\zeta(t_1) = 0$, then by continuity there exists a small neighborhood $[t_1, t_1 + \Delta t_1)$ such that $\vartheta(t) \geq 0$ on $[t_1, t_1 + \Delta t_1)$ ($\Delta t_1 > 0$) from the above analysis; if there exists some time constant $t_2 > 0$ such that $\zeta(t_2) > 0$ and $\vartheta(t_2) > (\frac{q\zeta(t_2)}{k_\vartheta})^{\frac{1}{h}} > 0$, then by continuity there exists a small neighborhood $[t_2, t_2 + \Delta t_2)$ such that $\vartheta(t) > 0$ on $[t_2, t_2 + \Delta t_2)$ ($\Delta t_2 > 0$); if there exists some time constant $t_3 > 0$ such that $\zeta(t_3) > 0$ and $\vartheta(t_3) = (\frac{q\zeta(t_3)}{k_\vartheta})^{\frac{1}{h}} > 0$, then by (9) it holds that $\dot{\vartheta}(t_3) = -k_\vartheta[(\frac{q\zeta(t_3)}{k_\vartheta})^{\frac{1}{h}}]^h + q\zeta(t_3) = 0$, and by continuity there exists a small neighborhood $[t_3, t_3 + \Delta t_3)$ such that $\vartheta(t) = (\frac{q\zeta(t_3)}{k_\vartheta})^{\frac{1}{h}} > 0$ on $[t_3, t_3 + \Delta t_3)$ ($\Delta t_3 > 0$); and if there exists some time constant $t_4 > 0$ such that $\zeta(t_4) > 0$ and $0 < \vartheta(t_4) < (\frac{q\zeta(t_4)}{k_\vartheta})^{\frac{1}{h}}$, it follows from (9) that $\dot{\vartheta}(t_4) > -k_\vartheta[(\frac{q\zeta(t_4)}{k_\vartheta})^{\frac{1}{h}}]^h + q\zeta(t_4) = 0$, and then there exists a small neighborhood $[t_4, t_4 + \Delta t_4)$ such that $\vartheta(t) > \vartheta(t_4) > 0$ on $[t_4, t_4 + \Delta t_4)$ ($\Delta t_4 > 0$) by continuity. By global continuity of the function $\vartheta(t)$, we have $\vartheta(t) \geq 0 \forall t \geq 0$ for any given bounded initial condition $\vartheta(0) \geq 0$ in this case. \square

D. Proof of Property 4

Proof. To proceed, we first prove an inequality that for $x, y \in R$, it holds that

$$x^h(y-x) \leq \frac{1}{1+h}(y^{1+h} - x^{1+h}) \tag{79}$$

with h being defined the same as in Property 3. In fact, let $g(x) = x^h(c-x) - \frac{1}{1+h}(c^{1+h} - x^{1+h})$, then taking the derivation of $g(x)$ with respect to x yields

$$\frac{d}{dx}g(x) = hx^{h-1}(c-x) - x^h + x^h = hx^{h-1}(c-x), \tag{80}$$

which implies that $g(x)$ arrives its maximum value at $x = c$, that is, $\forall x \in R g(x) \leq g(c) = 0$, thus (78) holds.

Upon using (78), it is deduced from Lemma 3 that

$$\begin{aligned} \tilde{\zeta}(\zeta - \tilde{\zeta})^{\frac{3s-1}{1+s}} &\leq \frac{1+s}{4s}[\zeta^{\frac{4s}{1+s}} - (\zeta - \tilde{\zeta})^{\frac{4s}{1+s}}] = \frac{1+s}{4s}[\zeta^{\frac{4s}{1+s}} - |\zeta - \tilde{\zeta}||\zeta - \tilde{\zeta}|^{\frac{3s-1}{1+s}}] \\ &\leq \frac{1+s}{4s}[\zeta^{\frac{4s}{1+s}} - 2^{\frac{2s-2}{1+s}}|\zeta - \tilde{\zeta}||\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}}|] \end{aligned} \tag{81}$$

Note that $s = \frac{4ln-1}{4ln+1}$, straightforward derivation yields that $\frac{3s-1}{1+s} = \frac{2ln-1}{2ln}$, and therefore $|\zeta - \tilde{\zeta}||\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}}| = (\zeta - \tilde{\zeta})(\zeta^{\frac{3s-1}{1+s}} - \tilde{\zeta}^{\frac{3s-1}{1+s}})$. By using Lemma 2, it then holds that

$$\begin{aligned} \tilde{\zeta}(\zeta - \tilde{\zeta})^{\frac{3s-1}{1+s}} &\leq \frac{1+s}{4s}[\zeta^{\frac{4s}{1+s}} - 2^{\frac{2s-2}{1+s}}\zeta^{\frac{4s}{1+s}} + 2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}} + 2^{\frac{2s-2}{1+s}}\zeta\tilde{\zeta}^{\frac{3s-1}{1+s}} + 2^{\frac{2s-2}{1+s}}\tilde{\zeta}\zeta^{\frac{3s-1}{1+s}} - 2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}}] \\ &\leq \frac{1+s}{4s}\left[\zeta^{\frac{4s}{1+s}} - 2^{\frac{2s-2}{1+s}}\zeta^{\frac{4s}{1+s}} + \frac{(1+s)(2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}})}{4s} + \frac{(3s-1)\zeta^{\frac{4s}{1+s}}}{4s} + \frac{(1+s)(2^{-4(\frac{s-1}{1+s})^2}\zeta^{\frac{4s}{1+s}})}{4s} + \frac{(3s-1)(2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}})}{4s} - 2^{\frac{2s-2}{1+s}}\tilde{\zeta}^{\frac{4s}{1+s}}\right] \\ &\leq \frac{1+s}{4s}\left[\left(2^{\frac{8s(s-1)}{(1+s)^2}} - 2^{\frac{2(s-1)}{1+s}}\right)\tilde{\zeta}^{\frac{4s}{1+s}} + \left(1 - 2^{\frac{2s-2}{1+s}} + \frac{3s-1}{4s} + \frac{(1+s)2^{\frac{-16s(s-1)^2}{(1+s)^2}}}{4s}\right)\zeta^{\frac{4s}{1+s}}\right]. \end{aligned} \tag{82}$$

\square

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