A Lambda Calculus for Real Analysis

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30 January 2008

Abstract

Abstract Stone Duality is a revolutionary theory that works *directly* with computable continuous functions, without using set theory, infinitary lattice theory or a prior theory of discrete computation. Every expression in the calculus denotes both a continuous function and a program, but the reasoning looks remarkably like a sanitised form of that in classical topology.

This paper is an introduction to ASD for the general mathematician, and applies it to elementary real analysis. It culminates in the Intermediate Value Theorem, *i.e.* the solution of equations $fx = 0$ for continuous $f : \mathbb{R} \to \mathbb{R}$. As is well known from both numerical and constructive considerations, the equation cannot be solved if f "hovers" near 0, whilst tangential solutions will never be found.

In ASD, both of these failures and the general method of finding solutions of the equation when they exist are explained by the new concept of "overtness". The zeroes are captured, not as a set, but by higher-type operators $\Box$ and $\Diamond$ that remain (Scott) continuous across singularities of a parametric equation.

Expressing topology in terms of continuous functions rather than sets of points leads to a very closely dual treatment of open and closed subspaces, without the double negations of intuitionistic approaches. In this, the dual of compactness is overtness, and whereas meets and joins in locale theory are asymmetrically finite and infinite, they have overt and compact indices in ASD.

Overtness replaces metrical properties such as total boundedness, and cardinality conditions such as having a countable dense subset. It is also related to locatedness in constructive analysis and recursive enumerability in recursion theory.

As a further application of connectedness, we also show that every open set of the line is uniquely expressible as a countable union of intervals, in a suitable constructive sense, which is not the case in Bishop's theory.

Contents

This paper was originally presented at "Computability and Complexity in Analysis" in Kyoto in August 2005, since when I have had the pleasure of discussing this work with numerous people.

The characterisation of open subsets of $\mathbb{R}$ as countable unions of disjoint open intervals, and some counterexamples, were added in April 2007; these are in (the present) Sections 13, 15 and 16. In this, the January 2008 version, Sections 3, 5, 6, 7, 9 and 10 have been completely (re)written to provide a brief "need to know" account of various foundational disciplines that underlie this work.

Introduction

This paper introduces a new calculus for constructive general topology, and in particular for analysis on the (Dedekind) real line. When using this calculus, there is no need to *prove* that functions are continuous, because from the start the types are *intrinsically* topological spaces, not sets with *imposed* structure, and all terms are automatically continuous with respect to the topology of the types. They also have a computational interpretation, at least in principle. Some basic facts about analysis, such as the Heine–Borel theorem (compactness of $[0, 1]$) are built in to the language. It enjoys a very strong open–closed duality, in contrast to the usual asymmetry between finite intersections and infinite unions, whilst avoiding the double negations that are a feature of intuitionism. With regard to the agenda of real analysis, our objectives are the *intermediate value theorem* and the characterisation of open subspaces as countable disjoint unions of intervals.

These goals are motivated respectively by applications and the need for a test of whether our axioms capture *the real* real line. They are manifestations of *connectedness*, which in turn depends on finding the *maximum* value in a non-empty compact subspace. These topics relate to questions that have been studied at length in the literature on constructive analysis.

As well as our new language, in this paper we also introduce a new topological *concept*, namely *overt subspaces*. This property is completely invisible in classical topology, because there *every* subspace has it. In Bishop's constructive analysis, its role is played by *locatedness* and *total boundedness*, but these are defined metrically, using lots of ϵs and δs, which we almost entirely avoid.

By analogy with the view that the important thing about a compact subspace is *which open sets cover it*, rather than *what points it contains*, we shall define overtness in terms of another property of open subspaces, namely whether they *touch* the subspace (intersect it non-trivially). Compact and overt subspaces are therefore determined by logical operators $\Box$ and $\Diamond$ that satisfy the rules of *modal logic*. Our motivating example of this is the collection of solutions that "interval halving" (and other) computational methods actually find for real-valued equations.

In order to give some impression of what overtness means, and why the usual language is inadequate to describe it, Section 2 provides a translation into the usual set-theoretic language of the way in which we shall deal with the intermediate value theorem in our new language later in this paper. Before that, Section 1 reviews the constructive critique of this result.

The modal operators $\Box$ and $\Diamond$ are continuous functions, not on the space $\mathbb{R}$, but on its topology (lattice of open subspaces). This means that we have to equip topologies with topologies, and the one that we choose is due to Dana Scott; it is related to local compactness. It appears in the background of analysis in the guise of semicontinuity and the ascending and descending real numbers, which will themselves play an important role in this paper. However, there seems to be no appropriate introduction that is actually aimed at analysts, so Section 3 outlines the ideas from topological lattice theory and theoretical computer science that lie behind our calculus.

Even in the classical language, the $\Box$ and $\Diamond$ operators give a new perspective on the *singular* case, *i.e.* the way in which the zeroes of a function (such as a polynomial) vary, merge and vanish as its parameters change. Whilst the *set* of zeroes changes discontinuously, $\Box$ and $\Diamond$ remain Scott-continuous across the singularity; the only thing that breaks is one of the equations that relates them. The abstract formulation in ASD makes the construction of $\Box$ and $\Diamond$ from the function an entirely symbolic one, interprets these operators as subspaces and offers a general (at least conceptual) structure in which to compute the zeroes.

Sections 1–3 are therefore not representative of either the paper or the new calculus — if you just want a "sample", please look at Sections 10, 13 and 14 instead.

In Section 4 we start to introduce the new calculus, in a relatively informal way, developing the intuition that $x < y$ and $x \neq y$ are properties of real numbers that we can observe computationally, whereas $x \geq y$ and $x = y$ are not. Section 5 sets out the restricted form of predicate logic that we shall use, including the existential quantifier and an important principle that underlies our dual treatment of open and closed subspaces. We use this fragment of the calculus to discuss Dedekind and Cauchy completeness in Section 6, adopting the former as an axiom and proving the latter from it.

In topology there is a distinction between open and general subspaces. Section 7 describes the λ-calculus that we use to define the former and the quasi-set-theoretic notation for the latter. Recall that $\lambda x.\, px$ is a notation for functions that formalises "$x \mapsto p(x)$", and so allows them to be treated as first class citizens in the mathematical world.

This formalism enables us to define compact subspaces in Section 8 *abstractly* in terms of their covers, developing the familiar results about closed or compact subspaces and direct images. However, the equivalence with closed and bounded subspaces depends on Scott- or semicontinuity properties that we formulate idiomatically as a "uniformity" principle in Section 9.

The calculus begins to look like real analysis in Section 10, where we show that every function $\mathbb{R} \to \mathbb{R}$ is continuous in the ϵ–δ sense, indeed uniformly so when the domain is compact. Although we do not study differential and integral calculus in this paper, we also indicate in that section how these can be expressed in our language.

Section 11 provides the formal account of overtness, closely following that of compactness in Section 8.

We begin to see the benefit of this lattice-dual *topological* axiomatisation of compact overt subspaces in Section 12, as there is an axiom-by-axiom translation into the Dedekind cut for its maximum. The idea is similar to the *constructive least upper bound principle.*

In Section 13 this duality also leads to two definitions of connectedness, each yielding an approximate version of the intermediate value theorem. The overt one agrees with the definition that is already known in constructive analysis. The *classical* proof of connectedness of the interval and the line is valid in our calculus.

Sections 14 and 15 accomplish our main goals: the intermediate value theorem and the characterisation of open subspaces as unions of intervals. Section 16 tests the boundaries of our ideas with some counterexamples, emphasising the role of overtness.

This calculus is called Abstract Stone Duality because it was inspired by Marshall Stone's results on the duality of algebra and topology, and his maxim that one should "always topologize". Whereas the algebra is expressed in locale theory by the notion of frame (a lattice with infinite joins, over which finite meets distribute), in which the carrier is still a set, ASD "topologises the topology" by saying that these carriers are themselves spaces of the kind being defined. This can be formulated using the notion of *monad* in category theory, although numerous lengthy papers have been needed to implement this idea.

The present paper, however, presents a slightly simplified version of ASD that relies solely on the basic intuitions and knowledge of a general mathematician. The category theory has gone, and we give a "need to know" tutorial for each of the other foundational techniques that we shall use.

The paper [I] is parallel to this one in that they provide introductions to the same material for different audiences. That one includes a more formal treatment that it uses to *construct* $\mathbb{R}$ using two-sided Dedekind cuts, whereas here we introduce the line axiomatically. It also discusses the constructive, conceptual and computational motivations of the ASD programme, and concludes with a discussion of the failure of compactness of the closed interval other purely recursive approaches to topology, and of how ASD overcomes this problem.

1 The constructive intermediate value theorem

The application to which we shall put our new λ-calculus for general topology, when we have defined it, is the solution of equations involving *continuous* functions $\mathbb{R} \to \mathbb{R}$ (differentiation and integration will be considered in future work). The result that guarantees the existence of such solutions is known as the *intermediate value theorem*, and is proved in very wide generality in classical analysis.

In this section we review the constructive critique of this theorem. It imposes an additional condition on the continuous function, where the classical result has greater generality. On the other hand, the constructive argument is based on an interval-halving method that gives just one extra bit of the solution for each iteration, whereas the well known Newton algorithm *doubles* the precision (number of bits) each time.

Whilst it is widely appreciated that constructivism emphasises similar issues to those that arise in computational practice [Dav05], classical analysts sometimes feel that constructivists want to rob them of their theorems, without replacing them with algorithms that are any better than those that numerical analysts already know.

When we look more closely into these complaints, we find that the two sides are talking at cross purposes. Some mathematicians consider the generality of classical theorems to be more important than their applications. Anyone who is genuinely interested in *solving* an equation, *i.e.* in finding a number, will probably already have some algorithm (such as Newton's) in mind, and will be willing to accept the pre-conditions that this imposes. We find, on examination, that these *imply* the extra property that constructivists require, which is in fact very mild. Indeed, it is satisfied by any example in which you might reasonably expect to be able to compute a zero.

Turning to the classical Newton algorithm, it is not always as clever as it claims: on the large scale it can run *away* from a nearby zero, and sometimes behaves chaotically. In fact, it only exhibits its rapid convergence after we have first *separated* the zeroes, which we must do by some discrete method such as interval-halving. Besides, this algorithm uses *differentiability*, whereas we only intend to study *continuity* in this paper.

Constructive mathematics is about proving theorems just as much as classical analysis is. What we gain from looking at the intermediate value theorem constructively is a more subtle understanding of the space of solutions in the singular and non-singular situations. In this paper, this will take the form of a new topological property of the space of "stable" zeroes, which are essentially those that can be found computationally. Nevertheless, the space of *all* zeroes (stable or otherwise) still plays an equally important role: we shall study the two together, in a way that is an example of the open–closed duality in topology.

Finally, whilst constructive arguments are of their nature like algorithms, they may be intended as correctness or "existence proofs" just as much as their classical analogues, subsequent *implementation* being based on further logical development. In particular, the calculus that we shall introduce in this paper is a mathematical one with a computational interpretation "in principle". More concrete ideas for the implementation of this language will be discussed in later work, which will use the concepts presented here but quite different algorithms. A particularly interesting one is Ramon Moore's *interval Newton* method [Moo66, Chapter 7], which behaves at *small* scales like its traditional form, but at larger ones like interval halving, so finding the initial approximation to the zero in a systematic way.

Let us begin, therefore, with the form in which the intermediate value theorem is taught to first year mathematics undergraduates:

Theorem 1.1 Let $f : \mathbb{I} \equiv [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) \le 0 \le f(1)$. Then there is some $x \in \mathbb{I}$ for which $f(x) = 0$.

Proof There are two well known proofs of this.

(a) Put $x \equiv \mathsf{sup}\,\{y \in \mathbb{I} \mid fy \le 0\}$ and suppose that $0 < \epsilon < f(x)$, so $x > 0$. Then, by ϵ–δ-continuity, there is some interval $(x \pm \delta) \equiv (x - \delta, x + \delta)$ on which $0 < f(y)$, so x was not the *least* upper bound of its defining set, contrary to its construction. A similar argument deals with the negative case, which leaves $f(x) = 0$.

(b) The other proof uses "interval halving". Let $c_0 \equiv 0$ and $v_0 \equiv 1$. By recursion, consider

$$x_n \equiv \tfrac{1}{2}(c_n + v_n), \quad \text{and put} \quad c_{n+1},\ v_{n+1} \ \equiv \ \begin{cases} c_n,\ x_n & \text{if } f(x_n) > 0 \\ x_n,\ v_n & \text{if } f(x_n) \le 0, \end{cases}$$

so by induction $f(c_n) \le 0 \le f(v_n)$. But c_n and v_n are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0, so they converge to a common value x. Using ϵ–δ-continuity in the last step again, $f(x) = 0$. $\square$

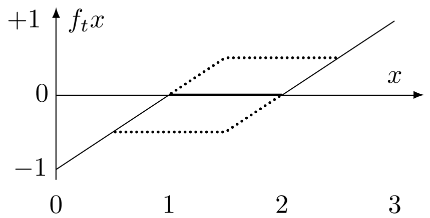
The first proof has traditionally been preferred by pure mathematicians, since it is a "closed formula". This phrase appears to mean that it only employs symbols that Euler might have used — a restriction that is similar to, but certainly not as clearly defined as, that to ruler and compasses in classical geometry. Whatever a "closed formula" is, the second proof is (apparently) not one, as it involves recursion. That is, it's a *program* that actually tells us *how* to compute the zero — which sup certainly doesn't.

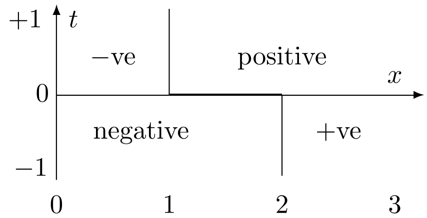
However, *neither* of these methods is "constructive" (or, as I prefer to say, "computational"), as anyone who has tried to solve equations numerically is well aware. Dismissing the first, let's see what's wrong with the second.

Example 1.2 Consider this parametric function, which ***hovers*** around 0:

$$\text{for } -1 \leq t \leq +1 \quad \text{and} \quad 0 \leq x \leq 3, \quad \text{let } f_t(x) \;\equiv\; \mathsf{min}\big(x-1,\, \mathsf{max}\,(t, x-2)\big).$$

The graph of $f_t(x)$ against x for $t \approx 0$ is shown on the left. The diagram on the right shows how $f_t(x)$ depends qualitatively on t and x, where the two regions are open, and the thick lines denote $f_t x = 0$. In particular, $f(1) = 0$ iff $t \geq 0$, $f(2) = 0$ iff $t \leq 0$ and $f(\frac{3}{2}) = 0$ iff $t = 0$.





Neither the classical theorem nor any numerical algorithm has much to say about *analysis* in this example. However, if any of them does yield a zero of f_t, *as a side-effect it will decide a question of logic*, namely how t stands in relation to 0.

Remark 1.3 As L.E.J. Brouwer observed in his ground-breaking work in 1907 [Bro75, Hey56], for an arbitrary numerical expression t, *we may not know* whether $t < 0$, $t = 0$ or $t > 0$. There are many different ways in which such indeterminate values may arise, depending on whether your reasons for using analysis come from experimental science, engineering, numerical computation or logic. So t may be

(a) a parameter that we intend to vary;

(b) an experimental measurement that we can make only to a certain precision;

(c) the result of a numerical computation of which we have (so far) only found so many digits;

(d) a constant defined in terms of some mathematical question that has (so far) resisted solution, such as the Riemann Hypothesis or the Goldbach Conjecture (Brouwer used patterns in the digits of $\pi \equiv 3.14159\cdots$ for this);

(e) a constant defined in terms of some logical question that is provably unanswerable, such as $t \equiv \sum_{n=0}^{\infty} 2^{-n} \cdot g_n$, where g_n is the primitive recursive sequence

$$g_n \;\equiv\; \begin{cases} 1 & \text{if } n \text{ encodes a proof that } (\vdash 0 = 1) \\ 0 & \text{otherwise,} \end{cases}$$

so $t = 0$ iff the calculus is consistent, which, according to Kurt Gödel [Göd31], it is unable to prove for itself.

As the Example is a monster from logic and not analysis, we bar it. It is also sometimes more convenient to suppose that the function is defined on the whole line.

Definition 1.4 We say that $f : \mathbb{R} \to \mathbb{R}$ ***doesn't hover*** if,

$$\text{for any } d < u, \qquad \exists x.\, (d < x < u) \;\wedge\; (fx \neq 0),$$

so $\{x \mid fx \neq 0\}$ is dense. A similar property, that f is "locally non-constant", is used in [BR87].

Example 1.5 Any nonzero polynomial of degree n doesn't hover, x being one of any given $n+1$ distinct points in the interval (d, u). $\square$

Remark 1.6 For Newton's algorithm to be applicable to solving the equation $f(x) = 0$, we must assume the derivative f' to exist, and preferably be continuous. Also, since we intend to divide by $f'(x)$, this should be nonzero, although it is enough that f' doesn't hover. So let $d < x' < u$ with $f'(x') \neq 0$. Then, by manipulating the inequalities in the ϵ–δ definition of $f'(x')$, *cf.* Definition 10.11, there must be some $d < x < u$ with $f(x) \neq 0$. This argument may be adapted to exploit any higher derivative that is nonzero instead. $\square$

So this condition is very mild when taken in the context of its practical applications. Using it, here is the usual constructive intermediate value theorem.

Theorem 1.7 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, has $f(0) < 0 < f(1)$ and doesn't hover. Then it has a zero, both constructively and numerically [TvD88, Theorem 6.1.5].

Proof In the interval halving algorithm (Theorem 1.1(b)), we may have $f(x_n) = 0$. This can be avoided by relaxing the choice of x_n to the x provided by Definition 1.4, to which we supply, say, $d_n \equiv \frac{1}{3}(2c_n + v_n)$ and $u_n \equiv \frac{1}{3}(c_n + 2v_n)$. Then we only have to test whether $f(x_n) < 0$ or > 0, which is allowed, both constructively and numerically. $\square$

This proof is better computationally than the previous version, in that it doesn't involve a test for equality. But it introduces a new problem: the meaning of $\exists$, to which we shall return in Remark 6.10. Here we characterise the solutions that this algorithm actually finds.

Definition 1.8 We call $x \in \mathbb{R}$ a ***stable zero*** of f if, for any $c < x < v$,

$$\exists du.\, (c < d < x < u < v) \;\wedge\; (fd < 0 < fu \;\vee\; fd > 0 > fu),$$

leaving you to check that a stable zero of a continuous function really is a zero.

On the other hand, even in such a nice situation as solving a polynomial equation, not all zeroes need be stable — in particular, double ones (where the graph of f touches the axis without crossing it) are unstable. As Example 1.2 shows, if f hovers, there need not be any stable zeroes.

Example 1.9 Consider $f_t x \;\equiv\; tx^2 - tx + 1$, so $f_t 0 = f_t 1 = 1$. There are two stable zeroes when $t > 4$, a single unstable one at $\frac{1}{2}$ when $t \equiv 4$, but no zeroes at all when $t < 4$. $\square$

Remark 1.10 The reason for the name is that, classically, x is a stable zero iff every nearby function (in the sup or ℓ_∞ norm) has a nearby zero:

$$\forall \delta > 0.\, \exists \epsilon > 0.\, \forall g.\, \big(\|f - g\| \leq \epsilon \;\Rightarrow\; \exists y.\, gy = 0 \wedge |y - x| < \delta\big).$$

Note that, because of the $\forall g$, this is *not* a well formed predicate in the calculus that we shall introduce.

This discussion may perhaps suggest that unstable zeroes are a bad thing, but we stress that both stable and arbitrary zeroes will play equally important roles in this paper. Also, it is possible to find unstable zeroes computationally, so long as they are *isolated* and *we know they're there*, but this is not part of our story.

Notation 1.11 Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ that doesn't hover, let $S \subset Z \subset \mathbb{R}$ be the subspaces consisting of, respectively, stable and all zeroes of f.

These coincide for ***non-singular*** values of the parameters (these being, for example, the coefficients of a polynomial), but in certain ***singular*** situations, S is smaller than Z.

2 Stable zeroes and straddling intervals

Now we shall look at the *topological* properties of the subspaces S and Z. We know, of course, that Z is *closed*, and therefore *compact* if we choose to bound the domain of the function $(f : \mathbb{I} \to \mathbb{R})$.

S is also closed when f doesn't hover. But the interesting thing is that it is *overt*. As we shall see, it is not possible to define overtness as a classical set of points, but there are several other ideas in constructive, computable and even classical analysis that are *similar* to it and suggest intuitions.

The way in which we shall define overtness in ASD is to use logic. In this section we show how the notions of zero and stable zero for a function give rise to "modal" predicates $\Box$ and $\Diamond$ that may or may not be satisfied by open subspaces. Since such subspaces are themselves predicates on points, the result of this discussion will be to represent compact and overt subspaces as *predicates on predicates*.

Proposition 2.1 An open subspace $U \subset \mathbb{R}$ ***touches*** S, *i.e.* it contains a stable zero, $x \in U \cap S$, iff U contains a ***straddling interval***,

$$[d, u] \subset U \quad \text{with} \quad fd < 0 < fu \quad \text{or} \quad fd > 0 > fu.$$

If a point is a stable zero then every open neighbourhood contains a straddling interval, and conversely if f doesn't hover.

Proof [$\Rightarrow$] Since the point x itself is in the interior of U, some interval $c < x < v$ is also contained in U. By Definition 1.8, this interval contains one that straddles. [$\Leftarrow$] The straddling interval is an intermediate value problem in miniature, for which Theorem 1.7 finds a stable zero. Finally, suppose that every interval $c < x < v$ around x contains a straddling interval; if the latter contains x then we're done. Otherwise, let it be $[d, e]$ with $c < d < e < x$, fd and fe having opposite signs; but f doesn't hover in the interval (x, v), so $fu \neq 0$ for some $x < u < v$; so either $[d, u]$ or $[e, u]$ is a straddling interval around x. $\Box$

Remark 2.2 If an interval $[d, u]$ straddles with respect to f then it also does so with respect to any nearby function g, *i.e.* with $\|f - g\| < \epsilon$, where $\epsilon \equiv \mathsf{min}(|fd|, |fu|)$, *cf.* Remark 1.10. Since the definition only *refers* to the endpoints, it is also invariant with respect to (long range) homotopies that fix the values there, in contrast to Example 1.9. $\Box$

Notation 2.3 We write $\Diamond U$ if the open subspace U contains a straddling interval. The hypothesis of the intermediate value theorem is that $\Diamond U_0$ is true for any open interval $U_0 \supset \mathbb{I}$, whilst $\Diamond \emptyset$ is obviously false.

Theorem 2.4 $\Diamond \left(\bigcup_{j \in J} U_j\right) \iff \exists j \in J.\ \Diamond U_j$. In particular, $\Diamond(U_1 \cup U_2) \Rightarrow \Diamond U_1 \vee \Diamond U_2$.

Proof Suppose that $\mathbb{I} \equiv [d, u] \subset \bigcup_{j \in J} U_j$ with $fd < 0 < fu$, and consider the open subspaces

$$V^{\pm} \ \equiv\ \{x : \mathbb{R} \mid \exists j{:}J.\ \exists y{:}\mathbb{R}.\ (x < y) \wedge (fy \gtrless 0) \wedge [x, y] \subset U_j\}.$$

For each $x \in \mathbb{I}$, there are $j \in J$ and $e, t \in \mathbb{I}$ such that $x \in (e, t) \subset [e, t] \subset U_j$, but since f doesn't hover in (x, t) there's some $x < y < t$ with $fy \neq 0$ and $[x, y] \subset U_j$. Then $x \in V^{\pm}$, according to the sign of fy. Hence $\mathbb{I} \subset V^+ \cup V^-$, whilst $d \in V^-$ and $u \in V^+$ by hypothesis.

Now, since $\mathbb{I}$ is *connected*, $V^+ \cap V^-$ is non-empty, so it contains some open interval, in which f doesn't hover, so $fx \neq 0$ for some $x \in V^+ \cap V^-$. If $fx < 0$ then (since $x \in V^+$) there is a straddling interval $[x, y] \subset U_j$ with $fy > 0$; similarly if $fx > 0$ we have $x \in V^-$ and $fy < 0$. $\Box$

Remark 2.5 This property, that $\Diamond$ preserves joins, is the one that we shall use from now on in this paper: it defines overtness.

We shall only consider the non-hovering condition again when we prove the same result in our new calculus in Section 14. Although it is the usual one in constructive analysis, this condition does not seem to be very appropriate, as its obvious generalisation is insufficient to prove the result in $\mathbb{R}^{\mathbf{2}}$. This is because there is a computable continuous endofunction of the square, $\mathbb{I}^2$, *none of whose classically defined fixed points is computable* [Bai85], whereas the classical mathematician would claim at least that *either* 1 or 2 is a zero in Example 1.2.

As we observed before, if we wanted to apply Newton's method, we would need the derivative of the function to be continuous and non-zero near the required solution. In that case, we have an ***open map***, *i.e.* one for which the direct image of any open subspace is open. The analogue of a straddling interval in $\mathbb{R}^{\mathbf{n}}$ is an ***enclosing sphere***, *i.e.* one whose image contains 0 in its interior, and it is not difficult to prove the join-preserving property from openness.

Notice also that our property is a (constructive) logical analogue of the arithmetical ***degree*** that L.E.J. Brouwer introduced in his *alter ego* as a (non-constructive) topologist.

We might imagine an overt subspace or a Brouwer degree like radioactivity that cannot be seen *itself*, but whose presence in any *region*, however small, can be detected. Such properties have a computational interpretation:

Theorem 2.6 Let $\Diamond$ be a property of open subspaces of $\mathbb{R}$ that takes unions to disjunctions and satisfies $\Diamond U_0$ for some open interval U_0. Then $\Diamond$ has an ***accumulation point*** $x \in U_0$, *i.e.* one of which every open neighbourhood $x \in U \subset \mathbb{R}$ satisfies $\Diamond U$. In the example of the intermediate value theorem for non-hovering functions, any such x is a stable zero.

Proof Interval halving again: let $c_0 \equiv 0$, $v_0 \equiv 1$ and, by recursion, $d_n \equiv \frac{1}{3}(2c_n + v_n)$ and $u_n \equiv \frac{1}{3}(c_n + 2v_n)$, so

$$\top \;\Leftrightarrow\; \Diamond(c_n, v_n) \;\equiv\; \Diamond\big((c_n, u_n) \cup (d_n, v_n)\big) \;\Leftrightarrow\; \Diamond(c_n, u_n) \vee \Diamond(d_n, v_n);$$

then at least one of the disjuncts is true, so let (c_{n+1}, v_{n+1}) be either (c_n, u_n) or (d_n, v_n).

Hence c_n and v_n converge to some limit x, respectively from below and above. If $x \in U$ then $x \in (c_n, v_n) \subset (x \pm \epsilon) \subset U$ for some $\epsilon > 0$ and n, but $\Diamond(c_n, v_n)$ is true by construction, so $\Diamond U$ also holds, since $\Diamond$ takes $\subset$ to $\Rightarrow$.

So the operator $\Diamond$ appears to capture interval-division algorithms, albeit as parallel non-deterministic processes. Let's see what classical point-set topology has to say about it.

Exercise 2.7 Classically, if we define $\Diamond U$ as $U \cap S \neq \emptyset$ or $\exists x \in S.\, (x \in U)$, for *any subset* $S \subset \mathbb{R}$ *whatever*, then the operator $\Diamond$ satisfies the property in Theorem 2.4. $\square$

Examples 2.8

(a) The existential quantifier to which we drew attention following the proof of Theorem 1.7 is defined by $\Diamond U \;\equiv\; \exists x \in S.\, (x \in U)$, where S is the open subspace $\{x \mid fx \neq 0\}$.

(b) The accumulation points (in the usual sense) of any sequence or net S are those of $\Diamond$ in the sense of Theorem 2.6.

Apparently, $\Diamond$ is merely a roundabout way of defining a closed subspace, or the closure of an arbitrary subspace:

Proposition 2.9 Let $\Diamond$ be an operator for which $\Diamond \bigcup_{i \in I} U_i$ iff $\exists i.\, \Diamond U_i$, and define

$$S \;\equiv\; \{x \in \mathbb{R} \mid \text{for all open } U \subset \mathbb{R}, \quad x \in U \Rightarrow \Diamond U\}.$$

Then
$$W \;\equiv\; \mathbb{R} \setminus S \;=\; \bigcup \{U \text{ open} \mid \neg \Diamond U\}$$

is open (making S closed) and has $\neg \Diamond W$ by preservation of unions. Since $\Diamond$ takes $\subset$ to $\Rightarrow$, $\Diamond U$ holds iff $U \not\subset W$, *i.e.* there some point $x \in U \cap S$. If $\Diamond$ had been derived from some S' as in Exercise 2.7 then $S = \overline{S'}$, its closure, since $\Diamond U \iff (U \cap S' = \emptyset)$. $\square$

Remark 2.10 We learn from this that

(a) since $\Diamond$-like properties are defined, like compactness (which we are about to consider), in terms of unions of open subspaces, they deserve to be called *general topology*, and we shall see that the analogy goes much deeper than this;

(b) the proof of Theorem 2.4, that the subspace of stable zeroes has such a $\Diamond$ in a *useful way*, uses a concept from *geometric topology* (connectedness) in the case of $\mathbb{R}^{\mathbf{1}}$;

(c) the operator $\Diamond$ is essentially the *bounded existential quantifier*, $\Diamond U \equiv \exists x \in S.\,(x \in U)$;

(d) there are very general *algorithms* that use such operators (abstracted from the original question) to solve many kinds of problem in a uniform way; so

(e) $\Diamond$-like properties stand exactly at the gateway between mathematical and computational topology; but

(f) classical point-set topology is too clumsy to take advantage of this.

In fact, the problem lies more with the sets of points than with classical logic.

Whereas *stable* zeroes are characterised by an operator $\Diamond$, there is another operator $\Box$ that describes the subspace $Z \subset \mathbb{I}$ of *all* zeroes. ($\Box$ is called "necessarily" and $\Diamond$ is called "possibly".)

Notation 2.11 Let Z be any compact subspace. For any open subspace U, we write $\Box U$ if U ***contains*** or ***covers*** Z (where $\Diamond$ was about touching). If Z is the complement of an open subspace $W \subset X$ of a compact Hausdorff space then

$$\Box U \quad \text{iff} \quad (U \cup W) = X.$$

The "finite open sub-cover" definition of compactness says exactly that $\Box \bigcup_{i \in I} U_i$ iff $\Box \bigcup_{i \in F} U_i$ for some *finite* $F \subset I$. This is the same as the defining property of $\Diamond$, except that in that case F consisted of a single index $\{i\} \subset I$.

We shall consider this *common infinitary* property of $\Box$ and $\Diamond$ in the next section. Here we look at their *contrasting finitary* properties:

Proposition 2.12 Let $W \subset X$ be an open subspace of a Hausdorff space X, and let $\Box$ and $\Diamond$ be operators defined as above, *i.e.*

$$\Box U \equiv (U \cup W = X) \quad \text{and} \quad \Diamond V \equiv (V \not\subset W)$$

for any open subspaces $U, V \subset X$. Then

(a) the operator $\Box$ preserves *finite intersections*,

$$\Box X \text{ is true} \quad \text{and} \quad \Box U \land \Box V \;\Rightarrow\; \Box(U \cap V),$$

(b) whereas $\Diamond$ preserve *finite unions*,

$$\Diamond \emptyset \text{ is false} \quad \text{and} \quad \Diamond(U \cup V) \;\Rightarrow\; \Diamond U \lor \Diamond V.$$

(c) The corresponding closed subspace $X \setminus W$ is nonempty iff $\Box \emptyset$ is false iff $\Diamond X$ is true,

(d) and it is a singleton iff $\Box$ preserves unions, iff $\Diamond$ preserves intersections.

(e) Both operators are Scott continuous (next section). $\Box$

Now recall the situation in which $\Box$ was defined in terms of the compact subspace Z of all zeroes, and $\Diamond$ using the overt subspace S of stable zeroes of a non-hovering continuous function $\mathbb{R} \to \mathbb{R}$. In the non-singular situation these coincide, but for singular cases of the parameters, S is properly contained in Z.

Proposition 2.13 If $\Box$ and $\Diamond$ arise from subspaces $S \subset Z$ of a Hausdorff space X, with $Z \equiv (X \setminus W)$ compact, then they satisfy the ***modal laws***: for all open $U, V \subset X$,

$$\Box U \land \Diamond V \;\Rightarrow\; \Diamond(U \cap V), \qquad \Box U \iff (U \cup W = X) \quad \text{and} \quad \Diamond V \;\Rightarrow\; (V \not\subset W),$$

whilst

$$\Box(U \cup V) \;\Rightarrow\; \Box U \lor \Diamond V \quad \text{and} \quad \Diamond V \;\Leftarrow\; (V \not\subset W),$$

hold iff S is dense in Z. $\Box$

Example 2.14 If $fx \equiv (x-1)^2(x+2) \equiv x^3 - 3x + 2$ then $S = \{-2\}$ and $Z = \{1, -2\}$. The last two laws fail for the intervals $U \equiv (-3, -1)$ and $V \equiv (0, 2)$. $\square$

Remark 2.15 Therefore, whilst the *subsets* S and Z agree in the non-singular situation, they provide a rather unsatisfactory description of the way in which the zeroes of a function (or even of a polynomial) depend on its parameters, because they change *abruptly* at singularities. Notice also that they do so on opposite sides of the singularity.

Whatever description or algorithm we use to solve equations, *something* has to break at singularities. Nevertheless, our operators $\Diamond$ and $\Box$ are defined from the function in a *uniform* way throughout the parameter space. The only things that go wrong are (some of) the modal *laws* that relate them.

Although we shall not discuss computation explicitly in this paper, Theorem 2.6 and Remark 6.5 indicate what the computational meaning of the calculus is intended to be. They provide a general method of finding stable zeroes, *even in the singular case*, but this is necessarily ***non-deterministic***.

We intend to introduce an abstract calculus in which *all operations* are regarded as *continuous* functions. Since $\Diamond$ and $\Box$ are applied to open subspaces of $\mathbb{R}$, and not to its points, we first have to explain in a concrete way how the topology (lattice of open subspaces) of a space carries its own topology.

3 The Scott topology

The topology that we impose on the topology of a space exploits the fact that $\Box$ and $\Diamond$ preserve directed joins. This is now well known in theoretical computer science and topological lattice theory, so, if you are already familiar with either of these subjects, you may safely omit this section, as it just collects the basic facts of which you should be aware in order to follow the rest of the paper. Indeed, it serves as *background* and not an *introduction*, as our calculus will *abstract* from these ideas, rather than *assume* them.

Although the Scott topology does arise in real analysis in the guise of *semicontinuity*, as we shall see, it is not as well known in more traditional mathematical disciplines as it deserves. This is probably because it is *not Hausdorff*. Whilst there is a compact Hausdorff topology (the *Lawson* topology) that one can put on lattices of open sets, it does not have the properties that we require.

The canonical textbook about these topologies and the *continuous lattices* on which they are particularly well behaved is [GHK+80]; its six authors represent the different areas in which these ideas arose. There are numerous accounts of the Scott topology in *domain theory*, but, unfortunately, hardly anything that concentrates on analysis.

Definition 3.1 Let $\mathcal{L}$ be a complete lattice. A subset $\mathcal{U} \subset \mathcal{L}$ is said to be ***Scott open*** if any subset $\mathcal{S} \subset \mathcal{L}$ for which $\bigvee \mathcal{S} \in \mathcal{U}$ already has some *finite* $\mathcal{F} \subset \mathcal{S}$ with $\bigvee \mathcal{F} \in \mathcal{U}$. The Scott open subsets $\mathcal{U} \subset \mathcal{L}$ form a topology. That is, $\emptyset, \mathcal{L} \subset \mathcal{L}$ are Scott-open, if $\mathcal{U}, \mathcal{V} \subset \mathcal{L}$ are Scott-open then so is $\mathcal{U} \cap \mathcal{V} \subset \mathcal{L}$, and any union of Scott-open subsets is Scott-open.

Proposition 3.2 Let $\mathcal{L}$ be the lattice of open subspaces of a locally compact space X. Then
(a) a subset $K \subset X$ is compact iff the family $\mathcal{U}_K \equiv \{U \in \mathcal{L} \mid K \subset U\}$ of open neighbourhoods of K is a Scott open subset of $\mathcal{L}$; and
(b) the families $\{\mathcal{U}_K \subset \mathcal{L} \mid K \subset X \text{ compact}\}$ provide a basis for the Scott topology on $\mathcal{L}$.

Proof The first part just re-states the usual "finite open sub-cover" definition of compactness. The second depends on the notion of local compactness (Definition 3.17). $\square$

If you are familiar with the ***compact–open topology*** on the set of continuous functions $X \to Y$, which was introduced by Ralph Fox in 1945 [Fox45], you will recognise this result as a special case of it, Y being the Sierpiński space (see below). Our case is much simpler than the general one, but Dana Scott identified it the crucial one in the study of topologies on function spaces [Sco72]. It had already become clear by then that the *neighbourhoods* of a compact subspace (which our $\square$ and $\mathcal{U}_K$ capture) are more important than its *points* [Wil70].

There are two other examples of the Scott topology that are useful in analysis and will play a major role in this paper. (In fact, they can both be seen as special cases of the topology on a topology on a space.)

Definition 3.3 The space $\underline{\mathbb{R}}$ of ***ascending reals*** consists,
(a) classically, of $\mathbb{R}$ together with $\pm\infty$, ordered arithmetically; or,
(b) constructively, of the *rounded lower* subsets $D \subset \mathbb{Q}$, *i.e.* for those which

$$d \in D \iff \exists e{:}\mathbb{Q}.\ d < e \in D,$$

ordered by inclusion,
endowed with the Scott topology defined by this order. The ***descending reals*** $\overline{\mathbb{R}}$ are defined in a similar way, but using the reverse arithmetical order, so $U \subset \mathbb{Q}$ is *rounded upper* if $u \in U \Leftrightarrow \exists t{:}\mathbb{Q}.\ u > t \in D$.

The constructive construction yields the same result if we replace $\mathbb{Q}$ by $\mathbb{R}$, *cf.* Corollary 10.4.

The significance of (this topology on) the space $\underline{\mathbb{R}}$ in traditional analysis is

Proposition 3.4 A function $f : X \to \underline{\mathbb{R}}$ is ***lower semicontinuous*** by definition if the inverse image of any open upper interval $(d, +\infty]$ is open in X. This happens iff f is continuous with respect to the Scott topology. $\square$

So, when we say that *all functions are continuous* in our calculus, we are not precluding the consideration of *semi*continuous functions: they just have to be seen as valued in $\underline{\mathbb{R}}$ or $\overline{\mathbb{R}}$ and not $\mathbb{R}$. Other simple forms of discontinuity such as jumps may be treated by using the *interval domain.* In particular, arithmetic negation makes $\underline{\mathbb{R}} \cong \overline{\mathbb{R}}$, but it is not continuous *endo*function of either space.

Remark 3.5 Constructively, these spaces are *not* obtained by *re*-topologising the extended set of reals. On the contrary, an ordinary ***Euclidean real number*** is defined by a pair (called a *Dedekind cut*, Definition 6.6) consisting of an ascending real (the set of its rational lower bounds) and a descending one (the upper bounds) that are compatible. In the computable setting, there are descending reals that have no ascending partner, and *vice versa* (Example 16.6).

The analysis of the ascending reals is very simple. In particular, *any* set of ascending reals has a supremum, given by union, and this is the limit of the set, so there is no difficulty with interchanging limits.

This offers two ways of forming the supremum of any set of Euclidean reals:
(a) as the *union* of their *lower* bounds, the result of this being an *ascending* real, or
(b) as the *intersection* of their *upper* bounds, yielding a *descending* real.
In our calculus, we shall find that we can do this when the set is overt or compact, respectively (Remarks 12.4 and 9.13).

Since there are two different constructions of the supremum of a set of reals, these might disagree. Constructively, we need an additional condition on the set in order to ensure that the supremum is a Euclidean real number.

Definition 3.6 A subset $S \subset \mathbb{R}$ obeys the ***constructive least upper bound principle*** if
(a) it is non-empty and bounded above, and
(b) for all real numbers x, z with $x < z$, either z is an upper bound of S, or there is some $s \in S$ with $x < s$.
This condition, which is probably due to L.E.J. Brouwer, is *necessary* to form $y \equiv \mathsf{sup}\, K$ because of the *locatedness* property (Axiom 4.9) with respect to $(x < z)$, *i.e.* $(x < y) \vee (y < z)$. We shall find in Section 12 that it follows from the mixed modal laws (*cf.* Proposition 2.13) and is sufficient to define a Dedekind cut, and therefore a Euclidean real number.

You may perhaps think that this "constructive" situation is rather complicated, and could be fixed by adding some extra axioms. However, it is not difficult to adapt the arguments of Sections 1 and 16 to show that any such axiom would provide an oracle that could solve unreasonable computational and logical problems like those in Remark 1.3. We shall come to see that these anomalous situations are just as natural as singularities in polynomial equations, and are indeed closely related to them. When we recognise that ascending and descending reals occasionally lead their own separate lives, we come to appreciate the symmetries that real analysis enjoys, instead of its pathological counterexamples.

Definition 3.7 Our second example of the Scott topology is the ***Sierpiński space***, which we call Σ. We define it as the lattice of open subspaces of the singleton. Classically, therefore,

$$\Sigma \text{ looks like } \begin{pmatrix}\odot\\ \bullet\end{pmatrix}, \qquad \text{not like } \mathbf{2} \equiv (\odot\odot),$$

having two points and three open sets. We shall call these points $\top$ and $\bot$, the former being open and the latter closed. Since Σ is a lattice, it also has $\wedge$ and $\vee$.

The space $\mathbf{2}$ is both discrete and Hausdorff, but Σ is neither. Whilst there is a continuous function that takes the two points of $\mathbf{2}$ to those of Σ, any continuous function $\Sigma \to \mathbf{2}$ is constant. Hence Σ is ***connected***, at least in the classical sense, and indeed in the constructive ones that we shall consider in Sections 13–15. It's even *path*-connected.

This means that Σ has "more than" two points — there is something in between $\bot$ and $\top$ that "connects" them. From a constructive point of view, this is because we defined the points of Σ as the open *subsets* of the singleton, which need not be *decidable* (complemented).

The Sierpiński space was treated with derision in classical topology, but it plays a *key* role in the subject that is even more important than Proposition 3.4 for the ascending reals.

Proposition 3.8 For any space X, there is a bijective correspondence amongst
(a) open subspaces $U \subset X$,
(b) continuous maps $\phi : X \to \Sigma$ and
(c) closed subspaces $C \subset X$,
where we shall say that ϕ ***classifies*** $U \equiv \phi^{-1}(\top)$ and ***co-classifies*** $C \equiv \phi^{-1}(\bot)$. □

Notice, in particular, that the correspondence between U and C is given by their common relationship to ϕ, and not by set-theoretic complementation. This is how we shall avoid the ubiquitous $\neg\neg$-statements that are found in intuitionistic analysis. Nevertheless, it is convenient to retain the word ***complementary*** for this relationship.

Remark 3.9 In the case where $X \equiv \Sigma$, continuous functions $\Sigma \to \Sigma$ correspond to open subsets of Σ. Three of these are definable: the identity and the constant functions with values $\bot$ and $\top$, corresponding to the singleton, empty and entire open subspaces respectively. Just as there was no arithmetical negation for the ascending reals,

there is no continuous function ("logical negation", $\neg$) *that interchanges* $\bot$ *and* $\top$.

More generally, Scott-continuous functions respect the order on the lattice. Indeed, *any* topological space X has a ***specialisation order***, with

$$x \leqslant y \text{ if every neighbourhood of } x \text{ also contains } y.$$

This is antisymmetric iff the space is T_0, discrete iff it is T_1 and agrees with the order on the underlying lattice when that is given the Scott topology. Notice that we distinguish this order relation $\leqslant$ from $\leq$ in real and integer arithmetic; they agree in the case of the ascending reals, whilst $\leqslant$ is $\geq$ or $=$ for the descending or Euclidean reals respectively. The key difference is that the order $\leqslant$ is ***intrinsic***, *i.e.* every continuous function $f : X \to Y$ preserves it, whilst $\leq$ is ***imposed*** on $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, in the sense that continuous functions may preserve, reverse or ignore it.

Scott continuity is stronger than just preserving order, but instead of talking about arbitrary joins and finite sub-joins, it is convenient to introduce a new definition.

Definition 3.10 A poset (partially ordered set) $(\mathcal{I}, \leq)$ is ***directed*** if it is inhabited (has an element) and, for any $i, j \in \mathcal{I}$, there is some $k \in \mathcal{I}$ with $i \leq k \geq j$. When we form a join or union indexed by $\mathcal{I}$ (taking $\leq$ to $\leqslant$), we use an arrow to indicate that it is directed: $\bigvee^{\uparrow}$ or $\bigcup^{\uparrow}$.

Examples 3.11 The following ordered sets are directed:
(a) any *total* order (or *chain*); in particular
(b) $\mathbb{N}$, $\mathbb{Q}$ or $\{q : \mathbb{Q} \mid q < a\}$ with the arithmetical order, where a is any (ascending) real number; and
(c) $\mathbb{Q}$ or $\{q : \mathbb{Q} \mid q > a\}$ with the reverse arithmetical order, where a is any (descending) real number;
(d) also, the set of finite subsets of any set, with the inclusion order.

We may reformulate the Scott topology using directedness: $\mathcal{U} \subset \mathcal{L}$ is open iff whenever $\bigvee^{\uparrow} x_i \in \mathcal{U}$, already $x_i \in \mathcal{U}$ for some $i \in I$. Hence one may define this topology on any ***dcpo*** (directed-complete partial order), *i.e.* a poset in which every *directed* subset but not necessarily every finite subset has a join.

Proposition 3.12 A function $F : \mathcal{L}_1 \to \mathcal{L}_2$ between complete lattices (or dcpos) is ***Scott continuous***, *i.e.* $F^{-1}(\mathcal{V})$ is Scott open in $\mathcal{L}_1$ whenever $\mathcal{V} \subset \mathcal{L}_2$ is Scott open in $\mathcal{L}_2$, iff F preserves directed unions, *i.e.*

$$F\Big(\bigvee_{i \in I}^{\uparrow} x_i\Big) \;=\; \bigvee_{i \in I}^{\uparrow} F(x_i)$$

for all directed $(x_i)_{i \in I} \subset \mathcal{L}_1$. □

Example 3.13 Our operators $\Box$ and $\Diamond$ are Scott-continuous functions from the lattice of open subspaces of X to Σ, since they preserve directed joins.

Moreover, if they are defined in terms of some (function with) parameters, they are jointly continuous with respect to both those parameters and to open subspaces of X, *throughout the parameter space.*

Remark 3.14 Using directed covers of compact spaces instead of general ones simplifies the idioms of analysis, because they are often *naturally* indexed by the rationals or reals.

Suppose, for example, that we want to find a bound for a function $f : K \to \mathbb{R}$. The subsets $U_u \equiv \{k \in K \mid fk < u\}$ are open and cover K, and also satisfy $u \leq v \Rightarrow U_u \subset U_v$. Since u ranges over a (totally ordered and so) directed poset, the *finite* open sub-cover need only have *one* member (named by the greatest u in the finite set), and we have $K \subset U_u$ for a single u. In other words, there is a ***uniform*** bound.

Looking at the requirements for this situation more closely, $f : K \to \overline{\mathbb{R}}$ need only be *upper semi*continuous. Such f correspond bijectively to directed families (U_u) indexed by $u \in \mathbb{Q}$, and also to ***directed relations*** like $\theta(k, u) \equiv (fk < u)$ that satisfy

$$(u \leq v) \;\wedge\; \theta(k, u) \;\Rightarrow\; \theta(k, v).$$

On the other hand, when studying continuity and differentiability (Section 10), we require $\delta > 0$ with a certain property. Underlying this is a *lower* semicontinuous function, a directed family of subsets with $\delta \leq \epsilon \Rightarrow U_\delta \supset U_\epsilon$ or a predicate with $(\delta \leq \epsilon) \wedge \theta(k, \epsilon) \Rightarrow \theta(k, \delta)$.

Is it θ or F that is being continuous here? Considering $\theta : \mathbb{R} \to \Sigma^X$, we are mixing the old idea of "approaching" a limit ($u \nearrow \infty$ or $\delta \searrow 0$), with the topological one, that F classifies an open subspace of Σ^X. For $F : \Sigma^X \to \Sigma$, on the other hand, this is *Scott* continuity, *i.e.* preservation of directed joins. When we need Scott continuity in our abstract calculus, in Section 9, we shall formulate it using directed relations.

Now let's think about Proposition 3.8 again.

Notation 3.15 Since open sets $U \subset X$ correspond to continuous maps $X \to \Sigma$, we write Σ^X for the lattice of them, equipped with the Scott topology. This correspondence also gives rise to the notation

$$\phi a \qquad \text{or} \qquad \phi a \Leftrightarrow \top \qquad \text{for} \qquad a \in U$$

for membership of this subspace.

Implicit in "ϕa" is a binary higher-type function, called (in different contexts) ***evaluation*** or ***application***, so $\mathsf{ev}(\phi, a) \equiv \phi a$. We want this, like everything else, to be continuous, but this requirement places a severe restriction on the compatibility between the ideas in this paper and those of traditional topology, which we shall discuss in Remark 7.17.

Proposition 3.16 The function $\mathsf{ev} : \Sigma^X \times X \to \Sigma$ given by $(\phi, a) \mapsto \phi a$ is jointly continuous (with respect to the Tychonov product topology defined from the given topology on X and the Scott topology on Σ^X and Σ) iff X is *locally compact*. In this case, Σ^X is also locally compact. □

As we are dealing with non-Hausdorff spaces (in particular Σ^X) here, we need to adjust the traditional definition of local compactness [HM81]:

Definition 3.17 A (not necessarily Hausdorff) space X is ***locally compact*** if, for any $x \in U \subset X$ with U open, there are compact K and open V with $x \in V \subset K \subset U$.

This relation between open subsets, written $V \ll U$ and called ***way below***, may be characterised without mentioning the compact subspace K between them: if $U \subset \bigcup^{\uparrow} W_i$ then already $V \subset W_i$ for some i. This is the point from which the theory of *continuous lattices* begins [GHK+80], but we shall not need to make much use of it, beyond observing the ubiquitous alternating inclusions of open and compact intervals in real analysis (Remark 10.1).

The result that justifies calling Σ^X a ***function-space*** is then

Theorem 3.18 Let X be locally compact and Γ any space. Then there is a bijection between continuous functions

$$\frac{\Gamma \times X \longrightarrow \Sigma}{\overline{\Gamma \longrightarrow \Sigma^X}}$$

that is given in the upward direction by composition with $\mathsf{ev} : \Sigma^X \times X \to \Sigma$. This correspondence is *natural* in the space Γ, *i.e.* it respects pre-composition with any continuous function $\Delta \to \Gamma$. □

The downward direction formalises the way in which we define an open subspace by prescribing its membership predicate. It is called ***λ-abstraction***, and we shall discuss it further in Section 7.

Remark 3.19 Scott's work led to the development of *domain theory* and *denotational semantics* in theoretical computer science. This gives topological meanings to programs as continuous functions, and is particularly valuable for *functional* programming languages, *i.e.* those in which functions may be defined as first class objects, *e.g.* [Plo77]. The latter are interpreted using λ-abstraction, whilst recursive definitions that need not necessarily terminate or be well founded are given a meaning by means of directed joins.

Denotational semantics was founded on an intuition of the analogy between continuity and computation that had earlier roots in recursion theory. In particular, the ***recursively enumerable*** subsets of $\mathbb{N}$ provide something like a topology, in so far as they admit all finite intersections and *some* infinite unions.

The connection can be put much more simply than this, in terms of computation with real numbers. We cannot make a *positive* (terminating) test for *equality* (*cf.* Remark 1.3), but we can do so for $\neq$, $>$ or $<$. More generally, we may observe membership of an *open* subspace, since that is determined by some *finite approximation* to (essentially, finitely many decimal places of) the number. Like open subsets, (parallel) observations admit finite intersections and (some) infinite unions.

Another basic intuition behind Scott continuity is that the result of a computation depends on only *finite* part of the data.

We shall begin the axiomatisation of our calculus in the next section from these remarks, but first we need to make some more foundational observations about general topology.

Remark 3.20 The Sierpiński space is particularly familiar in computation, as it provides the type of values of a program that may terminate ($\top$) or diverge ($\bot$) but generates no numerical output or other side-effect. This type is called `void` in C and Java, but `unit` in ML. An *input* of this type is a *signal* that may or may not ever arrive.

Then a program F of type $\Sigma \to \Sigma$, *i.e.* which takes a signal as input and then may terminate (output another signal) or diverge, may behave in one of three ways:

(a) it may always diverge ($\bot$), whether it obtains an input signal or not;

(b) it may transmit the signal (id), *i.e.* wait for its input and then terminate, perhaps after some further internal processing; or

(c) it may always terminate ($\top$), with or without a signal.

The one thing that it *cannot* do is to *negate* its input ($\neg$), *i.e.* terminate iff its signal never arrives; this is called the ***Halting problem*** [Tur35]. Σ is therefore quite different from a two-element or Boolean type.

This situation is just like that in Remark 3.9, except that we can now see *computationally* something that was perhaps a little ambiguous in constructive topology, namely

the general behaviour of any program $F : \Sigma \to \Sigma$ is determined by the specific cases $F\top$ and $F\bot$ in which it definitely does or does not receive an input signal.

As in topology, we must have $F\bot \leqslant F\top$, *i.e.* if the program terminates without receiving the input signal, it must also terminate if it does receive it. Using the lattice structure on Σ, we may use "linear interpolation" to define a function $F : \Sigma \to \Sigma$, given just $F\top$ and $F\bot$. Because of the previous remark, this recovers the original F:

Definition 3.21 The ***Phoa principle*** (pronounced "Pwah") [Hyl91] says that

$$\text{for any} \quad F : \Sigma \to \Sigma \quad \text{and} \quad x : \Sigma, \qquad Fx \;\Leftrightarrow\; F\bot \vee x \wedge F\top.$$

As it will lead to rules of inference in topology that may appear to be classical (Axiom 5.5), it is important to emphasise that this principle was discovered as a result of investigations in several *constructive* disciplines (albeit not real analysis). It was noticed independently as the Frobenius laws (*cf.* Propositions 8.2 and 11.2) for open and proper maps in intuitionistic locale theory [JT84, Ver94] (the formulation of general topology purely in terms of lattices, without mentioning points). It also falls out of the formulation of locales over any elementary topos using a monad [Joh82, §II 1.2]. Abstractly, it can be seen as exactly the condition that it required to ensure the extensional correspondence amongst open and closed subspaces and terms of type Σ^X in Proposition 3.8 [C].

Remark 3.22 There is a certain imprecision about the analogy between topology and recursion theory, since the former traditionally requires *arbitrary* unions, but the latter only *recursive* ones. In fact, the translation *from* programs *to* continuous functions is perfectly rigorous, and has been used very successfully to develop methods of demonstrating correctness of programs. One may also try to reformulate topology using *σ-frames*, *i.e.* lattices with *countable* unions, over which meets distribute (the σ follows the usage of probability theory, but is confusing in our notation). However, countability is just a mutilated form of set theory that does not get to grips with computation, and actually makes no difference for objects such as $\mathbb{N}$ and $\mathbb{R}$.

These issues may be addressed by considering topology and computation *in parallel*, *i.e.* by requiring the morphisms to be pairs consisting of a continuous function and a program that "agree" in a suitable sense. There are two such techniques that are well established. One uses ***logical***

relations and can show that, if the topological value of a program is $\top$ then its computation terminates [Plo77], *cf.* Remark 6.5. The other, called ***type two effectivity***, develops computational representations of many ideas in classical general topology and real analysis [Wei00].

We have already seen in Remark 2.9 that our possibility operator $\Diamond$ and the new notion of overtness are badly represented by the *arbitrary* unions of subsets that are used in traditional general topology (or locale theory). We want to replace these arbitrary unions by *recursive* ones, thereby legitimising the idea that the RE subsets of $\mathbb{N}$ form a topology. However, when we consider how complicated both topology and recursion theory are on their own, it seems to be a recipe for a dog's breakfast to try to *combine* them into a new theory.

But that's not what we are going to do: the revolutionary approach that we shall follow in this paper is quite different from any of these things. We begin by *abandoning* traditional topology and recursion theory *altogether*, and setting out some extremely basic intuitions about the primitive things that we can calculate about real numbers. We find that, *without ever introducing set theory*, we can recover the main results about continuity on the real line, including the issues regarding connectedness and solutions of equations that we introduced in the first two sections. On the other hand, it has been well known since the time of Church, Kleene and Turing that more or less anything that *looks* like computation is equivalent to the standard forms (at least for partial functions $\mathbb{N} \to \mathbb{N}$). In particular, there are enough ingredients amongst our topological ideas to do general computation.

Overtness plays a key role underlying all of this. In the previous section, we saw manifestations of it in open maps, the existential quantifier and the join-preserving property of $\Diamond$. It is also related to recursive enumerability, and specifies *which* joins exist in open set lattices and the ascending reals. However, despite the fact that it does so many jobs, we don't have to do anything to encode this behaviour into the system: it will all just fall out naturally.

In this section we have introduced some ideas from non-Hausdorff general topology because they underlie four decades' work in the theory of computation, but are largely unknown to analysts. In particular, they provide the background for our new calculus. However, I now ask you to *forget* them again, along with all of the other textbook accounts of general topology, and begin the next section with a fresh mind. Our new calculus will "stand on its own feet". It has a concrete *representation* that we have just sketched, but is not *defined* by it, just as groups may be *represented* by permutations or matrices, and integers by sheep or pebbles. Progress in mathematics is made by abstraction from such things.

4 Introducing the calculus

Now we start to present our new calculus, which is called ***Abstract Stone Duality***, in a relatively informal way. For a more formal treatment that is better suited to logicians please see [I] instead. The first few axioms are very familiar facts about the real line and other spaces.

Axiom 4.1 In this paper, $\emptyset$, $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$,..., $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{I} \cong [0,1] \subset \mathbb{R}$ and the Sierpiński space Σ will be assumed as base types. If X and Y are types then so are their product $X \times Y$ and sum (disjoint union) $X + Y$. There are also ways of introducing subspaces and function-spaces, which we discuss in Section 7, but we stress that *these do not use set theory.*

Every variable x or expression a has a type, and we write $x : X$ or $a : X$ to say what it is.

Along with $X \times Y$, $X + Y$ and Σ^X come projections, pairing, inclusions, case-analysis, function-application and abstraction operations. These are explained in any account of *simple type theory*, *e.g.*, [Tay99, §2.3].

You may substitute the phrase "locally compact topological space" for "type" if you wish, just as you understood elements of a group as matrices, and numbers as collections of beads earlier in your mathematical education. However, our terms are really generalisations of arithmetic expressions, and the role of types is to stop us from applying logical operators to numbers or *vice versa.*

The fundamental importance of $\mathbb{R}$ in science surely justifies introducing and axiomatising it independently, as we do in this paper. On the other hand, [I] *constructed* it using two-sided Dedekind cuts of $\mathbb{Q}$ in the ASD calculus. So you may say that the Axioms about $\mathbb{R}$ here were Theorems there, or that that paper *implements* the ideas of this one, or again that it shows that these axioms provide a *conservative extension*, *cf.* Remark 10.3. It thereby reduces the topology and arithmetic of the reals to λ-terms that only involve the rationals, their arithmetical operations and relations, and the quantifiers. Giving one implementation is not, however, an exclusive action: there may be better ones.

Axiom 4.2 Terms of any type may be defined by primitive recursion over $\mathbb{N}$, for example this has addition and multiplication. The objects $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ carry the structure of commutative rings, with the usual notation, inclusions and algebraic laws. We shall consider division and general recursion shortly.

Axiom 4.3 $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all carry the usual six binary relations

$$x = y, \quad x \neq y, \quad x < y, \quad x \leq y, \quad x > y \quad \text{and} \quad x \geq y,$$

but $\mathbb{R}$ and $\mathbb{I}$ only have $\qquad x \neq y, \quad x < y \quad \text{and} \quad x > y.$

Notice again that we distinguish $\leq$ in arithmetic from $\leqslant$ in logic and topology.

Axiom 4.4 These nine expressions are ***propositions***, as are

$$\top \text{ (true)}, \quad \bot \text{ (false)}, \quad \sigma \wedge \tau, \quad \sigma \vee \tau, \quad \exists x{:}S.\ \phi(x) \quad \text{and} \quad \forall x{:}K.\ \phi(x),$$

whenever σ, τ and $\phi(x)$ are, except that the quantifiers may only be formed when S is overt and K compact, as we shall explain later. Propositions are the same as terms of type Σ, the Sierpiński space, *cf.* Definition 3.7. Therefore, for the topological and computational reasons that we gave in the previous section, we cannot use $\neg$ or $\Rightarrow$ to form them.

However, such a logic would be too weak to be useful, whilst there are compact and other kinds of subspaces besides open or closed ones. So, where there are just subsets and propositions in set theory, in topology we need terminology that makes distinctions amongst species of *logical* properties that correspond to the various sorts of subspaces.

Definition 4.5 If σ and τ are propositional expressions and $a, b : X$ are any two expressions of the same (but *any*) type then

$$\sigma \;\Rightarrow\; \tau, \quad \sigma \;\Leftrightarrow\; \tau \quad \text{and} \quad a = b : X$$

are called ***statements***. The intended logical meaning of the first is that the open subspace represented by σ is contained in that represented by τ, and that if the program σ terminates then so does τ, whilst the containment between corresponding closed subspaces is the other way round. The second says that they coincide. All three can be seen as *equations between terms* of type Σ or X, but are not terms themselves.

We write & instead of $\wedge$ for the conjunction of statements; note that $\exists$, $\forall$, $\Rightarrow$ and $\Leftrightarrow$ *cannot* be used to combine statements into more complicated things, at least in the current version of the calculus. We shall, however, write

$$\alpha \Rightarrow \beta \Leftrightarrow \gamma \Rightarrow \delta \qquad \text{for} \qquad (\alpha \Rightarrow \beta)\ \&\ (\beta \Rightarrow \gamma)\ \&\ (\gamma \Rightarrow \beta)\ \&\ (\gamma \Rightarrow \delta),$$

just as we would with equations and inequalities of numbers. We shall see how statements can be used to define subspaces, in particular compact ones, in Sections 7–8.

A propositional expression ϕx that contains a parameter (free variable) $x : X$ defines a function $X \to \Sigma$, which is automatically continuous, and so (by Proposition 3.8) both an open subspace U and a closed one C, where

$$\phi x \Leftrightarrow \top \quad \text{means} \quad x \in U \quad \text{and} \quad \phi x \Leftrightarrow \bot \quad \text{means} \quad x \in C.$$

These are *statements*, which we sometimes write more briefly as ϕx and $\neg\phi x$ respectively. Since we are not allowed to write $\neg\neg\phi x$ or $\phi x \vee \neg\phi x$, the subspaces U and C are not "complementary" in anything like the set-theoretic sense

This also means that we should treat "$x \in U$", "$x \in C$" and more generally "$a : X$" (which says that a is a well defined term of type X) as other kinds of statement.

We may use statements as axioms, assumptions or conclusions, as we shall explain in the next section. Let's consider the meaning of statements that are formed directly from the propositional arithmetical relations on $\mathbb{N}$ and $\mathbb{R}$ that we introduced in Axiom 4.3. Recall that we said that $\mathbb{R}$ didn't have $=$, $\leq$ or $\geq$ as *propositions*, because the subspaces of $\mathbb{R} \times \mathbb{R}$ that these define are not open.

Examples 4.6 For $x, y : \mathbb{R}$, the *statements*

$$(x \neq y) \Rightarrow \bot, \qquad (x > y) \Rightarrow \bot \qquad \text{and} \qquad (x < y) \Rightarrow \bot$$

mean $x = y$, $x \leq y$ and $x \geq y$ respectively. We add these familiar symbols ($=$, $\leq$ and $\geq$) to our notation for statements, but of course the symbol $=$ has now become *overloaded.*

Definition 4.7 We call a type N (such as $\mathbb{N}$ or $\mathbb{Q}$ but not $\mathbb{R}$) ***discrete*** if the two *statements* in the rule on the left below are interchangeable. The top one is the *statement* of equality of two *expressions* (n, m) that we could make for *any* type, and the bottom one is a statement of equality of two *propositions*, $(n =_N m)$ and $\top$. A discrete space is special in that the *proposition* $(n =_N m)$ of equality is meaningful for it (as it is for $\mathbb{N}$ and $\mathbb{Q}$ but not $\mathbb{R}$) and makes this rule valid.

$$\frac{n \ = \ m : N}{\overline{(n =_N m) \ \Leftrightarrow \ \top}} \qquad\qquad \frac{h \ = \ k : H}{\overline{(h \neq_H k) \ \Leftrightarrow \ \bot}}$$

Definition 4.8 Similarly, we call a type H (such as $\mathbb{N}$ or $\mathbb{R}$) ***Hausdorff*** if the *statement* of equality of expressions (h, k) on the top right is interchangeable with the statement of equality of propositions below it, one of which is the *proposition* $(h \neq_H k)$ of *in*equality. In other constructive accounts of analysis, $\neq$ is sometimes called ***apartness*** and written $h \# k$.

Since propositions provide a logical way of describing open subspaces, these two properties say that the diagonal subspace $X \subset X \times X$ is respectively open or closed. In traditional topology, which has *arbitrary* unions, every discrete space is Hausdorff, but this is not so in ASD: $=_N$ may be defined whilst $\neq_N$ isn't (Example 16.5). When they are *both* defined, as they are for $\mathbb{N}$, they are automatically complementary:

$$(n =_N m) \vee (n \neq_N m) \ \Leftrightarrow \ \top \quad \text{and} \quad (n =_N m) \wedge (n \neq_N m) \ \Leftrightarrow \ \bot,$$

and then we say that N has ***decidable equality***. Lemma 5.8 collects some of the properties of equality and inequality.

We may now formulate some of the basic properties of $\mathbb{R}$ as axiomatic statements:

Axiom 4.9 $\mathbb{Q}$ and $\mathbb{R}$ are ***totally ordered*** Hausdorff fields, *i.e.*

$$(x \neq y) \iff (x < y) \vee (y < x) \quad \text{and} \quad x^{-1} \text{ is defined iff } x \neq 0.$$

The order relation $<$ is also ***transitive*** and ***located***,

$$\begin{array}{rcl} (x < y) \wedge (y < z) & \Rightarrow & (x < z) \\ (x < y) \vee (y < z) & \Leftarrow & (x < z). \end{array}$$

Beware that the word "located" has two meanings in constructive analysis, which are related but not directly so.

5 Formal reasoning

In our new calculus, you will be able to reason about continuous, computable functions in ways that are quite similar to those with which you are already familiar for giving proofs based on set theory. So the situation is like that of learning Italian when your native language is Spanish: it's possible to communicate quite effectively, but in order to learn the new language properly, you have to begin by defining the grammar of your own language in a more formal way. To put this another way, Hermann Weyl said that "logic is the hygiene the mathematician practices to keep his ideas healthy and strong". For us, "health" means continuity, which we guarantee by taking certain precautions that limit the applicability of the usual logical connectives.

The way in which we shall set out our restricted form of predicate calculus belongs to a tradition begun by Gerhard Gentzen [Gen35]. A textbook introduction that includes all of the predicate and λ-calculus that we introduce here and in Section 7 is provided by [GLT89].

Definition 5.1 A mathematical argument consists of a sequence (or tree) of *statements*, which for us will be as in Definition 4.5. However, each statement may contain *parameters*, so it is made in the ***context*** of these parameters, together with their *types* and certain *assumptions*, the latter also being statements. Some proofs proceed *directly* from fixed assumptions by developing successive conclusions. More usually, they are *indirect*: assumptions and variables need to be *introduced* and *discharged*, in particular when using the quantifiers, so the context may vary from one statement to the next.

In proof *theory*, where contexts are manipulated heavily, it is customary to use the Greek letters, Γ, Δ,... for them. We, on the other hand, only need the occasional reminder that parameters and assumptions may be present, so we shall usually write $\cdots$ instead of Γ for an unspecified context.

One formal way of showing the relevant context is to write it on the left of a "turnstile" $\vdash$. (This symbol was first used by Gottlob Frege, but has done many different jobs since his time.) Then a statement in context is called a ***judgement***, and may look like

$$\ldots,\ x:\mathbb{R},\ d \le x \le u,\ \phi x \Leftrightarrow \top,\ \psi x \Leftrightarrow \bot\ \vdash\ \theta x \wedge (x < u)\ \Rightarrow\ \exists t{:}\mathbb{I}.\ \phi t \wedge \psi t,$$

in which the various expressions (d, u, ϕ, ...) have themselves been introduced in previous judgements. The informal way of writing this is "let $x:\mathbb{R}$ and suppose that $d \le x \le u$, ..." in the text and to display the conclusion,

$$\theta x \wedge (x < u)\ \Rightarrow\ \exists t{:}\mathbb{I}.\ \phi t \wedge \psi t.$$

Even if we do not mention free variables explicitly, any (symbol for a) *term* in this paper denotes an expression that may contain parameters. That is, unless we specifically say otherwise, as in Remark 6.5. On the other hand, our *types* are fixed, without parameters.

Definition 5.2 In keeping with the need to allow parameters to pass all the way through an argument, when we talk about a ***function*** $f : X \to Y$, we mean an *expression* $\ldots,\ x : X\ \vdash\ f(x) : Y$ of type Y, in which we draw attention to the variable $x : X$ as one amongst many that might occur in it.

Exercise 5.3 Fill in the contexts ("$\Gamma \vdash$") in all of the formal assertions that we make in this paper. (Print out a fresh copy!)

Definition 5.4 This discussion may sound like mere symbol-pushing, but it has a direct topological meaning or ***denotation***. The context Γ denotes a big space (***universe of discourse***), sometimes written $[\![\Gamma]\!]$, that is a subspace of the product of the types of the parameters, carved out by the assumptions. In this paper, the denotation of a context will usually be locally compact, but in general it may be a more general kind of space (Section 7)

Terms and functions also have denotations, which are continuous functions between the denotations of the corresponding contexts. We have to construct these continuous functions by recursion over the language, with a scheme of recursion steps for each of the connectives; for example, Theorem 3.18 handles abstraction and application of functions.

The notion of "denotation" here is the same as that in Remark 3.19, *i.e.* the structure-preserving translation of a symbolic language into point-set topology. In fact, the denotational semantics of programming languages "factors through" the denotation of ASD, in that it can be rewritten to use our syntactic language for topology.

Whilst the denotation helps us to understand what the calculus is doing, the syntax stands on its own feet, and we shall develop *proofs* directly in it. These are *well formed* if their successive judgements obey one of a number of patterns called ***rules of inference***. When we state these, a single line indicates deduction downwards, and a double one equivalence (upwards too).

Axiom 5.5 The Phoa principle (Definition 3.21) may be formulated as a pair of rules that transfer statements across the $\vdash$, between the assumptions and the conclusion. One of these is the rule for $\Rightarrow$ in intuitionistic sequent calculus, but the other is like Gentzen's rule for classical negation:

$$\frac{\ldots,\ \sigma \Leftrightarrow \top\ \vdash\ \alpha\ \Rightarrow\ \beta}{\ldots\ \vdash\ \sigma \wedge \alpha\ \Rightarrow\ \beta} \qquad\qquad \frac{\ldots,\ \sigma \Leftrightarrow \bot\ \vdash\ \beta\ \Rightarrow\ \alpha}{\ldots\ \vdash\ \beta\ \Rightarrow\ \sigma \vee \alpha}$$

By Definition 5.4, "$\Gamma,\ \sigma \Leftrightarrow \top$" denotes an open subspace of the universe, and "$\alpha \Rightarrow \beta$" means that one relative open subspace of this is contained in another. Then both lines of the rule on the left say that the intersection of the *open* subspaces of Γ classified by σ and α is contained in that classified by β.

Because of Definition 4.5, the rule on the right is valid constructively, because it says exactly the same thing as that on the left, except that it concerns the intersection of *closed* subspaces.

However, to distinguish our theory of topology from (classical) set theory, we need another

Axiom 5.6 Every $F : \Sigma^\Sigma$ is ***monotone***: if $\sigma \Rightarrow \tau$ then $F\sigma \Rightarrow F\tau$, *cf.* Remarks 3.9 and 3.20. From this we recover the Phoa principle (Definition 3.21):

Proposition 5.7 $F\sigma \Leftrightarrow F\bot \vee \sigma \wedge F\top$.

Proof To give just one example of a proof *tree*, we have

$$\dfrac{\dfrac{\dfrac{\sigma \Leftrightarrow \bot \vdash F\sigma \Leftrightarrow F\bot}{F\sigma \Rightarrow F\bot \vee \sigma} \quad \dfrac{\sigma \Rightarrow \top}{F\sigma \Rightarrow F\top}}{F\sigma \Rightarrow (F\bot \vee \sigma) \wedge F\top} \qquad \dfrac{\dfrac{\bot \Rightarrow \sigma}{F\bot \Rightarrow F\sigma} \quad \dfrac{\top \Leftrightarrow \sigma \vdash F\top \Leftrightarrow F\sigma}{\sigma \wedge F\top \Rightarrow F\sigma}}{F\bot \vee (\sigma \wedge F\top) \Rightarrow F\sigma}}{F\sigma \Leftrightarrow (F\bot \vee \sigma) \wedge F\top \Leftrightarrow F\bot \vee (\sigma \wedge F\top)}$$

where the last step uses the *absorptive law* of the lattice Σ and the fact that $F\bot \Rightarrow F\top$. □

We can use the Gentzen-style rules to show that $=$ in a discrete space obeys the usual properties of substitution, reflexivity, symmetry and transitivity, whilst inequality obeys their lattice duals.

Lemma 5.8 Let $\phi(x)$ be a propositional expression on a discrete space N or a Hausdorff one H. Then

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\phi(m) \wedge (n =_N m) & \Rightarrow & \phi(n) & \phi(h) \vee (h \neq_H k) & \Leftarrow & \phi(k) \\
(n =_N m) & \Rightarrow & f(n) =_M f(m) & (h \neq_H k) & \Leftarrow & f(h) \neq_K f(k) \\
(n =_N n) & \Leftrightarrow & \top & (h \neq_H h) & \Leftrightarrow & \bot \\
(n =_N m) & \Leftrightarrow & (m =_N n) & (h \neq_H k) & \Leftrightarrow & (k \neq_H h) \\
(n =_N m) \wedge (m =_N k) & \Rightarrow & (n =_N k) & (h \neq_H k) \vee (k \neq_H \ell) & \Leftarrow & (h \neq_H \ell),
\end{array}$$

where $f : N \to M$ or $f : H \to K$ is a function (Definition 5.2) to another discrete space M or Hausdorff one K.

Proof The first part comes from the basic rule of substitution,

$$\ldots,\ n = m : N \quad \vdash \quad \phi n \Leftrightarrow \phi m,$$

where discreteness (Definition 4.7) allows us to replace the statement of equality of terms on the left by the equality proposition:

$$\ldots,\ (n =_N m) \Leftrightarrow \top \quad \vdash \quad \phi n \Leftrightarrow \phi m.$$

Then the first Gentzen rule transfers this across the $\vdash$,

$$\ldots \quad \vdash \quad (n =_N m) \wedge \phi n \ \Rightarrow\ \phi m.$$

The other results in the left-hand column follow from this.

If you think you understand the results for $=$, which apply to open subspaces, then you also understand those for $\neq$, since they say the same thing in the upside-down notation for closed subspaces. The formal proof is

$$\begin{array}{ll} \ldots,\ h = k : H & \vdash \quad \phi h \Leftrightarrow \phi k \\ \ldots,\ (h \neq_H k) \Leftrightarrow \bot & \vdash \quad \phi h \Leftrightarrow \phi k \\ \ldots & \vdash \quad \phi h \Rightarrow \phi k \vee (h \neq_H k) \end{array}$$

using Definition 4.8 and the second Gentzen rule. ◻

Whereas the Gentzen rules in topology (and those for $\Rightarrow$ in set theory) move *propositions* across the $\vdash$, the rules for the quantifiers move a *variable*. This is ***free*** in one line of the rule and ***bound*** in the other.

Axiom 5.9 The rule for the ***existential quantifier*** is

$$\frac{\ldots,\ x : X \ \vdash\ \phi(x) \ \Rightarrow\ \sigma}{\ldots \ \vdash\ \exists x.\, \phi(x) \ \Rightarrow\ \sigma,}$$

in which the propositional expression $\phi(x)$ may contain the variable, but σ must not.

The upward direction of this rule is equivalent to ***existential instantiation***,

$$\phi a \ \Rightarrow\ \exists x.\, \phi x.$$

The downward direction justifies the mathematical idiom ***there exists***, in which, when given $\exists x.\, \phi(x)$, we may *temporarily assume* that we have an x that satisfies $\phi(x)$, in order to deduce any conclusion σ, *so long as* σ doesn't depend on what x is. *No choice* is involved in doing this [Tay99, §1.6].

This existential quantification is the equivalent in logic of the infinitary unions in topology, but we have already said that *arbitrary* unions were the reason why Proposition 2.9 missed the point of the computational idea behind Theorem 2.4. The rule above is therefore not asserted in its full generality in ASD: the type X of the quantified variable x must be ***overt***.

However, this is not a severe handicap, as $\mathbb{N}$, $\mathbb{R}$ and the other base types that we listed in Axiom 4.1 are all overt. The one significant exception in this paper is that *closed subspaces* need not in general be overt (Section 16). We shall study overt (sub)spaces more fully in Section 11: the reason for introducing the quantifier here is that we need it to state some of the basic properties of $\mathbb{N}$ and $\mathbb{R}$.

Lemma 5.10 The order $<$ on $\mathbb{R}$ admits ***interpolation*** and ***extrapolation***:

$$(x < z) \ \Leftrightarrow\ \exists y.\ \ (x < y) \wedge (y < z) \qquad\qquad \top \ \Leftrightarrow\ \exists xz.\, (x < y) \wedge (y < z).$$

Proof [$\Rightarrow$] by existential instantiation, with $y \equiv \frac{1}{2}(x+z)$, $x \equiv y-1$ and $z \equiv y+1$.
[$\Leftarrow$] by the other direction of Axiom 5.9. $\square$

In the same way, from Lemma 5.8, we deduce

Lemma 5.11 For any propositional expression $\phi(x)$ on an overt discrete space N,

$$\phi n \;\Leftrightarrow\; \exists m{:}N.\, \phi m \wedge (n =_N m). \qquad \square$$

Theorem 8.12 gives the dual rules for the universal quantifier $\forall$, which ranges over a *compact* space, and Lemma 8.13 is the dual of Lemma 5.11.

6 Numbers from logic

The axioms that we have given so far generate numbers from numbers using the arithmetic operations, logic from numbers using relations, and logic from logic using the connectives and quantifiers. However, for the logic to be of any practical *use*, we must complete the circle by recovering numbers from it. The idea is to define an integer or a real number by giving some propositional expression that fixes it uniquely. We shall see that the various "limiting" operations in analysis can be derived from this, and that all computation is done in the arena of logic.

In the case of $\mathbb{N}$, we characterise those propositions that should be of the form $\phi n \equiv (n =_{\mathbb{N}} a)$, where $a : \mathbb{N}$ is the number to be defined. This is a rule of logic that is easily overlooked; it was first stated correctly by Giuseppe Peano, writing $\bar{\iota}$ for our "the" [Pea97, §22].

Definition 6.1 A ***description*** is a propositional expression $\phi(n)$ containing a variable of overt discrete type N (such as $\mathbb{N}$ or $\mathbb{Q}$ but not $\mathbb{R}$) such that

$$\top \;\Leftrightarrow\; \exists n.\, \phi(n) \quad \text{and} \quad \phi(n) \wedge \phi(m) \;\Rightarrow\; (n =_N m).$$

For example, for any given $a : \mathbb{N}$, the proposition $\phi(n) \equiv (n =_N a)$ is a description.

Axiom 6.2 For *any* description ϕ, there is term, written $a \equiv \mathsf{the}\, n.\, \phi(n)$, of type N, for which $\phi(n) \Leftrightarrow (n =_N a)$. For example, $\mathsf{the}\, n.\, (n^3 =_{\mathbb{N}} 8)$ is 2.

The following is a *syntactical* result about this new symbol:

Lemma 6.3 Any numerical expression $a : \mathbb{N}$ is provably equivalent to one of the form $\mathsf{the}\, n.\, \phi(n)$, in which the description operator is not used in the syntax of ϕ, whilst any term of any other type may be rewritten without using it at all.

Proof Any numerical term may be "wrapped" in a description operator,

$$a \;=\; \mathsf{the}\, n.\, (n =_{\mathbb{N}} a),$$

so we just have to show how to eliminate "the" from within propositional *sub*-expressions. From Lemma 5.11 and the Axiom we have

$$\theta\big(\mathsf{the}\, n.\, \phi n\big) \iff \exists m.\, \theta m \wedge \big(m =_{\mathbb{N}} \mathsf{the}\, n.\, \phi n\big) \iff \exists m.\, \theta m \wedge \phi m,$$

and we may use this to rewrite each description operator within any given expression. $\square$

Remark 6.4 In the case $N \equiv \mathbb{N}$, let $\theta(n)$ be a decidable formula that has the first property (existence), *i.e.* there is another propositional expression $\psi(n)$ with

$$\ldots,\, n : \mathbb{N} \;\vdash\; \theta(n) \wedge \psi(n) \Leftrightarrow \bot, \quad \ldots,\, n : \mathbb{N} \;\vdash\; \theta(n) \vee \psi(n) \Leftrightarrow \top \quad \text{and} \quad \ldots \;\vdash\; \exists n.\, \theta(n) \Leftrightarrow \top.$$

Then $\phi(n) \;\equiv\; \theta(n) \wedge \forall m < n.\, \psi(m)$ is a description, and $\mathsf{the}\, n.\, \phi(n)$ is the least n such that $\theta(n)$. Hence we have ***general recursion*** or ***minimalisation***, at least in this special case. As there is a general recursive function that grows faster than any that can be defined using primitive recursion (Axiom 4.2) alone, the Lemma cannot eliminate *all* descriptions — they properly extend the calculus. See [D] for the case involving partial functions, where θ and ψ need only satisfy the first of the three conditions.

These results are applicable to propositional expressions with parameters and may also be used to embed computation in our calculus. They are proved as constructions *within* the ASD calculus, and so hold in any model of it. However, the converse operation depends on a much more powerful result:

Remark 6.5 The calculus obeys the ***existence property***:

$$\text{if } \vdash \exists n.\, \phi(n) \Leftrightarrow \top \text{ is provable} \quad \text{then so is } \vdash \phi(m) \Leftrightarrow \top, \text{ for some numeral } \vdash m : \mathbb{N},$$

where $\vdash$ with nothing on the left means that *no parameters are allowed.* This is shown by methods of ***proof theory*** that go *outside* the calculus (*e.g.* logical relations, *cf.* Remark 3.22): the *proof* of $\vdash \phi(n) \Leftrightarrow \top$ can be manipulated to derive the numeral $\vdash m : \mathbb{N}$.

The search for a witness m can be made automatically, using ***logic programming***, and it doesn't even require the proof as an input: the search terminates (possibly after a *very* long time) if it exists, but diverges or aborts if it doesn't. However, the details of the computation depend on a careful syntactic analysis of the whole calculus, which is beyond the scope of this paper (see [A, §11] for a sketch of the translation into PROLOG).

Beware, also, that there is no lattice dual of this result: there is a propositional expression $\phi(n)$ that is *not* equivalent to $\top$, but such that $\vdash \phi(m) \Leftrightarrow \top$ for each numeral $\vdash m : \mathbb{N}$.

Turning to real numbers, we cannot define them by descriptions, since equality is not a proposition (Definition 4.7). We could use inequality instead, but it is more natural to characterise the propositional expressions of the form $\delta d \Leftrightarrow (d < a)$ and $\upsilon u \Leftrightarrow (a < u)$.

Definition 6.6 A ***Dedekind cut*** (δ, υ) is a pair of propositional expressions δd and υu, with real or rational arguments d and u, such that

$$\begin{array}{rcl@{\qquad}rcl}
\exists e.\, (d < e) \wedge \delta e & \Leftrightarrow & \delta d & \exists t.\, \upsilon t \wedge (t < u) & \Leftrightarrow & \upsilon u \\
\exists d.\, \delta d & \Leftrightarrow & \top & \exists u.\, \upsilon u & \Leftrightarrow & \top \\
\delta d \wedge \upsilon u & \Rightarrow & (d < u) & (\delta d \vee \upsilon u) & \Leftarrow & (d < u).
\end{array}$$

The top two statements say that δ and υ are *ascending* and *descending* reals respectively, *cf.* Definition 3.3. The middle pair say that they are *bounded*; and the bottom ones that they are *disjoint* and *located*, *cf.* Axiom 4.9. The real line itself is constructed from Dedekind cuts of $\mathbb{Q}$ in [I].

Axiom 6.7 $\mathbb{R}$ is ***Dedekind complete*** in the sense that every cut is of the form

$$\delta d \Leftrightarrow (d < a) \quad \text{and} \quad \upsilon u \Leftrightarrow (a < u) \quad \text{for some unique } a : \mathbb{R}.$$

Just as we introduced $\mathsf{the}\, n.\, \phi(n)$ as a new piece of syntax for the integer defined by a description, we sometimes also write $\mathsf{cut}(\delta, \upsilon)$ for this new real number a. However, the analogue of Lemma 6.3 for cuts relies on additional axioms, and will be given in Corollary 10.4.

In order to *print* the result of a real-valued expression in the traditional decimal or binary notation, we first need another axiom to say that it is equivalent to a Dedekind cut of the (decimal or dyadic) *rationals*:

Axiom 6.8 The ***Archimedean principle*** excludes infinities and infinitesimals, for $x, \epsilon : \mathbb{Q}$ or $\mathbb{R}$:

$$\epsilon > 0 \;\Rightarrow\; \exists m{:}\mathbb{Z}.\;\; \epsilon(m-1) < x < \epsilon(m+1).$$

We often use $\epsilon \equiv 2^{-n}$ in this:

Lemma 6.9 Between any two real numbers lies a dyadic rational [I, Proposition 14.3]:

$$x < y \quad \Rightarrow \quad \exists n, k{:}\mathbb{Z}. \quad x < k \cdot 2^{-n} < y,$$

and on either side of any real number there are dyadic rationals that are arbitrarily close together, *cf.* [I, Proposition 11.14]:

$$x : \mathbb{R},\ n : \mathbb{Z}\ \vdash\ \exists k{:}\mathbb{Z}. \quad (k-1) \cdot 2^{-n} < x < (k+1) \cdot 2^{-n}. \qquad \square$$

Remark 6.10 Applying the existence property (Remark 6.5) to k, we may therefore evaluate any expression $\vdash a : \mathbb{R}$ without parameters, to any prescribed number $\vdash n : \mathbb{Z}$ of binary digits: $a \approx k \cdot 2^{-n}$. In the same way, given a possibility operator $\vdash \lozenge : \Sigma^{\mathbb{R}} \to \Sigma$ (without parameters) that is true on some interval, $\vdash \lozenge(a, b) \Leftrightarrow \top$, we may find $k : \mathbb{Z}$ such that

$$\vdash \lozenge\big((k-1) \cdot 2^{-n}, (k+1) \cdot 2^{-n}\big) \Leftrightarrow \top,$$

cf. Theorem 2.6. The query that we raised about the existential quantifier $\exists x_n$ in Theorem 1.7 may also be settled in this way.

We stress that this proof-theoretic computation operates on an entire expression, and that it needs to know the required precision n explicitly (say, 1000). Our sketchy mathematical discussion does not claim to say *anything* about the practical implementation on a computer, so you cannot judge how fast or slow it is. There is a syntactic translation that eliminates real numbers from our calculus in favour of dyadic intervals [K], which can then be manipulated using *constraint logic programing*. I envisage adapting John Cleary's algorithm [Cle87], which can solve equations as rapidly as Newton's algorithm does so.

Also, even in the case of a single real number a, it is subject to the usual caveat that there may be an error of as much as one unit (not just a half) in the last place, essentially because of Remark 1.3 again. In other words, the result k is ***non-determinate***. This explains how the computation can jump from one solution to another when we vary the coefficients of, for example, a cubic equation.

Nevertheless, returning from computation to analysis, we may abstract from this result by replacing the numeral n with a (variable) parameter, and $k \cdot 2^{-n}$ with a real-valued expression a_n that depends in a definite way on n:

Definition 6.11 A ***Cauchy sequence*** is an expression $\ldots,\ n : \mathbb{N}\ \vdash\ a_n : \mathbb{R}$ containing a variable n (albeit subscripted) of type $\mathbb{N}$ (and maybe other parameters from the list $\cdots$) such that

$$\ldots,\ n, m : \mathbb{N}\ \vdash\ n \leq m\ \Rightarrow\ |a_n - a_m| < 2^{-n}.$$

This sequence ***converges*** to a ***limit*** $a : \mathbb{R}$ if

$$\ldots,\ n : \mathbb{N}\ \vdash\ \top\ \Leftrightarrow\ (|a_n - a| < 2^{1-n})\ \equiv\ (a_n - 2^{1-n} < a < a_n + 2^{1-n}).$$

Remark 6.12 Here 2^{-n} is called the ***modulus of convergence***, and may be replaced by another function of n, for example Bishop used $\frac{1}{n+1}$. Specifying it makes the difference between

(a) the *classical* definition, in which any finite number of terms may be altered or deleted without affecting the limit, and we only find this out retrospectively; and

(b) our *constructive* one, for which the limit is determined to any prescribed accuracy (2^{-n}) by finitely many (n) terms, where n is known in advance.

The constructive definition is consistent with the fact that the result of any computation can only depend on a finite part of the data (Remark 3.19), but the classical definition conflicts with this.

Addition, multiplication and substitution of sequences incur an administrative overhead, because they need to be re-parametrised in order to satisfy the modulus of convergence. So, commonly occurring sequences such as the sums of *power* series don't march obediently towards their limits at the exponential pace that we have ordered. But analysts who regularly deal with them are skilled in re-bracketing and re-arranging series: there is no reason why the nth member of the *sequence* need be the sum of exactly n terms of the *series*.

We shall now find the limit of any Cauchy sequence. Given that we are about to prove our first Theorem of analysis in ASD, we adopt a more formal style, to illustrate the way in which the rules of the previous section are used. Nevertheless, plenty of logical details have been abbreviated, so it would be a very good exercise to make them explicit, in particular formalising the idiomatic use of "there exists" using Axiom 5.9 and [Tay99, §1.6].

Theorem 6.13 Every Cauchy sequence in $\mathbb{R}$ converges to a unique limit.

Proof The limit a will be defined as a Dedekind cut (δ, υ), where

$$\delta d \;\equiv\; \exists n{:}\mathbb{N}.\ (d < a_n - 2^{-n}) \quad \text{and} \quad \upsilon u \;\equiv\; \exists n{:}\mathbb{N}.\ (a_n + 2^{-n} < u).$$

These are easily seen to be lower and upper respectively (using transitivity of $<$ and the Frobenius law) and rounded (using interpolation). They are bounded by any $d < a_0 - 1 < a_0 + 1 < u$. For disjointness, given d, u, we have

$$\delta d \wedge \upsilon u \;\equiv\; \exists n n'{:}\mathbb{N}.\ (d < a_n - 2^{-n}) \;\wedge\; (a_{n'} + 2^{-n'} < u),$$

so let $m \equiv \mathsf{min}(n, n')$, whence $|a_n - a_{n'}| < 2^{-m} < 2^{-n} + 2^{-n'}$ and $u - d > 0$.

Locatedness is usually the most difficult thing to prove, as we found when constructing $\mathbb{R}$ itself and defining addition and multiplication on it [I]. We need the Archimedean principle (Axiom 6.8), in the form that

$$(d < u) \;\Rightarrow\; \exists n{:}\mathbb{N}.\ (u - d > 2^{2-n}).$$

Then, using Example 4.6 and Axioms 5.5 and 5.9, we deduce successively that

$$\begin{array}{lll}
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot & \vdash & (\exists n.\, d < a_n - 2^{-n} \vee a_n + 2^{-n} < u) \;\Rightarrow\; \bot \\
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot,\ n : \mathbb{N} & \vdash & \qquad (d < a_n - 2^{-n} \vee a_n + 2^{-n} < u) \;\Rightarrow\; \bot \\
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot,\ n : \mathbb{N} & \vdash & d \geq a_n - 2^{-n} \quad \text{and} \quad a_n + 2^{-n} \geq u \\
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot,\ n : \mathbb{N} & \vdash & u - d \;\leq\; (a_n + 2^{-n}) - (a_n - 2^{-n}) \;<\; 2^{2-n} \\
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot,\ n : \mathbb{N} & \vdash & \qquad\qquad\qquad (u - d > 2^{2-n}) \;\Rightarrow\; \bot \\
\ldots,\ (\delta d \vee \upsilon u) \Leftrightarrow \bot & \vdash & (d < u) \;\Rightarrow\; (\exists n.\, u - d > 2^{2-n}) \;\Rightarrow\; \bot \\
\ldots & \vdash & (d < u) \;\Rightarrow\; (\delta d \vee \upsilon u).
\end{array}$$

Hence, by Dedekind completeness, $\delta d \Leftrightarrow (d < a)$ and $\upsilon u \Leftrightarrow (a < u)$ for some unique $a : \mathbb{R}$. This is a limit because

$$\ldots,\ n, m : \mathbb{N} \;\vdash\; n < m \;\Rightarrow\; a_n - a_m < 2^{-n} < 2^{1-n} - 2^{-m} \;\Rightarrow\; a_n - 2^{1-n} < a_m - 2^{-m},$$

so, putting $m \equiv n + 1$,

$$\ldots,\ n : \mathbb{N} \;\vdash\; \top \;\Leftrightarrow\; (\exists m.\, a_n - 2^{1-n} < a_m - 2^{-m}) \;\equiv\; \delta(a_n - 2^{1-n}) \;\equiv\; (a_n - 2^{1-n} < a),$$

and similarly $a < a_n + 2^{1-n}$. Finally, it is unique because, if $\ldots \vdash a, b : \mathbb{R}$ satisfy

$$\ldots,\ n : \mathbb{N} \;\vdash\; \top \;\Leftrightarrow\; a_n - 2^{1-n} < a, b < a_n + 2^{1-n} \;\Rightarrow\; |a - b| < 2^{2-n},$$

then

$$\ldots \;\vdash\; (\exists n.\, |a - b| > 2^{2-n}) \;\Leftrightarrow\; \bot,$$

so $a = b$ by the Archimedean principle and Definition 4.6. $\square$

Remark 6.14 Other treatments of constructive analysis, most notably Errett Bishop's [BB85], *define* $\mathbb{R}$ in terms of Cauchy sequences of rationals, but this approach leads to a heavily *metrical* treatment of analysis. By contrast, in ASD we can do things *topologically*, as we shall see. Indeed, in the drafting of this paper and [I], I had developed the *analysis* as far as the intermediate value theorem using the *order* on $\mathbb{Q}$ and $\mathbb{R}$ before considering their *arithmetic* at all.

We can also take a more topological view than Remark 6.10 of the translation from Dedekind cuts to Cauchy sequences. Bearing in mind the unavoidable non-determinacy, one popular representation, called ***signed binary***, is $\sum_n d_n 2^{-n}$, where $d_n \in \{-1, 0, +1\} \equiv \mathbf{3}$. We want the evaluation map $\mathbf{3}^{\mathbb{N}} \to [-2, +2]$ to be *surjective*, rather than an isomorphism. In fact, it is a *proper surjection*, *i.e.* the inverse image of any $a : [-2, +2]$ is an *occupied compact* subspace (Definition 8.5 and Remark 8.9) of $\mathbf{3}^{\mathbb{N}}$. But the calculus does not provide function-spaces like this (essentially, ***Cantor space***) for free, so we have first to *construct* it. This will be done in future work.

Remark 6.15 Returning to computation, why is every computable continuous function $f : \mathbb{R} \to \mathbb{R}$ representable in ASD? Any such *continuous* function is determined by the open subset of rationals (d, q, u) such that $d < f(q) < u$. Any *program* that approximates f in any of the senses that we have discussed in this section can be adapted to one that terminates on input (d, q, u) iff $d < f(q) < u$. Since general recursion, and indeed the denotational semantics of a modern programming language, can be expressed in ASD, so can this modified program. As it yields Dedekind cuts, it defines a term $g : \mathbb{R} \to \mathbb{R}$, whose denotation is the given f. $\square$

The other two limiting operations of elementary analysis, derivatives and integrals, are also naturally defined as Dedekind cuts, and will be expressed in our calculus in Section 10. Section 12 shows that $\mathbb{R}$ is complete in an order-theoretic sense by constructing the maximum of a nonempty compact overt subspace. Definition by description and Dedekind completeness may be seen as examples of a more general idea called *sobriety*: see [I, Section 14] and [A].

7 Open and general subspaces

Whereas there is only one kind of "subset" in set theory, in topology we need to distinguish between open and other kinds of subspace, and this section introduces our notation for them. Following Proposition 3.8, we treat open subspaces of X as *functions* $X \to \Sigma$, whilst we adapt the familiar notation of set theory to *general* subspaces.

As with Section 5, you may wonder why it is necessary to spell out this kind of formalism when you were already perfectly capable of reasoning with open subspaces in analysis. It is because, as we saw in Sections 2 and 3, we need to treat them as first class objects, indeed as points of another topological space, which we called Σ^X and endowed with the Scott topology. Also, setting out the rules of our notation explicitly is the first step towards using computation instead of set theory as the foundations for analysis.

This notation, called the ***typed*** λ ***(lambda)-calculus***, exploits the correspondence in Theorem 3.18. Whilst this is little more than a syntactical operation in set theory or functional programming, in topology this Theorem depends crucially on the correct definitions of local compactness and of the Scott topology. These ideas are therefore implicitly built in to the syntax.

When ordinary mathematicians talk about "functions", they write them as expressions $f(x)$ in which one particular variable x plays a distinguished role, although this may actually occur many times, or not at all, whilst other variables (parameters) may also be present. So far, we have followed this informal custom, using "propositional expressions" like $\phi(x)$, even in the more formal settings of the defining Axioms 5.9, 6.2 and 6.6 for the existential quantifier, descriptions and Dedekind cuts, and of Lemmas 5.8 and 5.11 about (in)equality in discrete and Hausdorff spaces. However, this custom will no longer be adequate in the next section, where we start to develop the modal operators $\square$ and $\lozenge$ from Section 2 in our new calculus, since they take ϕ as an *argument*, regarding it *as a thing in itself*, either as a variable or defined from a particular expression.

The notation $\lambda x.\, f(x)$ *abstracts* the function f as a thing in itself from the expression $f(x)$. Then, whereas x may have been scattered throughout the expression $f(x)$, it occurs exactly once in $\big(\lambda x.\, f(x)\big)x$, at the end, so we may write fx without the brackets instead. However, we shall only use this for *propositional* expressions, not for functions of general type.

Axiom 7.1 For any *proposition*

$$\ldots,\ x_1 : X_1, \ldots, x_n : X_n \ \vdash\ \sigma,$$

possibly containing variables $x_1, \ldots, x_n$ of any type, and maybe other variables ("parameters") from the list $\ldots$, we may form the ***predicate***

$$\ldots\ \vdash\ \lambda x_1, \ldots, x_n.\, \sigma,$$

in which $x_1, \ldots, x_n$ are ***bound*** by ***lambda abstraction***, but the parameters (the other variables in the list $\cdots$) remain ***free***. Binding of variables is not new with the λ-calculus: it is already familiar in both the existential quantifier $\exists x.\, \phi(x)$ (Axiom 5.9) and the definite integral $\int_a^b f(x)\,\mathsf{d}x$; see [Tay99, §1.1] for an introduction. As you already know from these cases, we may give the variable x some new name throughout the expression, without changing its meaning; this is called ***α-equivalence***.

Usage of the λ-calculus is different from the informal notation for functions. Consider, for example, the expression $\phi \ \equiv\ \lambda y.\, |fy - fx| < \epsilon$ that we shall use in Theorem 10.5. In the first stage of that argument, we shall want to draw attention to the role of y, whilst the parameters x, ϵ and f will become relevant later. In the informal usage we might write $\phi_{x,\epsilon,f}(y)$ for this expression, but this doesn't make it clear that the first stage is about the function $y \mapsto \big(|fy - fx| < \epsilon\big)$ as a thing in itself. In the λ-calculus, ϕ stands for the function in itself, rather than an expression involving y. However, "ϕ" itself is not a variable: it stands for an expression in which the variables x and ϵ may occur, as may copies of the expression for which f stands.

Axiom 7.2 If $\lambda\vec{x}.\, \sigma$ is to be a first class citizen then it must have a *type*. This is

$$\Sigma^{X_1 \times \cdots \times X_n} \qquad \text{or} \qquad X_1 \to \cdots \to X_n \to \Sigma,$$

where the first notation is used in mathematics and the second in computer science. The convention is that $X \to Y \to \Sigma$ means $X \to (Y \to \Sigma)$.

Turning from symbolic manipulation to topology, the next result is crucial to the claim that every expression in the calculus denotes a continuous function.

Proposition 7.3 If the denotation (in the sense of Definition 5.4) of

$$\Gamma,\ x_1 : X_1, \ldots, x_n : X_n \ \vdash\ \sigma \quad \text{is a continuous function} \quad [\![\Gamma]\!] \times [\![X_1]\!] \times \cdots \times [\![X_n]\!] \longrightarrow \Sigma$$

then, using the downward correspondence in Theorem 3.18, the denotation of

$$\Gamma\ \vdash\ \lambda x_1 \ldots x_n.\, \sigma \quad \text{is a continuous function} \quad [\![\Gamma]\!] \longrightarrow \Sigma^{[\![X_1]\!] \times \cdots \times [\![X_n]\!]},$$

where the products are given the usual Tychonov topology and $\Sigma^{[\![X_1]\!] \times \cdots \times [\![X_n]\!]}$ the Scott topology. □

The name that we give to the upward direction is ***application***:

Axiom 7.4 Given any term ϕ of type $\Sigma^{X_1 \times \cdots \times X_n}$, along with terms ("arguments") $a_1 : X_1$, ..., $a_n : X_n$ of the corresponding types, any of which may contain parameters, we may ***apply*** the predicate to the arguments. This yields the proposition $\phi a_1 a_2 \ldots a_n$, where ϕab means $(\phi a)b$.

Proposition 7.5 If the denotations of ϕ, a_1, ..., a_n are continuous functions then so is that of $\phi a_1 a_2 \ldots a_n$.

Proof By Proposition 3.16, ev is jointly continuous. $\square$

Finally, we need to say — in the syntax itself, and not just its denotation — that the two directions of the correspondence are mutually inverse.

Axiom 7.6 When a lambda-abstraction $\phi \equiv \lambda \vec{x}.\, \sigma$ is applied to arguments, the result is equivalent to substitution of those arguments for the abstracted variables:

$$(\lambda x_1{:}X_1, \ldots, x_n.\, \sigma)a_1 a_2 \ldots a_n \;\Leftrightarrow\; [a_1/x_1, \ldots, a_n/x_n]^*\sigma.$$

This is called the ***β-rule***. In ***functional programming***, it has proved to be a very profitable idea to give this *equation* a *direction*, so it is a *computation step*, known as ***β-reduction***.

The (meta)notation $[\cdots]^*\sigma$ for substitution is a *syntactic transformation*; [Tay99] explains what the star means. Beware that, if you don't take care over renaming bound variables, substituting an argument with a free variable x within a sub-expression bound by λx leads to nonsense; this problem may be avoided by renaming bound variables. With the same caveat, substitution commutes with application and abstraction. These issues are explained in any textbook on the λ-calculus, such as [Bar81], [GLT89] or [Tay99, §4.7], but in this paper we shall just treat β as an equation, as we get away with doing very little actual substitution.

Axiom 7.7 The converse is the ***η-rule***: we recover the given function or predicate if we apply it to *fresh* variables and then abstract them:

$$\lambda x_1, \ldots, x_n.\, \phi x_1 \ldots x_n \;=\; \phi,$$

this being an equational statement of type $\Sigma^{X_1\times\cdots\times X_n}$ (Definition 4.5).

Proposition 7.8 The correspondence in Theorem 3.18 obeys the β- and η-rules, so it is an isomorphism. $\square$

Although we have said that we may only λ-abstract propositions (terms of type Σ), we have not yet made use of the logical structure of Σ.

Notation 7.9 If the propositions $\sigma, \tau : \Sigma$ are related by $\sigma \Rightarrow \tau$ or $\sigma \Leftrightarrow \tau$ then (as we have just done in the previous Axiom) we write $\lambda\vec{x}.\,\sigma \leqslant \lambda\vec{x}.\,\tau$ or $\lambda\vec{x}.\,\sigma = \lambda\vec{x}.\,\tau$ for the corresponding relationships between the predicates. So $\leqslant$ is really the same as $\Rightarrow$: we just use two symbols for clarity, distinguishing between propositions (type Σ) and predicates (type Σ^X).

Semantically, $\leqslant$ and $=$ are *inclusion* or equality of open subspaces, but *containment* or equality of the closed ones. The different directions of the containment of subspace is one reason for not using the symbol $\subset$, another being that some logicians use $\supset$ in place of $\Rightarrow$.

The logical connectives ($\wedge$ and $\vee$) and quantifiers ($\exists$ and $\forall$) also extend from propositions to predicates, and they commute with abstraction and application. Semantically, $\wedge$ and $\vee$ correspond to intersection and union respectively of open subspaces, and *vice versa* for closed subspaces.

The order on predicates induces the ***specialisation order*** on the underlying spaces, *cf.* Remark 3.9:

$$x \leqslant y \quad\equiv\quad \lambda\phi.\,\phi x \leqslant \lambda\phi.\,\phi y.$$

This order is important in domain theory, where it is often written $\sqsubseteq$, but it will not be used in this paper, as it is trivial for discrete and Hausdorff spaces.

Remark 7.10 The language of introduction, elimination, β- and η-rules may be used to describe the rules of inference for the other logical symbols, and even for other "universal properties" in mathematics [Tay99]. For example, $=_N$ and "the" for descriptions play the role of application and abstraction in the λ-calculus, with β- and η-rules

$$\phi m \Leftrightarrow (m =_N \mathsf{the}\; n.\, \phi n) \qquad\text{and}\qquad (\mathsf{the}\; n.\, n =_N m) \;=\; m : N.$$

Dedekind completeness is similarly based on $<, >$ and the operator "cut", with rules

$$\delta d \wedge \upsilon u \Leftrightarrow \big(d < \mathsf{cut}(\delta, \upsilon) < u\big) \qquad \text{and} \qquad \mathsf{cut}(\lambda d.\, d < a, \lambda u.\, a < u) \;=\; a : \mathbb{R}.$$

Notice that, in all three calculi, the β-rule substitutes part of the surrounding expression for the bound variable(s).

Since we plan to develop real analysis, rather than formal logic or programming languages, that is as far as we need to go with the syntax of the λ-calculus.

As predicates (λ-expressions) may be used to define both open and closed subspaces, we need further notation that distinguishes between these, and that can also be used for compact and other kinds of subspace. We shall adopt the familiar symbolism of set theory to do this, although it is important to appreciate that we are merely using the notation and not the axioms or methods of set theory. Also, in this paper, we shall only make very light use of this notation.

Notation 7.11 We define a ***subspace*** of a space X by putting a *statement* (Definition 4.5) on the right of the divider in the notation

$$\{x : X \mid \phi x \Rightarrow \psi x\}.$$

Then if $a : X$ is a term of type X, we say that it is a ***member*** or ***element*** of the subspace,

$$a \in \{x : X \mid \phi x \Rightarrow \psi x\}, \quad \text{if the statement} \quad \phi a \Rightarrow \psi a$$

holds, as in the familiar idiom of ***comprehension***. This is yet another example of Remark 7.10.

Membership or elementhood of a general subspace is therefore another kind of *statement*, as we would expect given that the distinction between open and general subspaces is the same as that between propositions and statements.

If $\phi, \psi : \Sigma^X$ are predicates without parameters, the *denotation* of $\{x : X \mid \phi x \Rightarrow \psi x\}$ is the same as that of the context $[x : X,\ \phi x \Rightarrow \psi x]$ in Definition 5.4. The notation may therefore be extended to allow any list $x_1 : X_1, \ldots, x_n : X_n$ of variables on the left of the divider. In this case we obtain a subspace of the product $X_1 \times \cdots \times X_n$, so tuples $(a_1, \ldots, a_n)$ are eligible for membership of the subspace. Other kinds of statements ($\phi x \Leftrightarrow \psi x$, $\phi x \leqslant \psi x$ and $fx = gx$) may also be used on the right of the divider.

As with contexts, this denotation need not be a locally compact space, so the function-space $\Sigma^{\{x:X \mid \phi x \Rightarrow \psi x\}}$ need not exist. For this reason,

we do not introduce $\{x : X \mid \phi x \Rightarrow \psi x\}$ as a new *type* in the calculus.

In this paper, we shall just use the set notation (and the associated $\subset$ relation) in a *suggestive* way, in those specific cases where we have some other justification for introducing the subspace as a type, for example because it is (isomorphic to) an open or closed one. In all cases, the subspace notation can be eliminated syntactically.

If its meaning is so weak, why do we both introducing the notation at all? It is so that, in Sections 8 and 11, we can say how open or closed subspaces defined by predicates and compact or overt ones defined by modal operators are "the same subspace". The formal content of this that that any expression is a member (*i.e.* satisfies the defining statement) of one subspace iff it does so for the other. In particular, we shall want to say in ASD that any compact subspace of a Hausdorff space such as $\mathbb{R}$ is also closed (Proposition 8.6).

The one potential problem that this usage will create in this paper is with *overt* subspaces that are neither open nor closed, arising from singular $\lozenge$-operators (Section 2). However, the worst example that I have been able to find (Remark 16.14) is locally closed, *i.e.* an open subspace of a closed one.

So, let's see how open and closed subspaces are expressed.

Definition 7.12 The predicate $\phi : \Sigma^X$ gives rise to both

$$\text{the } \textbf{\textit{open subspace}} \ \{x : X \mid \phi x \Leftrightarrow \top\} \quad \text{and} \quad \text{the } \textbf{\textit{closed subspace}} \ \{x : X \mid \phi x \Leftrightarrow \bot\},$$

which we sometimes abbreviate to $\{x : X \mid \phi x\}$ and $\{x : X \mid \neg\phi x\}$.

For any term $a : X$, we then have, as in Definition 4.5,

$$\begin{aligned} a \in \{x : X \mid \phi x \Leftrightarrow \top\} \quad &\text{iff} \quad \phi a \ \Leftrightarrow \ \top \\ a \in \{x : X \mid \phi x \Leftrightarrow \bot\} \quad &\text{iff} \quad \phi a \ \Leftrightarrow \ \bot. \end{aligned}$$

Remark 7.13 Whilst we cannot introduce *general* subspaces as *types* in the calculus, we can do so for open or closed subspaces. This is because their open subsubspaces correspond in a canonical way to open subspaces of the original space X. With $U \equiv \{x : X \mid \phi x \Leftrightarrow \top\}$, any open $V \subset U \subset X$ is already open in X, whilst relatively open $V \subset \{x : X \mid \phi x \Leftrightarrow \bot\}$ are represented by open $U \cup V \subset X$. Hence the topologies $\Sigma^{\{x:X|\phi x\}}$ and $\Sigma^{\{x:X|\neg\phi x\}}$ are retracts of Σ^X. See [I, Section 5] for more detail.

As we shall usually work inside $\mathbb{R}$ as a "universal" space, it is convenient to represent open subspaces of other subspaces as $\psi : \Sigma^{\mathbb{R}}$, subject to an equivalence relation that says when $\psi, \theta : \Sigma^{\mathbb{R}}$ restrict to the same thing on the subspace:

$$\psi \sim \theta \quad \equiv \quad \psi \wedge \phi = \theta \wedge \phi \qquad \text{in} \qquad \Sigma^{\{x:X|\phi x\}}$$

or

$$\psi \sim \theta \quad \equiv \quad \psi \vee \phi = \theta \vee \phi \qquad \text{in} \qquad \Sigma^{\{x:X|\neg\phi x\}},$$

which we shall use in Remark 8.15 and Definition 13.1. $\square$

As a typical example of the way in which we shall use the general subspace notation in this paper, we can show that any *function* (Definition 5.2) is *continuous* in the sense that the inverse image of any open or closed subspace is again open or closed respectively.

Definition 7.14 The ***inverse image*** of a predicate $\psi : \Sigma^Y$ along a function $f : X \to Y$ is given by composition. There are several notations for this, coming from different disciplines:

$$f^*\psi \ \equiv \ f^{-1}\psi \ \equiv \ \Sigma^f\psi \ \equiv \ (f\,;\psi) \ \equiv \ (\psi \cdot f) \ \equiv \ \lambda x.\, \psi(fx) : \Sigma^X.$$

The corresponding open and closed subspaces of X are

$$f^{-1}\{y : Y \mid \psi y\} \ \equiv \ \{x : X \mid \psi(fx)\} \quad \text{and} \quad f^{-1}\{y : Y \mid \neg\psi y\} \ \equiv \ \{x : X \mid \neg\psi(fx)\}.$$

Lemma 7.15 These are indeed the "inverse images" in a sense that is consistent with our definitions of membership for $a : X$:

$$\frac{a \in f^{-1}\{y : Y \mid \psi y\}}{fa \in \{y : Y \mid \psi y\}} \quad \text{and} \quad \frac{a \in f^{-1}\{y : Y \mid \neg\psi y\}}{fa \in \{y : Y \mid \neg\psi y\}} \qquad \square$$

Whilst open and closed subspaces have *inverse* but not direct images under general continuous functions, we shall see in Theorem 8.8 and Notation 11.11 respectively that it is the other way round for compact and overt ones.

Remark 7.16 We may also use the subspace notation to define $\mathbb{N}^X$ and $\mathbb{R}^X$ (and in particular $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{R}}$). The idea is to encode numbers in logic as descriptions or Dedekind cuts:

$$\mathbb{N}^X \ \equiv \ \{\phi : \Sigma^{X\times\mathbb{N}} \mid \top \ = \ \lambda x.\, \exists n.\, \phi xn \quad \& \quad \lambda xnm.\, (\phi xn \wedge \phi xm) \ \leqslant \ \lambda xnm.\, (n =_{\mathbb{N}} m)\}$$

and similarly $\mathbb{R}^X \ \equiv \ \{\delta, \upsilon : \Sigma^{X\times\mathbb{Q}} \mid (\delta, \upsilon) \text{ is a Dedekind cut}\}$.

If X is definable as a type in our calculus, so its denotation is locally compact, then these subspaces obey the rules that we expect for exponentials, and their denotations carry the usual compact–open topology. However, the latter is *not* locally compact (unless X is finite), so we cannot form $\Sigma^{\mathbb{N}^X}$ or $\Sigma^{\mathbb{N}^X}$.

Remark 7.17 At first sight, these limitations of the calculus arise from Proposition 3.16, and are insuperable.

However, that assumes that the denotation must take values within the *traditional* category of objects with finite intersections and infinite unions. We want to study *topology*, not these objects that have usurped the title of "topological spaces". In fact, various other categories have been described in the literature, for example [BBS04, Hyl79, MS02, Ros00], that have all subspaces and function-spaces (*cf.* Theorem 3.18), whilst containing the traditional category, or at least its countably based objects.

In these categories, $\mathbb{N}$ and $\mathbb{R}$ have all of the properties (algebra, general recursion, definition by description, Dedekind completeness and the Heine–Borel property) that we discuss in this paper. However, they do not enjoy the *generalisations* of these ideas (sobriety [A], and monadicity [B], of which the Heine–Borel property is a special case [I]) that the more abstract parts of ASD advocate. On the other hand, these may be added by a further construction [B], although it is not clear at the moment that this preserves subspaces and function-spaces.

Finding the right extension of the existing calculus beyond local compactness is therefore very much an open question. I believe that this should be tackled by pursuing the *logic* that is developed in this paper, leaving behind the traditional categories and their modifications if these conflict with it. In particular, the *subspaces* introduced above should all be legitimate types, and the *statements* that are used to form them should allow nested implications. However, the most difficult question is to ensure that subspaces have "the right" topology (power of Σ), whatever that means. If we don't settle that, we shall have a higher-order repetition of the failure of the Heine–Borel theorem in recursive analysis.

8 Abstract compact subspaces

Although our abstract re-axiomatisation is not quite complete, we now have enough tools to fashion an account of the usual properties of compact subspaces based on the modal operator $\Box$ from Section 2.

As in the traditional development, it seems to be easier to begin with compact *subspaces* of a *Hausdorff* space, and then consider compact *spaces* as a special case. Recall from Definition 4.8 that a space H is Hausdorff if it carries an "inequality" predicate $\neq$.

In Section 6 it was enough to refer to the two open subsets δ and υ as *propositional formulae.* However, following Notation 2.11, we shall now need to use a "general" or "arbitrary" open subspace ϕ in a test of the form $\Box\,\phi$ to examine compact subspaces. This is why we needed λ-abstraction.

Previously, we wrote $\Box\,U$ for the *property* that a compact subspace K is contained in an open one U. Now we shall allow $\Box$ to stand for a *expression* that may quite naturally contain variable parameters. This will enable us to treat abstract compact sub*spaces* as intermediate expressions in the course of calculations that have real-valued inputs and outputs, such as the solution of polynomial equations. Working with higher order expressions is much simpler, both notationally and conceptually, than trying to understand what it means for a compact *set* K to depend *continuously* on parameters.

Definition 8.1 An ***abstract compact subspace*** of a Hausdorff space H is defined by its ***necessity modal operator***. This an expression $\Box$, possibly containing parameters, that must yield a proposition (term of type Σ) when applied to an open subset (term of type Σ^H). Hence by Axiom 7.2 its own type has to be Σ^{Σ^H}.

As in Notation 2.11, it is to satisfy, for any predicate-expression $\phi : \Sigma^H$,

$$\frac{\phi \vee \omega \;=\; \top}{\Box\,\phi \;\Leftrightarrow\; \top} \qquad \text{where} \quad \omega x \;\equiv\; \Box(\lambda y.\, x \neq y).$$

If we want to name the subspace, we write $[K]\phi$ instead of $\Box\,\phi$. The intended meaning is that the compact subspace K represented by $\Box$ is *contained in* (or *covered by*) the open subspace classified by ϕ. We shall explain ω shortly; it corresponds to W in Notation 2.11.

Proposition 8.2 Necessity operators satisfy, for $a : H$, $\sigma : \Sigma$, $\phi, \psi : \Sigma^H$ and $\theta : \Sigma^{H \times H}$,
(a) preservation of meets: $\Box\,\top \;\Leftrightarrow\; \top$ and $\Box(\phi \wedge \psi) \;\Leftrightarrow\; \Box\,\phi \wedge \Box\,\psi$,
(b) the (dual) ***Frobenius law***, $\Box(\lambda x.\, \sigma \vee \phi x) \;\Leftrightarrow\; \sigma \vee \Box\,\phi$,
(c) ***relative instantiation***, $\Box\,\phi \;\Rightarrow\; \phi a \vee \omega a$,
(d) substitution inside any expression to which $\Box$ is applied,
(e) commutativity, $[K_1]\big(\lambda x.\, [K_2](\lambda y.\, \theta xy)\big) \;\Leftrightarrow\; [K_2]\big(\lambda y.\, [K_1](\lambda x.\, \theta xy)\big)$, and
(f) uniqueness: if $\Box$ and $\blacksquare$ both satisfy the definition for the same ω then $\Box = \blacksquare$.

Proof [a] $\top \vee \omega = \top$, whilst $\phi \vee \omega = \psi \vee \omega = \top$ iff $(\phi \vee \omega) \wedge (\psi \vee \omega) = \top$ iff $(\phi \wedge \psi) \vee \omega = \top$ by distributivity. [b] Putting $F\sigma \equiv \Box(\lambda x.\, \sigma \vee \phi x)$ in Proposition 5.7, we have

$$F\top \;\equiv\; \Box(\lambda x.\, \top \vee \phi x) \;\Leftrightarrow\; \Box(\lambda x.\, \top) \;\equiv\; \Box\,\top \;\Leftrightarrow\; \top$$

and
$$F\bot \;\equiv\; \Box(\lambda x.\, \bot \vee \phi x) \;\Leftrightarrow\; \Box(\lambda x.\, \phi x) \;\equiv\; \Box\,\phi,$$

so
$$\Box(\sigma \vee \phi) \;\equiv\; F\sigma \;\Leftrightarrow\; F\bot \vee \sigma \wedge F\top \;\Leftrightarrow\; \Box\,\phi \vee \sigma \wedge \top \;\Leftrightarrow\; \Box\,\phi \vee \sigma.$$

[c] Relative instantiation is the upward direction of the defining rule and [d] the substitution property is simply that for λ-application (Axiom 7.4). [e] Either side of the commutative law is $\top$ iff

$$x, y : H,\ \omega_1 x \Leftrightarrow \omega_2 y \Leftrightarrow \bot \;\vdash\; \theta xy \;\Leftrightarrow\; \top.$$

[f] For any ϕ, by hypothesis $\Box\,\phi \Leftrightarrow \top \dashv\vdash \blacksquare\phi \Leftrightarrow \top$, so $\Box\,\phi \Leftrightarrow \blacksquare\phi$, by Axiom 5.5. $\Box$

Definition 8.3 If $\Box$ is covered by ϕ then ϕ is called a ***neighbourhood*** of $\Box$. A point $a : H$ (by which we mean an expression that may contain parameters) belongs to all such neighbourhoods ϕ if

$$\Box\,\phi \;\Rightarrow\; \phi a, \quad \text{or} \quad \Box \;\leqslant\; \lambda\phi.\, \phi a : \Sigma^{\Sigma^H}$$

using Notation 7.9. This agrees with

$$a \in K, \qquad \text{where} \qquad K \;\equiv\; \{x : H \mid \Box \leqslant \lambda\phi.\, \phi x\}$$

is defined using Notation 7.11.

Example 8.4 Any point $a : H$ defines a compact subspace $K \equiv \{a\}$, with $[K]\phi \equiv \phi a$ and $\omega x \equiv (x \neq a)$. (This uses Lemma 5.8.) Then $[K]$ preserves meets and joins, *cf.* Proposition 2.12(d). $\Box$

Definition 8.5 We say that a compact subspace K is ***occupied*** if $[K]\bot \Leftrightarrow \bot$, whereas $[K]\bot \Leftrightarrow \top$ iff $K \cong \emptyset$, *cf.* Proposition 2.12(c).

Any compact subspace that has a definable point is occupied, but not conversely. For example, we shall see that any function $\mathbb{I} \to \mathbb{R}$ attains its bounds (or any particular intermediate value) on an *occupied subspace* (Propositions 12.13 and 13.3), but not necessarily at any *point* of it (Example 16.16). To appreciate the choice of word, imagine going into a hotel room to find someone else's luggage already there, but not the actual person. The more familiar notion of *inhabitedness* says that a point *exists*, so it will be related to $\exists$ and $\Diamond$ for overt subspace in Definition 11.7.

Proposition 8.6 Any compact subspace of a Hausdorff space is closed.

Proof Since Notation 7.11 does not give an *â priori* notion of general subspace that can be closed or compact as a secondary property, what we mean by this is that: any expression $a : H$ is a member of $\Box$ (in the sense that we have just given) iff it is a member (in the sense of Definition 7.12) of the closed subspace co-classified by ω.

In English, the formula $\omega x \equiv \Box(\lambda y.\, x \neq y)$ in Definition 8.1 reads: "x belongs to the complementary open subspace (called W in Section 2) iff, necessarily for any y in the subspace, x is separated from y". The symbol $\Box$ says "necessarily, in the compact subspace", whilst $\neq$ denotes separation in a Hausdorff space (Definition 4.8).

Suppose that $a \in \Box$. Putting $\phi \equiv \lambda x.\, (x \neq a)$ in the first form of Definition 8.3,

$$\omega a \;\equiv\; \Box(\lambda x.\, x \neq a) \;\Rightarrow\; (\lambda x.\, x \neq a)a \;\Leftrightarrow\; (a \neq a) \;\Leftrightarrow\; \bot$$

(using β-reduction, Axiom 7.4). Conversely, if $\omega a \Leftrightarrow \bot$ then $\Box\phi \;\Rightarrow\; \phi a$ for any $\phi : \Sigma^H$, by Proposition 8.2(c). $\Box$

Examples 8.7 The empty subspace is compact, as are binary unions and intersections of compact subspaces of a Hausdorff space.

K	ωx	$[K]\phi$
$\emptyset$	$\top$	$\top$
$K_1 \cup K_2$	$\omega_1 x \wedge \omega_2 x$	$[K_1]\phi \wedge [K_2]\phi$
$K_1 \cap K_2$	$\omega_1 x \vee \omega_2 x$	$[K_1]\big(\phi \vee \omega_2\big) \;\equiv\; [K_2]\big(\phi \vee \omega_1\big)$

Again, the lack of a *general* notion of subspace with which to compare them obliges us to be careful when talking about "unions and intersections of subspaces" (*cf.* Section 7). Doing so amongst *closed* subspaces, this is Definition 4.5, and in view of the equivalence of closed and compact subspaces (Theorem 8.14 below) of either K_1 or K_2, we can equally well say this amongst *compact* subspaces. The "intersection" is also justified in that a term $a : X$ is a *member* of $K_1 \cap K_2$ iff it is a member of both K_1 and K_2. The next result justifies the name of $K_1 \cup K_2$, since it is the direct image of $K_1 + K_2$. $\Box$

Theorem 8.8 Let $f : X \to Y$ be a function (Definition 5.2) between Hausdorff spaces, and let $[K]$ be a compact subspace of X. Then

$$[fK]\psi \;\equiv\; [K](f^*\psi) \;\equiv\; [K]\big(\lambda x.\, \psi(fx)\big)$$

is a compact subspace of Y, called the ***direct image*** of $[K]$. If $a : X$ is a member of $K \subset X$ then $fa : Y$ is a member of $fK \subset Y$. If K is occupied, so is fK.

Proof Since X and Y are Hausdorff, we may define

$$\xi x \;\equiv\; [K](\lambda x'.\, x \neq x') \quad \text{and} \quad \omega y \;\equiv\; [fK](\lambda y'.\, y \neq y') \;\equiv\; [K](\lambda x.\, y \neq fx).$$

By hypothesis, $[K]\phi \Leftrightarrow \top$ iff $\phi \vee \xi = \top : \Sigma^X$, and we have to show that $[fK]\psi \Leftrightarrow \top$ iff $\psi \vee \omega = \top : \Sigma^Y$.

$$\begin{array}{rcll}
(\psi \vee \omega)(fx) & \equiv & \psi(fx) \vee [K](\lambda x'.\, fx \neq fx') & \text{def } \omega \\
& \Rightarrow & \psi(fx) \vee [K](\lambda x'.\, x \neq x') & \text{Lemma 5.8} \\
& \equiv & (\psi \cdot f \vee \xi)x & \text{def } \xi \\
(\psi \vee \omega)(y) & \Leftrightarrow & \psi y \vee [K](\lambda x.\, fx \neq y) & \text{def } \omega \\
& \Leftrightarrow & [K](\lambda x.\, \psi y \vee fx \neq y) & \text{Proposition 8.2(b)} \\
& \Leftrightarrow & [K]\big(\lambda x.\, \psi(fx) \vee fx \neq y\big) & \text{Lemma 5.8} \\
& \Leftarrow & [K](\lambda x.\, \psi(fx)\big) \;\equiv\; [fK]\psi & \text{Axiom 5.6}
\end{array}$$

From the first part, if $\psi \vee \omega = \top : \Sigma^Y$ then $(\psi \cdot f \vee \xi) = \top : \Sigma^X$, so $[K](\psi \cdot f) \equiv [fK]\psi \Leftrightarrow \top$. Conversely, from the second, if $[fK]\psi \Leftrightarrow \top$ then $(\psi \vee \omega) = \top : \Sigma^Y$. If $a : X$ with $[K]\phi \Rightarrow \phi a$ then $[fK]\psi \equiv [K](\psi \cdot f) \Rightarrow (\psi \cdot f)a \equiv \psi(fa)$. Finally,

$$[fK]\bot \;\equiv\; [K](\bot \cdot f) \;\equiv\; [K]\bot \;\Leftrightarrow\; \bot. \qquad \square$$

Remark 8.9 Recall that Definition 7.14 for the *inverse* image of an open or closed subspace was a two-way rule, whereas we only have one direction here, This says that there is a function $p : K \to fK$, but we also want to say that p is *surjective*. This is true in the sense that there is another function $p_\Box : \Sigma^K \to \Sigma^{fK}$ such that $p^* \cdot p_\Box = \mathsf{id}_{\Sigma^{fK}}$. However, for a member $\ldots \vdash b \in fK$, whilst the closed subspace $p^{-1}(b) \subset K$ is *occupied* (Definition 8.5), it need not have any *points* or definable members.

We now turn from subspaces to spaces.

Definition 8.10 A *space* K is ***compact*** if it has a ***universal quantifier***, written $\forall_K$, that satisfies the rule for $\Box$, together with the ***absolute instantiation rule*** for $a : K$ and $\phi : \Sigma^K$,

$$(\forall_K \phi) \;\Rightarrow\; \phi a.$$

In other words, K is a compact *sub*space of itself, of which *every* $x : K$ is a member, in the sense of Definition 8.3, so it has $\omega x \equiv \bot$.

We look at the logical meaning of this before the topological one.

Notation 8.11 We shall write $\forall k{:}K.\ \phi k$ for $\forall_K \phi$ when K is a compact *space*, and by extension also when it is a compact *subspace*, *i.e.* for $[K]\phi$. This simply means that we replace

$$\Box(\lambda k{:}K.\ -) \quad \text{by} \quad \forall k{:}K.\ -.$$

This use of logical notation is justified by Proposition 8.2 and the following ***rules of inference***.

Beware that $\forall$ *does not mean "for every definable point"*, for example when K is an occupied space without points, or even when $K \equiv \mathbb{I}$ ([I, Section 15]).

Theorem 8.12 In a compact space, $\forall_K$ satisfies the two-way rule on the left:

$$\frac{\ldots,\ x : K \;\vdash\; \sigma \;\Rightarrow\; \phi x}{\ldots \;\vdash\; \sigma \;\Rightarrow\; \forall x{:}K.\ \phi x.} \qquad\qquad \frac{\ldots,\ x : H,\ \omega x \Leftrightarrow \bot \;\vdash\; \sigma \;\Rightarrow\; \phi x}{\ldots \;\vdash\; \sigma \;\Rightarrow\; \Box\phi.}$$

where $\sigma : \Sigma$ doesn't involve the variable $x : K$. This is the dual of Axiom 5.9 for $\exists$. Similarly, in a Hausdorff space H, the modal operator $\Box$ and co-classifier ω for a compact *sub*space satisfy the rule on the right.

Proof By Axiom 5.5, the rule on the right above is

$$\frac{\ldots,\ x : H,\ \sigma \Leftrightarrow \top \;\vdash\; \top \;\Rightarrow\; \phi x \vee \omega x}{\ldots,\ \sigma \Leftrightarrow \top \;\vdash\; \top \;\Rightarrow\; \Box\phi} \quad \text{or} \quad \frac{\ldots,\ \sigma \Leftrightarrow \top \;\vdash\; \top \;=\; \phi \vee \omega : \Sigma^H}{\ldots,\ \sigma \Leftrightarrow \top \;\vdash\; \top \;\Leftrightarrow\; \Box\phi}$$

which is Definition 8.1. The other rule is the special case where $\omega x \;\equiv\; \bot$. $\square$

Lemma 8.13 In a compact Hausdorff space, $\phi x \;\Leftrightarrow\; \forall y.\ x \neq y \vee \phi y$. This is the dual of Lemma 5.11 for overt discrete spaces.

Proof

$$\begin{array}{rlll}
\ldots,\ x, y : K \;\vdash\; \phi x & \Rightarrow & x \neq y \vee \phi y & \text{Lemma 5.8} \\
\ldots,\ x \quad : K \;\vdash\; \phi x & \Rightarrow & \forall y.\ x \neq y \vee \phi y & \text{Theorem 8.12} \\
 & \Rightarrow & x \neq x \vee \phi x \;\Leftrightarrow\; \phi x & y \equiv x,\ \text{Definition 8.10}\ \square
\end{array}$$

From this we deduce the familiar topological result.

Theorem 8.14 The closed and compact spaces of a compact Hausdorff space are in bijection, with

$$\omega x \;\equiv\; \Box(\lambda y.\, x \neq y) \quad \text{and} \quad \Box\,\phi \;\equiv\; \forall_K(\phi \vee \omega) \;\equiv\; \forall x{:}K.\; \phi x \vee \omega x.$$

Proof When the ambient space is compact, the top line of the rule in Definition 8.1 is equivalent to $\forall_K(\phi \vee \omega) \Leftrightarrow \top$, so these two equations restate that Definition. Proposition 8.6 showed that the two notions of membership agree.

The translation $\omega \mapsto \Box \mapsto \omega$ is the identity by the Lemma:

$$\omega' x \;\equiv\; \Box(\lambda y.\, x \neq y) \;\equiv\; \forall_K(\lambda y.\, x \neq y \vee \omega y) \;\Leftrightarrow\; \omega x.$$

So too is that $\Box \mapsto \omega \mapsto \Box$ because

$$\Box'\,\phi \Leftrightarrow \top \;\dashv\vdash\; \forall_K(\phi \vee \omega) \Leftrightarrow \top \;\dashv\vdash\; \phi \vee \omega = \top \;\dashv\vdash\; \Box\,\phi \Leftrightarrow \top. \qquad \Box$$

Remark 8.15 Returning to logic, the significance of the deduction rule for *subspaces* (on the right in Theorem 8.12) is that we may treat $\Box$ as a ***bounded universal quantifier***.

For example, in Section 10 we shall consider, for $d, u : \mathbb{R}$ and $\phi : \Sigma^{\mathbb{R}}$,

$$\Box\,\phi \;\equiv\; [d,u]\phi \;\equiv\; \forall x{:}(d \leq x \leq u).\; \phi x.$$

That is, we reason about it as if the subspace

$$K \;\equiv\; \{x : H \mid \neg\omega x\} \;=\; \{x : H \mid \Box \leqslant \lambda\phi.\, \phi x\}$$

were actually a space in its own right (even though it may depend on parameters).

However, the ϕs that test compactness must be open subspaces of the *ambient* space H, not just of the smaller space K. This is OK, because the compact subspace K is also closed, so its open subspaces are given by Remark 7.13.

Remark 8.16 This account of compact subspaces has relied on the Hausdorffness assumption, which ought to be enough, given that we intend to study $\mathbb{R}$ and $\mathbb{R}^{\mathbf{n}}$. However, when we introduce (closed) overt subspaces in Section 11, we would like to do so in the form of the lattice ("de Morgan") dual of the theory of (open) compact subspaces (*cf.* Axiom 5.5), whereas $\mathbb{R}$ does not enjoy the *dual* property to Hausdorffness, which is discreteness. A small detour from the topology of $\mathbb{R}^{\mathbf{n}}$ is therefore appropriate.

There are ample supplies of *open* compact subspaces in

(a) Cantor space, with the "middle third" construction, where they are finite unions of "whole segments" of diameter 3^{-n} for some n;

(b) Cantor space, considered as the space of paths through an infinite binary tree, where they are determined by a finite sub-tree;

(c) $\Sigma^{\mathbb{N}}$, whose denotation is $\mathcal{P}(\mathbb{N})$ with the Scott topology, where they are determined by finite subsets.

Definition 8.17 A ***compact open subspace*** of a (not necessarily Hausdorff) space X is defined by the pair $(\Box, \alpha)$, where $\Box : \Sigma^{\Sigma^X}$ and $\alpha : \Sigma^X$ satisfy

$$\frac{\alpha \;\leqslant\; \phi}{\overline{\overline{\Box\,\phi \;\Leftrightarrow\; \top}}}$$

for any $\phi : \Sigma^X$. In particular $\Box\,\alpha \;\Leftrightarrow\; \top$.

Exercise 8.18 Show that

(a) $\Box$ satisfies Proposition 8.2, but the relative instantiation rule is $\Box\phi \wedge \alpha a \;\Rightarrow\; \phi a$, and each side of the commutative law is equivalent to $x, y : X,\ \alpha_1 x \Leftrightarrow \alpha_1 y \Leftrightarrow \top \vdash \phi xy \Leftrightarrow \top$.

(b) any term $a : X$ is a member of $\Box$ in the sense of Definition 8.3 ($\Box \leqslant \lambda\phi.\ \phi a$) iff it is a member of the open subspace classified by α ($\alpha a \Leftrightarrow \top$);

(c) the rule of inference in Theorem 8.12 becomes

$$\frac{\ldots,\ x : X,\ \alpha x \Leftrightarrow \top\ \vdash\ \sigma\ \Rightarrow\ \phi x}{\ldots\ \vdash\ \sigma\ \Rightarrow\ \Box\phi.} \qquad \Box$$

(d) if $\Box$ and $\blacksquare$ both satisfy the definition for the same α then $\Box = \blacksquare$.

Remark 8.19 You may be wondering why we have only defined compact subspaces that are also either closed or open. One reason is that, in this paper, we are primarily interested in subspaces of the Hausdorff spaces $\mathbb{R}$ and $\mathbb{R}^{\mathbf{n}}$, and, as in the classical theory, this simplifies the study of compactness. In the non-Hausdorff theory, for example, the intersection of two compact subspaces need not be compact, whilst our notion of *membership* of $\Box$ in Definition 8.3 does not necessarily recover a given compact subspace, but its *saturation*. Also, compact subspaces of locally compact but non-Hausdorff spaces need not be locally compact, *i.e.* exist in the present version of the calculus [G, §5].

Turning to the abstract characterisation of $\Box$, it is not (*â priori*) enough to say that it preserves $\top$ and $\wedge$. As we have seen, if K is a compact space with $i : K \to X$ then $\Box \equiv \forall_K \cdot \Sigma^i$ is a necessity operator on X. From the absolute instantiation rule for $\forall_K$, we deduce that $\Box$ satisfies

$$\Box\big(\lambda x.\ \Theta(x, \lambda\phi.\ \Box\phi)\big) \Rightarrow \Box\big(\lambda x.\ \Theta(x, \lambda\phi.\ \Box\phi \wedge \phi x)\big)$$

for all $\Theta : X \times \Sigma^{\Sigma^X} \to \Sigma$.

As it happens, this property follows from preservation of meets in locally compact spaces, using Scott continuity and bases. However, this route complicates the logical development of the theory, with the result that the correspondence between closed and compact subspaces of a compact Hausdorff space is stated in Section 5 of [G], but the proof is only completed in Section 9. The approach chosen above avoids these difficulties.

The extended calculus in Remark 7.17 would allow the formation of the subspace $K \equiv \{x : X \mid \Box \leqslant \lambda\phi.\ \phi x\}$, but is K actually compact (with its own $\forall_K$), even assuming the additional requirement?

9 Compactness and uniformity

What has happened to the "finite open subcover" property? Somehow, the underlying ideas of ASD are robust enough to allow us to develop the basic *results* about compactness without mentioning finiteness. On the other hand, we can only define some rather trivial *examples* of compact spaces.

We need two more axioms to complete our theory: that all functions are Scott-continuous, and that $[0,1] \subset \mathbb{R}$ is compact in the sense of the previous section. These ideas were motivated by the infinitary and finitary parts respectively of the discussion of compactness in Notation 2.11. Their abstract formulations are the subject of this section, and our goal is to show that the compact subspaces of $\mathbb{R}$ are exactly the closed and bounded ones.

Axiom 9.1 For any type X, every term $F : \Sigma^{\Sigma^X}$ is ***Scott continuous***, *i.e.* it preserves ***directed joins***.

Remark 9.2 As we explained following Axiom 5.9, a *join* indexed by $i : I$ in the space Σ^X is an existentially quantified predicate, $\lambda x.\ \exists i.\ \theta(x, i)$. For this, the object I must be overt.

Recall from Definition 3.10 that a join is *directed* if its indexing set I carries a partial order with a certain property. However, it is not, on the face of it, at all clear how to formulate this idea in ASD, since there is *no underlying set theory.* It turns out that *overt discrete* spaces play the role of sets. As we want to *impose* an order such as the arithmetical one on the indexing space I with a view to defining directedness, we don't want its topology or *intrinsic* (specialisation) order to get in the way, so I should also be discrete as a space. See [I, Definition 4.19], [G, §§3–4] and [E] for further discussion of this definition.

In fact, we don't need to worry about the general definition of directedness in this paper because, for many of the issues of real analysis, it is enough to consider joins indexed by the arithmetical order on $\mathbb{N}$ or $\mathbb{Q}$.

Definition 9.3 A ***directed relation*** on a space X is a propositional expression $\theta(x,i)$, in the sense of Definition 4.4, with two variables $x : X$ and $i : \mathbb{N}, \mathbb{Q}$ or $\mathbb{R}$, that preserves or reverses the arithmetical order in the second argument, in one or other of the senses below, which we shall call by the notation on the right:

$$\begin{array}{lcll} (i < j < \infty) \land \theta(x,i) & \Rightarrow & \theta(x,j) & \qquad i \nearrow \infty \\ (0 < i < j) \land \theta(x,j) & \Rightarrow & \theta(x,i) & \qquad i \searrow 0 \end{array}$$

As we noted in Remark 3.14, such a relation corresponds to a semicontinuous function from X, or a $\mathbb{Q}$-indexed family of open subsets of it. We shall see that, when the "finite open sub-cover" property of a compact space K is used in familiar ways in real analysis, a relation like this lies at the core of the argument. The order-preserving property is usually obvious from the form of θ, which, by the way, typically also depends on other parameters.

Since the purpose of making this definition is to *specialise* to the idioms of analysis, we leave the interested reader to generalise it back to other directed structures such as finite powersets. A simpler generalisation, to $i \nearrow a$ or $i \searrow a$, may be derived arithmetically (*cf.* Proposition 9.12) or order-theoretically ([I, Definition 4.20]).

,

Proposition 9.4 Let $\theta(x,i)$ be a directed relation on a space X, where $i : \mathbb{N}$ or $\mathbb{Q}$. Then any $F : \Sigma^{\Sigma^X}$ satisfies the rule

$$\frac{\ldots,\ x : X,\ i, j\ \vdash\ (i < j) \land \theta(x,i)\ \Rightarrow\ \theta(x,j)}{\ldots\ \vdash\ F\big(\lambda x.\, \exists i.\, \theta(x,i)\big)\ \Rightarrow\ \exists i.\, F\big(\lambda x.\, \theta(x,i)\big)}\ (i \nearrow \infty)$$

or

$$\frac{\ldots,\ x : X,\ i, j\ \vdash\ (0 < i < j) \land \theta(x,j)\ \Rightarrow\ \theta(x,i)}{\ldots\ \vdash\ F\big(\lambda x.\, \exists i > 0.\, \theta(x,i)\big)\ \Rightarrow\ \exists i > 0.\, F\big(\lambda x.\, \theta(x,i)\big)}\ (i \searrow 0).$$

Proof In the first case, the indexing space is $\mathbb{N}$ or $\mathbb{Q}$ with the arithmetic order, which the binary operation max makes directed. The second case used $\{q : \mathbb{Q} \mid q > 0\}$ with the reverse order and min. $\square$

We shall see in Corollary 10.4 that these rules are also valid with real instead of rational values of i.

Although this result is valid for any functional F on any space X, the most important application is to the necessity operator $\Box$ or quantifier $\forall$ for a compact (sub)space K in the sense of the previous section. Then we have the following ***uniformity principle***.

Corollary 9.5 For any directed relation $\theta(k,i)$ on a compact space K,

$$\begin{array}{llcll} & \forall k{:}K.\, \exists i.\, \theta(k,i) & \Rightarrow & \exists i.\, \forall k{:}K.\, \theta(k,i) & \qquad i \nearrow \infty \\ \text{or} & \forall k{:}K.\, \exists i > 0.\, \theta(k,i) & \Rightarrow & \exists i > 0.\, \forall k{:}K.\, \theta(k,i) & \qquad i \searrow 0\ \square \end{array}$$

In order to use this principle in real analysis, we need to state the final axiom:

Axiom 9.6 The closed interval, $\mathbb{I} \equiv [0,1] \subset \mathbb{R}$, is compact. What we mean by this is that the closed subspace $[0,1]$ that is co-classified by $\omega x \equiv (x < 0) \vee (1 < x)$ also has a modal operator $\Box$ for which the pair $(\Box, \omega)$ satisfies Definition 8.1. It is unique by Proposition 8.2(f).

There's nothing special about 0 and 1 in this:

Proposition 9.7 The closed interval $[d,u] \subset \mathbb{R}$, whose endpoints $d, u : \mathbb{R}$ may be variables or expressions with parameters, is the *closed* subspace co-classified by

$$\omega x \;\equiv\; x < d \vee u < x,$$

and is a *compact* subspace, with

$$[d,u]\phi \;\equiv\; \forall x{:}(d \le x \le u).\ \phi x \;\equiv\; \forall t{:}\mathbb{I}.\ \phi\big(d(1-t) + ut\big).$$

Proof This is as an application of Theorem 8.8 and Remark 8.15. $\Box$

Remark 9.8 We already know that all compact subspaces of $\mathbb{R}$ are closed (Proposition 8.6), so the next question is whether closed subspaces are compact iff they are bounded. However, the meaning of "boundedness" is problematic, especially when we allow parameters.

By definition of $[K]$, we know that the compact subspace is contained in an open interval (d,u) — where d and u may be either variables or general real-valued expressions, but nevertheless *specified* ones — iff

$$\forall x{:}K.\ (d < x < u) \;\equiv\; [K](\lambda x.\, d < x < u). \tag{a}$$

Since this is a *proposition*, we may quantify d and u (if they're variables rather than expressions). So K is bounded with *unspecified* bounds iff

$$\exists du.\, \forall x{:}K.\ (d < x < u). \tag{b}$$

Then (a$\Rightarrow$b), and Remark 6.5 makes them equivalent, but only when the parameters in the definition of $[K]$ are (at worst) integers or rationals, not real or logical.

Since properties (a,b) use the universal quantifier, we can only state them if we already know that the subspace is compact. On the other hand, the closed subspace co-classified by ω is contained in the open interval (d,u) iff the interval and the complementary open subspace cover $\mathbb{R}$, *i.e.*

$$\ldots,\ x : \mathbb{R} \;\vdash\; (d < x < u) \vee \omega x \;\Leftrightarrow\; \top, \quad \text{or} \quad \lambda x.\,(d < x < u) \vee \omega x \;=\; \top : \Sigma^{\mathbb{R}}. \tag{c}$$

It is contained in the closed interval $[d,u]$ iff its complement contains the complement of the interval (*cf.* Proposition 3.8), *i.e.*

$$\ldots,\ x : \mathbb{R} \;\vdash\; (x < d) \vee (u < x) \;\Rightarrow\; \omega x, \quad \text{or} \quad \lambda x.\,(x < d) \vee (u < x) \;\leqslant\; \omega : \Sigma^{\mathbb{R}}. \tag{d}$$

The versions of (c,d) on the left (with x free) are *judgements* (Axiom 5.1), whilst those on the right (with x bound by λ) are *statements* (Definition 4.5), to neither of which does the ASD calculus in Section 5 allow us to apply an existential quantifier.

Exercise 9.9 Show that (a$\vdash$c$\dashv\vdash$d). For [a$\vdash$d], show that

$$(\forall x{:}K.\ d < x < u) \;\wedge\; (y < d \vee u < y) \;\Rightarrow\; \omega y \;\equiv\; (\forall x{:}K.\ x \ne y).$$

For [c$\vdash$d], use Axiom 5.5. The converse, [d$\vdash$c], is more difficult, since we need to enlarge the interval: choosing $c < d$ and $u < v$; see [I, Exercises 6.16] for how to mix $<$ and $\le$ in $\mathbb{R}$, and in particular how to deal with $c < d \le x \le u < v$. $\Box$

Once again, we can solve this problem by generalising the Euclidean reals d and u in property (d) to ascending and descending ones, δ and υ:

Definition 9.10 The closed subspace co-classified by ω is said to be ***bounded*** if there is an ascending real δ and a descending one υ with

$$\exists d < u.\, \delta d \wedge \upsilon u \quad \text{and} \quad \delta \vee \upsilon \leqslant \omega : \Sigma^{\mathbb{R}}, \tag{e}$$

i.e. δ and υ are rounded and bounded in the sense of Definition 6.6, and co-classify a closed subspace that contains the given one, in the sense of Definition 4.5.

Proposition 9.11 Any such rounded, bounded pair (δ, υ) co-classifies a compact subspace of $\mathbb{R}$, with modal operator

$$\Box\phi \;\equiv\; \exists d < u.\, \forall k{:}[d,u].\; \delta k \vee \phi k \vee \upsilon k.$$

We call this subspace the ***compact interval*** $[\delta, \upsilon]$.

Proof Using the introduction rules for both quantifiers, we have

$$\begin{array}{rcll}
\delta x \vee \upsilon x & \Rightarrow & \delta y \vee (x \neq y) \vee \upsilon y & \text{Lemma 5.8} \\
\delta x \vee \upsilon x & \Rightarrow & \forall k{:}[x-1, x+1].\; \delta k \vee (x \neq k) \vee \upsilon k & \text{Theorem 8.12} \\
& \Rightarrow & \exists d < u.\, \forall k{:}[d,u].\; \delta k \vee (x \neq k) \vee \upsilon k & \text{Axiom 5.9} \\
& \equiv & \Box(\lambda k.\, x \neq k) \;\equiv\; \omega x, &
\end{array}$$

the last step being a definition, as in Definition 8.1.

Since δ and υ are rounded lower/upper and bounded, the same is true of the body of ωx, regarded as a predicate in d and u. Hence ωx implies the existence of

$$d < e < t < u \qquad \text{with} \qquad \delta e, \qquad \forall k{:}[d,u].\; \delta k \vee (x \neq k) \vee \upsilon k \qquad \text{and} \qquad \upsilon t.$$

By Axiom 4.9, $(x < e \vee d < x) \wedge (x < u \vee t < x)$, whence

$$(x < e) \vee (d < x < u) \vee (t < u), \qquad \text{so} \qquad \delta x \vee (\delta x \vee x \neq x \vee \upsilon x) \vee \upsilon x,$$

using the properties above. Hence $\omega x \Rightarrow \delta x \vee \upsilon x$ as required.

Finally, we have to check that $\omega \equiv \delta \vee \upsilon$ and $\Box$ satisfy Definition 8.1 for $\phi : \Sigma^{\mathbb{R}}$.

$$\begin{array}{rcll}
\Box\phi & \Rightarrow & \Box(\lambda k.\, \phi x \vee k \neq x) & \text{Lemma 5.8} \\
& \Leftrightarrow & \phi x \vee \Box(\lambda k.\, k \neq x) \;\equiv\; \phi x \vee \omega x, & \text{Proposition 8.2(b)}
\end{array}$$

which justifies the upward direction. Conversely, if $\top = \phi \vee \omega = \delta \vee \phi \vee \upsilon$ then, for *any* $d < u$,

$$\top \;\Leftrightarrow\; \forall k{:}[d,u].\; \delta k \vee \phi k \vee \upsilon k,$$

so $\Box\phi \Leftrightarrow \top$. $\Box$

Proposition 9.12 Every compact subspace of $\mathbb{R}$ is compact is bounded in senses (b,e).

Proof Given the modal operator $\Box$ of a compact subspace K, let

$$\omega x \;\equiv\; \Box(\lambda k.\, x \neq k), \qquad \delta d \;\equiv\; \Box(\lambda k.\, d < k) \qquad \text{and} \qquad \upsilon u \;\equiv\; \Box(\lambda k.\, k < u),$$

so $\delta \vee \upsilon \leqslant \omega$, whilst δ is lower and υ is upper. (We shall see in Lemma 13.10 that they are disjoint iff K is occupied.)

In order to show that $\exists u.\, \upsilon u$, we use uniformity with respect to the directed relation $\theta(x,u) \equiv (x < u)$ as $u \nearrow \infty$; the left-hand side holds with $u \equiv k+1$:

$$\top \;\Rightarrow\; \Box(\lambda k{:}K.\; \exists u{:}\mathbb{R}.\; k < u) \;\Rightarrow\; \exists u{:}\mathbb{R}.\; \Box(\lambda k{:}K.\; k < u).$$

It is rounded because

$$\begin{aligned} \upsilon u \;\equiv\; \Box(\lambda k.\, k < u) &\;\Leftrightarrow\; \Box(\lambda k.\, \exists\epsilon > 0.\, k < u - \epsilon) && \text{interpolation} \\ &\;\Leftrightarrow\; \exists\epsilon > 0.\, \Box(\lambda k.\, k < t - \epsilon) && \epsilon\searrow 0 \\ &\;\equiv\; \exists\epsilon > 0.\, \upsilon(t - \epsilon) \;\Leftrightarrow\; \exists t.\, (t < u) \wedge \upsilon t \end{aligned}$$

Similarly, δ is rounded lower and bounded, and we may choose d and u so that $\delta d \wedge (d < u) \wedge \upsilon u$. These make K bounded in senses (b,e). $\Box$

Remark 9.13 The descending real υ is the intersection of the families of upper bounds of the elements of K. We may form such an infinitary intersection of open subspaces because it is indexed by a compact set. Since, for descending reals, the intrinsic order with which we form this intersection is the opposite of the arithmetical order (Remark 3.9), we have the supremum of K, as in Remark 3.5(b).

Since any closed subspace of a compact (sub)space is compact by Theorem 8.14, we have

Theorem 9.14 A closed subspace of $\mathbb{R}$ is compact iff it is bounded. $\Box$

Notice that each direction of the proof uses one of the two axioms introduced in this section, so let's say something about their necessity.

Remark 9.15 The classical category of posets and monotone functions satisfies the rest of the theory apart from Scott continuity, where Σ^X is the lattice of upper subsets of X. In this model, all objects are compact and overt in the formal senses of Sections 8 and 11.

The category of dcpos (directed-complete partial orders) and Scott-continuous functions, on the other hand, satisfies everything apart from compactness of $[0, 1]$. This is also the situation in Russian Recursive Analysis and Bishop's Constructive Analysis, so if you are familiar with either of these theories you may be surprised that ASD has both the Heine–Borel theorem and a computable interpretation. This is explained in [I, Section 15].

Beware that we are only saying that these two categories obey all but one of the axioms *in this paper*: the full (current) ASD theory is stronger than this.

10 Continuity on the real line

Now that the calculus is complete, we can express some familiar ideas in elementary analysis, such as continuity, Riemann integration and differentiability, in it. Notice that we use the universal quantifier to say that a closed interval is contained in an open subspace; this will be a common idiom in later results.

Remark 10.1 Classically, a point x is said to be in the ***interior*** of a subset $U \subset \mathbb{R}$ if $x \in (d, u) \subset U$ for some $d, u : \mathbb{R}$. But, with any $d < e < x < t < u$, this is equivalent to

$$\exists et.\;\; x \in (e, t) \subset [e, t] \subset (d, u) \subset U,$$

which states local compactness of $\mathbb{R}$ (Definition 3.17). This is something that we can express in the ASD λ-calculus, because we can use the *universal quantifier* to say when a closed interval is contained in an open subspace. (The bounded *existential* quantifier $\exists\delta > 0.\ \cdots$ means $\exists\delta{:}\mathbb{R}.\ \delta > 0 \wedge \cdots$, as we shall see in the next section.)

Theorem 10.2 Every point $a : \mathbb{R}$ that is a member of the open subspace classified by $\phi : \Sigma^{\mathbb{R}}$ is in its interior, in the sense that

$$\phi a \;\Leftrightarrow\; \exists du.\;\, (d < a < u) \;\wedge\; \forall k{:}[d, u].\ \phi k \;\equiv\; \exists\delta > 0.\, \forall k{:}[a \pm \delta].\ \phi k.$$

Proof This uses the uniformity principle (Corollary 9.5) that we derived from Scott continuity of the universal quantifier. However, since in this case the existentially quantified variable δ is a *parameter* of the universal quantifier $\forall k : [a \pm \delta]$ itself, we reformulate the problem, using the directed relation $\theta(k, \delta) \equiv \big(|k - a| > \delta\big) \vee \phi k$ with $\delta \searrow 0$. Then

$$
\begin{array}{rcll}
\ldots,\ k : [a \pm 1] \ \vdash\ \phi a & \Rightarrow & (k \neq a) \vee \phi k & \text{Lemma 5.8} \\
& \Rightarrow & \exists \delta > 0.\ \big(|k - a| > \delta\big) \vee \phi k & 0 < \delta < |k - a| \\
& \equiv & \exists \delta > 0.\ \theta(k, \delta) & \text{def } \theta \\
\ldots \vdash\ \phi a & \Rightarrow & \forall k{:}[a \pm 1].\ \exists \delta > 0.\ \theta(k, \delta) & \text{Theorem 8.12} \\
& \Rightarrow & \exists \delta > 0.\ \forall k{:}[a \pm 1].\ \theta(k, \delta) & \delta \searrow 0 \\
& \Rightarrow & \exists \delta > 0.\ \forall k{:}[a \pm 1].\ \big(|k - a| > \delta\big) \vee \phi k & \text{def } \theta \\
& \Rightarrow & \exists \delta > 0.\ \forall k{:}[a \pm \delta].\ \phi k & \text{Theorem 8.14} \\
& \Rightarrow & \phi a & k \equiv a \ \square
\end{array}
$$

Remark 10.3 The same result was also derived in [I, Corollary 9.1] from the *representation* of real numbers by Dedekind cuts of rationals, whereas the proof above starts from the use of $\forall_K$ and its uniformity principle, as do the corollaries in this section. The proof that we have just given is part of the argument in [I, Section 14] that any model of the *axioms* for $\mathbb{R}$ that we have given in this paper is uniquely isomorphic to the *representation* as Dedekind cuts.

We shall devote most of this section to applications of this key result in basic real analysis, but it is also important in the foundations of ASD, in which we first tie up some loose ends.

Corollaries 10.4 For any predicate $\phi : \Sigma^{\mathbb{R}}$,
(a) $\exists x{:}\mathbb{R}.\ \phi x \iff \exists x{:}\mathbb{Q}.\ \phi x$;
(b) ϕ is rounded: $\phi y \Rightarrow \exists x < y < z.\ \phi x \wedge \phi z$;
(c) $\phi\big(\mathsf{cut}(\delta, \upsilon)\big) \iff \exists d < u.\ \delta d \wedge \upsilon u \wedge \forall x{:}[d, u].\ \phi x$.

Hence it doesn't matter whether we use $\mathbb{Q}$ or $\mathbb{R}$ when defining ascending or descending reals, or invoking uniformity in Proposition 9.4. Also, Dedekind cuts may be eliminated in the same way as descriptions in Lemma 6.3. $\square$

It's time to do some analysis.

Theorem 10.5 Every function $f : \mathbb{R} \to \mathbb{R}$ (Remark 5.2) is continuous in the ϵ–δ sense.

Proof For $x : \mathbb{R}$ and $\epsilon > 0$, put $\phi y \ \equiv\ |fy - fx| < \epsilon$. Then Theorem 10.2 provides the input tolerance, but in the form of a *closed* interval $[x \pm \delta]$ instead of the traditional open one:

$$\top \ \Leftrightarrow\ \phi x \ \Leftrightarrow\ \exists \delta > 0.\ \forall y{:}[x \pm \delta].\ \phi y,$$

so
$$x : \mathbb{R},\ \epsilon > 0 \ \vdash\ \top \ \Leftrightarrow\ \exists \delta > 0.\ \forall y{:}[x \pm \delta].\ \big(|fy - fx| < \epsilon\big). \qquad \square$$

Remark 10.6 This only differs from the familiar statement in that
(a) we put x and ϵ on the left of a turnstile $\vdash$ (Remark 5.1), since the calculus does not allow us to quantify them (as their range is not compact); but
(b) we use the *non-strict* inequality $|y - x| \leq \delta$ to bound the quantifier $\forall y$, since this has to range over a compact subspace (Section 8), instead of writing a strict inequality on the left of an $\Rightarrow$.

Remark 10.7 Similarly, a function of two variables is jointly continuous in the sense that

$$x, y : \mathbb{R},\ \epsilon > 0 \ \vdash\ \exists \delta > 0.\ \forall x'{:}[x \pm \delta].\ \forall y'{:}[y \pm \delta].\ \big(|fx'y' - fxy| < \epsilon\big),$$

i.e. with the sup metric on $\mathbb{R} \times \mathbb{R}$, so its topology $\Sigma^{\mathbb{R} \times \mathbb{R}}$ is the usual Tychonov product. $\square$

More or less the same argument gives a stronger result. Uniformity (of continuity here, but also of convergence, differentiability, *etc.*) is exactly the interchange of quantifiers as in Corollary 9.5. This is used in situations in analysis where limits need to be interchanged.

Theorem 10.8 Any function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly* continuous on any compact subspace $K \subset \mathbb{R}$.
Proof Consider the predicate $x, y : \mathbb{R},\ \epsilon > 0 \ \vdash\ \phi(x, y, \epsilon) \ \equiv\ (|fx - fy| < \epsilon)$. Then, as before

$$x : K,\ \epsilon > 0 \ \vdash\ \top \ \Leftrightarrow\ \phi(x, x, \epsilon) \ \Leftrightarrow\ \exists \delta > 0.\, \forall y{:}[x \pm \delta].\ \phi(x, y, \epsilon).$$

By the same argument as in Theorem 10.2, with $\delta \searrow 0$,

$$\begin{array}{llll} \epsilon > 0 & \Rightarrow & \forall x{:}K.\ \exists \delta > 0.\, \forall y{:}[x \pm \delta].\ \phi(x, y, \epsilon) & \text{Theorem 8.12} \\ & \Leftrightarrow & \exists \delta > 0.\, \forall x{:}K.\ \forall y{:}[x \pm \delta].\ \phi(x, y, \epsilon) & \delta \searrow 0 \\ \quad Corollary\ 9.5 & & & \\ & \Leftrightarrow & \exists \delta > 0.\, \forall x, y{:}K.\ (|x - y| > \delta)\ \vee\ (|fx - fy| < \epsilon). & \text{Thm. 8.14} \quad \square \end{array}$$

Remark 10.9 Here is the translation of our argument into traditional language; it appears to be essentially the one that G.H. Hardy used in [Har08, §107]. Our ϕ classifies

$$U \ \equiv\ \{(x, y) \mid |fx - fy| < \epsilon\} \ \subset\ K \times K.$$

This is a neighbourhood of the diagonal, every point of which lies inside an open δ-ball within U. As the diagonal is compact, finitely many such balls suffice, and we let δ be their minimum size, which is positive.

Another, much more complicated, argument appears in many textbooks, *e.g.* [Dug66, Theorem XI 4.5]. The image of the domain K is a compact subspace of the codomain, so finitely many ϵ-balls cover its image, and their inverse images cover the domain. Now calculate the *Lebesgue number*, *i.e.* the size of the smallest non-empty intersection; this is the required δ. $\square$

Although the remaining sections of this paper will study *continuous* functions, we pause briefly to illustrate how the definitions of (Riemann) integration and differentiation can be stated in our language, and that they naturally yield Dedekind cuts rather than Cauchy sequences.

Archimedes obtained his famous estimate $3\frac{10}{71} < \pi < 3\frac{1}{7}$ for the area of a circle by inscribing and circumscribing regular 96-gons and calculating their area. As is usual in ancient Greek mathematics, the generalisation to $3 \cdot 2^n$-gons is implicit, for lack of suitable notation: Archimedes even had to invent ways of writing large and fractional numbers. Nevertheless, the principle is clear: for any $d < \alpha < u$, where α is the claimed area or volume of the curved figure, there are enclosed and enclosing figures whose magnitudes are more easily calculated (for example, because the figures are rectilinear) and lie within these bounds.

Riemann integration generalises Archimedes' method, but the details are usually over-specified, raising the obligation to check that different methods of division yield the same answer. Using Dedekind cuts completely avoids this.

Theorem 10.10 Any function $f : \mathbb{I} \to \mathbb{R}$ is Riemann-integrable.

Proof To show that d and u are under- and over-estimates of $\int_0^1 fx\, \mathsf{d}x$, we simply have to find functions f^- and f^+ such that

$$d < \textstyle\sum_0^1 f^- x\, \mathsf{d}x \quad \wedge \quad \forall x \in [0, 1].\ f^- x < fx < f^+ x \quad \wedge \quad \textstyle\sum_0^1 f^+ x\, \mathsf{d}x < u,$$

and the integrals $\sum_0^1 f^\pm x\, \mathsf{d}x$ can be computed in some easier way. For example, they may be step "functions", where the lack of continuity is not an issue, because the meaning of the integral is clear, although we may use sawtooth functions instead. Putting this into our notation is simply a matter of coding inequalities.

That d and u admit such $f^{\pm}$ is then a *propositional formula*, $\delta d \wedge \upsilon u$. Clearly δ is lower and υ upper, and they are disjoint since $f^- < f < f^+$. They are also rounded by Corollary 10.4(b).

This means that they form a Dedekind cut that defines the integral, so long as we can also show that they are bounded and located. We do this by defining step functions $f^{\pm}$ whose integrals $\sum_0^1 f^{\pm}x \, \mathsf{d}x$ differ by exactly any given $\epsilon > 0$.

By uniform continuity of f, we have $\exists n{:}\mathbb{N}.\ \forall x, y{:}\mathbb{I}.\ |y - x| > 2^{-n-1} \vee |fy - fx| < \frac{1}{2}\epsilon$.

For $k \cdot 2^{-n} \leq x \leq (k+1) \cdot 2^{-n}$, we define $f^{\pm}(x) \equiv f(y) \pm \frac{1}{2}\epsilon$, where y is the mid-point of this sub-interval. This satisfies the requirements. To show that the integral is scalable, and additive with respect to functions and intervals simply involves pasting step functions together. □

Definition 10.11 Turning to differentiability, the better classical accounts use the expression

$$f(x + h) \;=\; f(x) + hf'(x) + o(h),$$

and constructive authors have found it most convenient to define the derivative as the *pair*, (f, f'). In order to give a definition using Dedekind cuts, we must therefore bound both the derivative and the original function,

$$e_0 < f(x) < t_0 \quad \text{and} \quad e_1 < f'(x) < t_1.$$

We can express this using bounds on the function over some interval around x,

$$\begin{aligned} \exists \delta > 0.\ \forall h{:}[0, \delta]. \qquad & e_1 + e_1 h < f(x + h) < t_0 + t_1 h \\ \wedge \quad & e_1 - t_1 h < f(x - h) < t_0 - e_1 h, \end{aligned}$$

which confines the function to a region in the shape of a *bow tie*, *cf.* [EL04].

Considered as a predicate in (e_0, t_0) or (e_1, t_1), this formula

(a) defines a Dedekind cut in (e_0, t_0) since $f : \mathbb{R} \to \mathbb{R}$ is a function;

(b) is a Dedekind cut in (e_1, t_1) if f is ***differentiable*** at x; but

(c) is a compact interval in (e_1, t_1) if f is ***Lipschitz*** at x. □

Proposition 10.12 Any function that is differentiable everywhere in a compact domain K is *uniformly* differentiable there.

Proof Inside all of the quantifiers in the definition of derivative above is a propositional expression $\phi(h)$ that does not depend on δ. For *any* such formula, we have

$$(0 < \delta < \delta') \;\wedge\; \forall h{:}[0, \delta'].\ \phi(h) \;\Rightarrow\; \forall h{:}[0, \delta].\ \phi(h)$$

and therefore, as $\delta \searrow 0$,

$$\forall k{:}X.\ \exists \delta > 0.\ \forall h{:}[0, \delta].\ \phi(h) \;\Rightarrow\; \exists \delta > 0.\ \forall k{:}X.\ \forall h{:}[0, \delta].\ \phi(h). \qquad \square$$

Remark 10.13 In summary, to what extent does $\Sigma^{\mathbb{R}}$ resemble the usual ("Euclidean") topology on the real line?

(a) Any open *interval* (d, u) is an open *subspace* $\{x : \mathbb{R} \mid \phi x\}$, where $\phi x \equiv (d < x < u)$.

(b) $\Sigma^{\mathbb{R}}$ has $\top$, $\bot$, $\wedge$, $\vee$ and $\exists_N$, for any overt object N, in particular for $\mathbb{N}$.

(c) It does not, however, have "arbitrary" joins; in particular we cannot form the *interior* of an arbitrary subset, *i.e.* the union of *all* open subspaces that this contains (Example 16.6).

(d) If a point belongs to an open subspace, then some interval around that point is also contained in the subspace (Theorem 10.2).

We shall complete this characterisation in Section 15, showing that any open subspace is a countable union of disjoint open intervals. An "interval" here is a convex subspace that doesn't necessarily have endpoints, whilst by "countable" we mean that the union is indexed by a quotient of $\mathbb{Q}$ by an open partial equivalence relation.

11 Overt subspaces

Now we are ready to give a formal introduction to the new concept of overtness, of which we saw an example in Section 2. The theory of open or overt subspaces of (overt) *discrete* spaces is the dual of that of closed or compact subspaces of (compact) Hausdorff spaces in Section 8.

Since the ambient space that interests us in this paper is *Hausdorff*, not discrete, things are not exactly the same, and we must start from a different place. Nevertheless, we shall try after that to follow the same plan as closely as possible. We begin with the dual of Definition 8.17 for a compact *open* subspace, because Proposition 2.9 suggested that the "usual" situation for an overt subspace of a Hausdorff space is that it be *closed* too. You should be sure that you thoroughly understand how the formalism in Section 8 relates to your own knowledge of compact subspaces before studying what follows, as we now rely entirely on the new calculus.

Definition 11.1 A ***closed overt subspace*** of a space X is defined by the pair $(\Diamond, \omega)$, where $\omega : \Sigma^X$ is its co-classifier *quâ* closed subspace and $\Diamond : \Sigma^{\Sigma^X}$ is its ***possibility modal operator*** *quâ* overt one. These must be related by the rule that, for any expression $\phi : \Sigma^X$,

$$\frac{\phi \;\leqslant\; \omega}{\overline{\Diamond\,\phi \;\Leftrightarrow\; \bot}}$$

so in particular $\Diamond\,\omega \;\Leftrightarrow\; \bot$. If we want to name the subspace, we write $\langle S\rangle$ for $\Diamond$. The intended meaning of $\Diamond\,\phi$ is that ϕ *touches* the subspace represented by $\Diamond$, *cf.* Proposition 2.1, whereas in Section 8, ϕ *covered* $\Box$.

The downward direction of this rule says that the overt subspace is contained in the closed one. In Section 2 we saw that the ***singular case***, in which this containment is proper, arose, for example, from double zeroes of polynomials.

Proposition 11.2 Possibility operators satisfy, for $a : H$, $\sigma : \Sigma$, $\phi, \psi : \Sigma^H$ and $\theta : \Sigma^{H\times H}$,

(a) preservation of joins: $\Diamond\,\bot \;\Leftrightarrow\; \bot$ and $\Diamond(\phi \vee \psi) \;\Leftrightarrow\; \Diamond\,\phi \vee \Diamond\,\psi$,

(b) the ***Frobenius law***, $\sigma \wedge \Diamond\,\phi \;\Leftrightarrow\; \Diamond(\sigma \wedge \phi)$,

(c) ***relative instantiation***, $\phi a \;\Rightarrow\; \omega a \vee \Diamond\,\phi$ (though this fails in the singular case),

(d) substitution inside any expression to which $\Diamond$ is applied,

(e) commutativity, $\langle S_1\rangle\big(\lambda x.\, \langle S_2\rangle(\lambda y.\, \theta xy)\big) \;\Leftrightarrow\; \langle S_2\rangle\big(\lambda y.\, \langle S_1\rangle(\lambda x.\, \theta xy)\big)$, and

(f) uniqueness: if $\Diamond$ and $\blacklozenge$ both satisfy the definition for the same ω then $\Diamond = \blacklozenge$.

Proof It would be a valuable exercise to transcribe the proof of Proposition 8.2 and Exercise 8.18, replacing each symbol with its lattice dual. Thus the Frobenius law and uniqueness follow from Proposition 5.7, the substitution property is that for λ-application (Axiom 7.4), and relative instantiation is the upward direction of the defining rule. Either side of the commutative law is $\bot$ iff

$$x, y : H,\; \omega_1 x \Leftrightarrow \omega_2 y \Leftrightarrow \bot \;\vdash\; \phi xy \;\Leftrightarrow\; \bot. \qquad \Box$$

Remark 11.3 Since $\Diamond$ automatically preserves directed joins (it is Scott continuous, Axiom 9.1), it actually preserves *all* joins, as in Theorem 2.4. That's not quite right: it preserves joins *indexed by other overt objects*, and the same is true of inverse image maps (f^{-1}, Definition 7.14). Another way of putting this is that possibility operators *commute*, which is what we have just said. Unlike in the lattice-theoretic or localic language, we are able to make identical statements about meets and joins.

Example 11.4 The empty subspace is overt, as is the union of any two overt subspaces, but the intersection may fail to be overt.

S	ωx	$\langle S\rangle\phi$
$\emptyset$	$\top$	$\bot$
$S_1 \cup S_2$	$\omega_1 x \wedge \omega_2 x$	$\langle S_1\rangle\phi \vee \langle S_2\rangle\phi$
$S_1 \cap S_2$	$\omega_1 x \vee \omega_2 x$	Example 16.4

$\square$

In the remainder of this paper we shall see that the interaction between overtness and compactness is extremely powerful in constructive topology. Logically, it also allows the definitions to be expressed as *statements* rather than *rules*, in the sense of Section 5. The analogous result for overt compact subspaces of discrete spaces uses $\Box(\lambda x.\ \Diamond(\lambda y.\ x = y) \Leftrightarrow \top$ for the third equation [E].

Proposition 11.5 In a Hausdorff space H, let $\Diamond$ and $\Box$ be any terms of type Σ^{Σ^H}, and put $\omega x \equiv \Box(\lambda y.\ x \neq y)$. Then $(\Box, \omega)$ and $(\Diamond, \omega)$ satisfy Definitions 8.1 and 11.1 respectively iff

$$\Box\top \Leftrightarrow \top \qquad \Diamond\,\omega \equiv \Diamond\big(\lambda x.\ \Box(\lambda y.\ x \neq y)\big) \Leftrightarrow \bot$$

$$\Box\,\phi \wedge \Diamond\,\psi \Rightarrow \Diamond(\phi \wedge \psi) \qquad \Box\,\phi \vee \Diamond\,\psi \Leftarrow \Box(\phi \vee \psi)$$

for $\phi, \psi : \Sigma^H$, *cf.* Proposition 2.13.

Proof [$\Rightarrow$] In any distributive lattice, if $\phi \vee \omega = \top$ and $\phi \wedge \psi \leqslant \omega$ then $\psi \leqslant \omega$. But since $\Box\,\phi \dashv\vdash (\phi \vee \omega = \top)$ and $\neg\Diamond\,\theta \dashv\vdash (\theta \leqslant \omega)$ by Definitions 8.1 and 11.1, this is

$$\Box\,\phi, \quad \neg\Diamond(\phi \wedge \psi) \quad \vdash \quad \neg\Diamond\,\psi,$$

from which we deduce the third law by Axiom 5.5. Similarly, we obtain the fourth from

$$\neg\Diamond\,\psi \quad \dashv\vdash \quad \psi \leqslant \omega \quad \vdash \quad \Box(\phi \vee \psi) \Rightarrow \Box(\phi \vee \omega) \Leftrightarrow \Box\,\phi.$$

[$\Leftarrow$] The downward directions of the two definitions are

$$\top = \phi \vee \omega \ \vdash \ \top \Leftrightarrow \Box(\phi \vee \omega) \Rightarrow \Box\,\phi \vee \Diamond\,\omega \Leftrightarrow \Box\,\phi$$

$$\phi \leqslant \omega \ \vdash \ \Diamond\,\phi \Rightarrow \Diamond\,\omega \Leftrightarrow \bot.$$

For the upward directions, note first that

$$\begin{array}{rcll} \omega x \vee \phi x & \equiv & \Box(\lambda y.\ x \neq y) \vee (\phi x \wedge \Box\top) & \\ & \Leftrightarrow & \Box(\lambda y.\ x \neq y \vee \phi x) & \text{Proposition 5.7} \\ & \Leftrightarrow & \Box(\lambda y.\ x \neq y \vee \phi y), & \text{Lemma 5.8} \end{array}$$

whilst $$\Box\,\phi \Rightarrow \Box(\lambda y.\ x \neq y \vee \phi y) \Rightarrow \Box(\lambda y.\ x \neq y) \vee \Diamond\,\phi.$$

Then $$\Box\,\phi \Leftrightarrow \top \vdash \omega \vee \phi = \top \qquad \text{and} \qquad \Diamond\,\phi \Leftrightarrow \bot \vdash \phi \leqslant \omega.$$ $\square$

Example 11.6 Any point $a : H$ defines a compact overt subspace $\{a\}$, with $\Diamond\,\phi \equiv \Box\,\phi \equiv \phi a$ and $\omega x \equiv (x \neq a)$. Conversely, any term $\bigcirc : \Sigma^{\Sigma^H}$ that obeys the rules of both modal operators (it is enough for it to preserve $\top$, $\bot$, $\wedge$ and $\vee$) is called ***prime***, and arises in this way from a unique point [G, §10]. $\square$

Definition 11.7 We say that $a : X$ is a ***member*** of the overt subspace, $a \in \Diamond$, if, for any $\phi : \Sigma^X$,

$$\phi a \Rightarrow \Diamond\,\phi, \quad \text{or} \quad \lambda\phi.\ \phi a \leqslant \Diamond : \Sigma^{\Sigma^X}.$$

An overt subspace is said to be ***inhabited*** if $\Diamond\top \Leftrightarrow \top$ (*cf.* Proposition 2.12(c)).

Considering Theorem 2.6 and Example 11.16, it is really more appropriate to call a an ***accumulation point*** rather than a member of $\Diamond$. This is the analogue of the fact that $a \in \Box$ in Definition 8.3 actually defined the *saturation* of a compact subspace of a non-Hausdorff space.

The formula $S \equiv \{x : X \mid \lambda\phi.\, \phi x \leqslant \Diamond\}$ is the one that we had classically in Proposition 2.9, and it is another example of the general subspace notation that we proposed in Notation 7.11.

Proposition 11.8 For any closed overt subspace of a Hausdorff space H, the following notions of membership are equivalent for $a : H$:

$$\lambda\phi.\, \phi a \leqslant \Diamond \qquad\qquad \omega a \Leftrightarrow \bot \qquad\qquad \Box \leqslant \lambda\phi.\, \phi a,$$

where the third applies if the subspace is also compact.

Proof If $\omega a \Leftrightarrow \bot$ then $\Box\,\phi \Rightarrow \phi a \Rightarrow \Diamond\,\phi$ by relative instantiation. Conversely,

$$\text{if} \quad (\lambda\phi.\, \phi a) \leqslant \Diamond \quad \text{then} \quad \omega a \;\Leftrightarrow\; (\lambda\phi.\, \phi a)\omega \;\Rightarrow\; \Diamond\,\omega \;\Leftrightarrow\; \bot.$$

Finally, Proposition 8.6 related membership for ω and $\Box$. $\square$

Theorem 11.9 Let K be a compact overt subspace. Then it is *decidable* whether K is empty. If it's not, it's both occupied and inhabited, and $\Box \leqslant \Diamond$.

Proof If K is empty then $\Box\,\bot \Leftrightarrow \top$ and $\Diamond\,\top \Leftrightarrow \bot$, whereas if it's occupied then $\Box\,\bot \Leftrightarrow \bot$, and if it's inhabited then $\Diamond\,\top \Leftrightarrow \top$. These situations are complementary, because $\top \Leftrightarrow \Box(\bot \vee \top) \Rightarrow \Box\,\bot \vee \Diamond\,\top$ and $\bot \Leftrightarrow \Diamond(\bot \wedge \top) \Leftarrow \Box\,\bot \wedge \Diamond\,\top$. Also

$$\Box\,\phi \;\Leftrightarrow\; \Box(\bot \vee \phi) \;\Rightarrow\; \Box\,\bot \vee \Diamond\,\phi \;\Leftrightarrow\; \Diamond\,\phi,$$

or

$$\Box\,\phi \;\Leftrightarrow\; \Diamond\,\top \vee \Box\,\phi \;\Rightarrow\; \Diamond(\top \vee \phi) \;\Leftrightarrow\; \Diamond\,\phi.$$

Conversely, $\Box \leqslant \Diamond \vdash \Box\,\bot \Rightarrow \Diamond\,\bot \equiv \bot$ or $\top \equiv \Box\,\top \Rightarrow \Diamond\,\top$. $\square$

Remark 11.10 The dichotomy is in the strict logical sense, the topological equivalent of which (*cf.* Remark 4.6) is that the parameter space Γ is a disjoint union of two *clopen* subspaces, not just an open and a closed one. Therefore, if Γ is connected, for example $\mathbb{R}^{\mathbf{n}}$, something in this short argument has to break at any singularities.

As we saw informally in Proposition 2.12, it is the modal law $\Box(\phi \vee \psi) \Rightarrow \Box\,\phi \vee \Diamond\,\psi$. Even in this singular case, Proposition 11.8 still ensures that any accumulation point of $\Diamond$ belongs to the closed subspace co-classified by ω. We return to this point in Remark 12.14.

Turning to the direct image, we don't have an exact analogue of Theorem 8.8, *i.e.* for $f : X \to Y$ with X overt and Y Hausdorff rather than discrete, but the situation still has an extremely important example. It suggests that $\Diamond$ is related to the *closure* of its defining data, as in Proposition 2.9.

Notation 11.11 Let $\langle S\rangle$ be an overt subspace of X, and $f : X \to Y$. Then

$$\langle fS\rangle\psi \;\equiv\; \langle S\rangle\big(\lambda x.\, \psi(fx)\big)$$

is called the ***direct image*** of $\langle S\rangle$. If $a : X$ is a member of $\langle S\rangle$ then $fa : Y$ is a member of $\langle fS\rangle$, whilst $\langle fS\rangle$ preserves joins. In fact, $\langle fS\rangle$ is the *closure* of the image, in the sense that it is contained in any closed subspace whose inverse image contains $\langle S\rangle$, since $\langle fS\rangle\omega \Leftrightarrow \bot$ iff $\langle S\rangle(f^*\omega) \Leftrightarrow \bot$. If S is inhabited, so is fS because

$$\langle fS\rangle\top \;\Leftrightarrow\; \langle S\rangle(\top \cdot f) \;\Leftrightarrow\; \langle S\rangle\bot \;\Leftrightarrow\; \top.$$

This doesn't completely satisfy Definition 11.1, because we haven't defined ω_{fS}, but this deficit is rectified when we have compactness too:

Proposition 11.12 Let $f : X \to Y$ be a function between Hausdorff spaces, and $K \subset X$ a compact overt subspace. Then its image $fK \subset Y$ is also compact overt.

Proof We prove the modal laws in Proposition 11.5 for $fK \subset Y$.

$$\begin{array}{rcll}
[fK]\top_Y & \equiv & [K]\top_X \;\Leftrightarrow\; \top & \\
\langle fK\rangle\omega_{fK} & \equiv & \langle fK\rangle\big(\lambda y_1.\,[fK](\lambda y_2.\,y_1 \neq y_2)\big) & \\
& \equiv & \langle K\rangle\big(\lambda x_1.\,[fK](\lambda y_2.\,fx_1 \neq y_2)\big) & \text{Notation 11.11}\\
& \equiv & \langle K\rangle\big(\lambda x_1.\,[K](\lambda x_2.\,fx_1 \neq fx_2)\big) & \text{Theorem 8.8}\\
& \Rightarrow & \langle K\rangle\big(\lambda x_1.\,[K](\lambda x_2.\,x_1 \neq x_2)\big) & \text{Lemma 5.8}\\
& \Leftrightarrow & \langle K\rangle\omega_K\bot \;\Leftrightarrow\; \bot &
\end{array}$$

$$\begin{array}{rcl}
[fK](\phi\vee\psi) \;\equiv\; [K](\phi\cdot f\vee\psi\cdot f) & \Rightarrow & [K](\phi\cdot f)\vee\langle K\rangle(\psi\cdot f) \;\equiv\; [fK]\phi\vee\langle fK\rangle\psi\\
\langle fK\rangle(\phi\wedge\psi) \;\equiv\; \langle K\rangle(\phi\cdot f\wedge\psi\cdot f) & \Leftarrow & [K](\phi\cdot f)\wedge\langle K\rangle(\psi\cdot f) \;\equiv\; [fK]\phi\wedge\langle fK\rangle\psi.
\end{array}$$ ☐

Definition 11.13 A *space* S is ***overt*** if it has an ***existential quantifier***, written $\exists_S$, that satisfies the equations for $\Diamond$, together with the ***absolute instantiation rule***, for $a : S$ and $\phi : \Sigma^S$,

$$\phi a \;\Rightarrow\; \exists_S\phi.$$

In other words, S is a closed overt *sub*space of itself, of which *all* expressions of type S are members in the sense of Definition 11.7, so it has $\omega x \Leftrightarrow \bot$.

Notation 11.14 We write $\exists x{:}S.\ \phi x$ for $\exists_S\phi$ when S is an overt *space*, so

$$\exists x{:}S.\ \phi x \qquad \text{means} \qquad \langle S\rangle(\lambda x.\,\phi x),$$

and by extension also when it is an overt *subspace*, *i.e.* for $\langle S\rangle\phi$. This use of logical notation is justified by Proposition 11.2 and the rules of inference that follow it.

Examples 11.15 The spaces $\mathbb{N}$ and $\mathbb{R}$ are overt by Axiom 5.9.

Example 11.16 The modal operator for the direct image of any map $f : \mathbb{N} \to X$ is

$$\Diamond\phi \;\equiv\; \exists n{:}\mathbb{N}.\ \phi(fn).$$

Let $a : X$ be a member of $\Diamond$ (Definition 11.7), and let $\phi : \Sigma^X$ be a neighbourhood of a, so

$$\top \;\Leftrightarrow\; \phi a \;\Rightarrow\; \Diamond\phi \;\equiv\; \exists n.\,\phi(fn).$$

Since ϕ was an *arbitrary* neighbourhood of a, this says that a is an ***accumulation point*** of the sequence, *cf.* Example 2.8(b). ☐

Remark 11.17 This depends only on the *topological* properties of $\mathbb{N}$, not on its arithmetic, recursion or cardinality. So the idea generalises to accumulation points of *nets* of whatever size and shape (if they are definable in the underlying calculus). ☐

Theorem 11.18 In an overt space, $\exists_S$ satisfies the two-way rule on the left:

$$\frac{\ldots,\ x:S \;\vdash\; \phi x \;\Rightarrow\; \sigma}{\ldots \;\vdash\; \exists x{:}S.\ \phi x \;\Rightarrow\; \sigma} \qquad\qquad \frac{\ldots,\ x:X,\ \neg\omega x \;\vdash\; \phi x \;\Rightarrow\; \sigma}{\ldots \;\vdash\; \Diamond\phi \;\Rightarrow\; \sigma}$$

where $\sigma : \Sigma$ doesn't involve the variable x. Similarly, $\Diamond$ and ω for a closed overt *sub*space of X satisfy the rule on the right.

Proof By Axiom 5.5, the rule on the right is

$$\frac{\ldots,\ x:X \;\vdash\; \phi x \;\Rightarrow\; \sigma\vee\omega x}{\ldots \;\vdash\; \Diamond\phi \;\Rightarrow\; \sigma} \qquad \text{or} \qquad \frac{\ldots \;\vdash\; \phi \;\leqslant\; \sigma\vee\omega : \Sigma^X}{\ldots \;\vdash\; \Diamond\phi \;\Rightarrow\; \sigma}$$

which is Definition 11.1. The rule on the left is the special case where $\omega x \equiv \bot$. $\square$

Remark 11.19 As in Remark 8.15, $\lozenge$ is a bounded existential quantifier, with which we can reason about members of the subspace as if this were itself a space, *cf.* Exercise 2.7. There are also duals of Theorems 8.8 and 8.14, but these apply to discrete spaces [E], not to Hausdorff ones.

Exercise 11.20 Any open subspace of an overt space is overt, *cf.* Example 2.8(a), Definition 8.1 and Proposition 8.6. In detail, let X be an overt space (with $\exists_X$) and let $\alpha : \Sigma^X$ classify an open subspace of it. Then $\lozenge\,\phi \equiv \exists x{:}X.\ \phi x \wedge \alpha x$ satisfies Proposition 11.2 and the rules

$$\frac{\phi \wedge \alpha \;=\; \bot}{\overline{\lozenge\,\phi \;\Leftrightarrow\; \bot}} \qquad \text{and} \qquad \frac{\ldots,\ x : X,\ \alpha x \;\vdash\; \phi x \;\Rightarrow\; \sigma}{\overline{\ldots \;\vdash\; \lozenge\,\phi \;\Rightarrow\; \sigma}}$$

Also, if $a \in \alpha$ ($\alpha a \Leftrightarrow \top$) then $a \in \lozenge$ ($\lambda\phi.\ \phi a \leqslant \lozenge$), but not necessarily conversely. $\square$

Example 11.21 In view of the link between overt and closed subspaces, it's not surprising that the open and closed intervals with given endpoints $d < u$ (are both overt and) have the same modal operator or bounded quantifier:

$$\langle d, u\rangle\phi \;\equiv\; \exists x{:}[d,u].\ \phi x \;\Leftrightarrow\; (d,u)\phi \;\equiv\; \exists x{:}(d,u).\ \phi x \;\Leftrightarrow\; \exists x{:}\mathbb{R}.\ (d < x < u) \wedge \phi x.$$

In fact, $[d,u]$ is also overt when $d = u$ (with $[d,d]\phi \equiv \phi d$) or $d \leq u$ [I, Section 10]. $\square$

12 Maximum of a compact overt subspace

Classically, any nonempty subset $K \subset \mathbb{R}$ that is bounded above has a supremum, and there's no loss of generality in stating this only when K is also closed and bounded below, *i.e.* compact. In our calculus, the supremum of a compact subspace K does exist, but in general it's only a *descending* real number (Remark 9.13).

In this section we shall show that, when K is also *overt*, the supremum as a Euclidean real number, defined by a Dedekind cut, and is in fact the *maximum* of K. The construction was inspired by the constructive least upper bound principle (Definition 3.6), except that we shall interpret the quantifiers $\forall$ and $\exists$ in our topological sense:

Notation 12.1 As in the previous section, the compact overt "subspace" K will be encoded by its pair of modal operators, $\square$ and $\lozenge$, which are expressions of type $\Sigma^{\Sigma^{\mathbb{R}}}$ that may include parameters. Of course, when we define $\mathsf{max}\,K$, it will be an expression with the same parameters as those in $\square$ and $\lozenge$.

The duality between these concepts is precisely reflected in a symmetry of the proof between the upper and lower halves of the Dedekind cut that defines the supremum. By Theorem 11.9, it is decidable whether K is empty or inhabited, so for the moment we assume the latter. Lemma 12.7 will provide a definable element of it.

Notation 12.2 Let $\delta d \equiv \lozenge(\lambda k.\ d < k)$ and $\upsilon u \equiv \square(\lambda k.\ k < u)$.

The proof that (δ, υ) is a Dedekind cut (Axiom 6.7) uses each of the modal laws in turn. Proposition 9.12 has already shown that υ is a bounded descending real (rounded upper).

Lemma 12.3 δ is lower, rounded and bounded.

Proof We leave the details as an easy exercise. It's lower by transitivity of $<$ and the Frobenius law for $\lozenge$. It's rounded by interpolation and Frobenius or by Theorem 10.2. As $\lozenge$ is inhabited, δ is bounded by extrapolation of $<$ and commutation of $\exists$ and $\lozenge$. $\square$

Remark 12.4 The ascending real δ is the union of the families of lower bounds of the elements of K. We may form this union because it is indexed by an overt set. Since, for ascending reals, the intrinsic and arithmetical orders agree (Remark 3.9), we have the supremum of K, as in Remark 3.5(a).

The next task is to show that these two suprema agree.

Lemma 12.5 Using both compactness and overtness of K, and Proposition 11.5, (δ, υ) are disjoint and located, so the two approximations agree.

Proof The proofs are dual, using the mixed modal laws and $\Diamond\,\sigma \Rightarrow \sigma \Rightarrow \Box\,\sigma$. The first also uses transitivity and the second locatedness of the order on $\mathbb{R}$ (Axiom 4.9).

$$(\delta d \wedge \upsilon u) \;\equiv\; \Diamond(\lambda k.\, d < k) \wedge \Box(\lambda k.\, k < u) \;\Rightarrow\; \Diamond(\lambda k.\, d < k \wedge k < u) \;\Rightarrow\; (d < u)$$

$$(\delta d \vee \upsilon u) \;\equiv\; \Diamond(\lambda k.\, d < k) \vee \Box(\lambda k.\, k < u) \;\Leftarrow\; \Box(\lambda k.\, d < k \vee k < u) \;\Leftarrow\; (d < u). \qquad \Box$$

Proposition 12.6 (δ, υ) is a Dedekind cut, so by Axiom 6.7 there is $a : \mathbb{R}$ with

$$\delta d \;\Leftrightarrow\; (d < a) \;\Leftrightarrow\; \Diamond(\lambda k.\, d < k) \quad \text{and} \quad \upsilon u \;\Leftrightarrow\; (a < u) \;\Leftrightarrow\; \Box(\lambda k.\, k < u). \qquad \Box$$

Hence K has a supremum a, but it's better than this:

Lemma 12.7 $a \in K$.

Proof This uses Propositions 11.5 and 11.8 and the last modal axiom, $\Diamond\,\omega \Leftrightarrow \bot$. Recall that K is the closed subspace co-classified by $\omega x \equiv \Box(\lambda k.\, x \neq k)$, so we must show that $\omega a \Leftrightarrow \bot$. But by Axiom 4.9 and one of the mixed modal laws,

$$\begin{aligned}
\omega a \;\equiv\; \Box(\lambda k.\, a \neq k) \;&\Leftrightarrow\; \Box(\lambda k.\, a < k) \;\vee\; (k < a) \\
&\Rightarrow\; \Diamond(\lambda k.\, a < k) \;\vee\; \Box(\lambda k.\, k < a) \\
&\equiv\; \delta a \vee \upsilon a \;\Leftrightarrow\; (a < a) \vee (a < a) \;\Leftrightarrow\; \bot. \qquad \Box
\end{aligned}$$

Exercise 12.8 Show that $\Box\,\phi \Rightarrow \phi a \Rightarrow \Diamond\,\phi$ directly. $\Box$

Theorem 12.9 Any overt compact subspace $K \subset \mathbb{R}$ is either empty or has a greatest member $\mathsf{max}\,K \equiv a \in K$. This satisfies, for $x : \mathbb{R}$,

$$\begin{array}{rcl}
(x < \mathsf{max}\,K) & \Leftrightarrow & (\exists k{:}K.\; x < k) \\
(\mathsf{max}\,K < x) & \Leftrightarrow & (\forall k{:}K.\; k < x) \\
k : K & \vdash & k \leq \mathsf{max}\,K
\end{array}
\qquad \text{and} \qquad
\frac{\ldots,\; k : K \;\vdash\; k \leq x}{\ldots \;\vdash\; \mathsf{max}\,K \leq x}$$

Proof The first two properties restate Proposition 12.6, which also gives, for $k : K$,

$$(\mathsf{max}\,K < k) \;\equiv\; \Box(\lambda k'.\, k' < k) \;\Rightarrow\; (k < k) \;\Leftrightarrow\; \bot,$$

so $k \leq \mathsf{max}\,K$ by Example 4.6. The rule on the right is substitution of $\mathsf{max}\,K$ for $k : K$. $\Box$

Remark 12.10 Using unbounded cuts, we could also define

$$\mathsf{max}\,\emptyset \;\equiv\; -\infty \;\equiv\; (\lambda d.\, \bot, \lambda u.\, \top) \quad \text{and} \quad \mathsf{min}\,\emptyset \;\equiv\; +\infty \;\equiv\; (\lambda d.\, \top, \lambda u.\, \bot),$$

but this loses information, because it does not let us say positively that $K \cong \emptyset$. $\Box$

Exercise 12.11 Let $a, b : \mathbb{R} \rightrightarrows \mathbb{R}$ be two functions, so there is no question of using a case analysis on whether $a \leq b$ or $a \geq b$. Regarding them as terms $a, b : \mathbb{R}$ with a real parameter, define the parametric modal operators $[K]$ and $\langle K \rangle$ that make the subspace $K \equiv \{a, b\} \subset \mathbb{R}$ compact and overt. Show that

$$\mathsf{max}(a,b) \;\equiv\; \mathsf{max}\, K \;\equiv\; (\delta, \upsilon) \;\Leftrightarrow\; (\delta_a \vee \delta_b,\; \upsilon_a \wedge \upsilon_b).$$

The properties listed in Theorem 12.9 are those for binary max in [I, Proposition 9.8]. □

Corollary 12.12 Any overt compact subspace $K \subset \mathbb{R}$ either
(a) is observably empty, in which case $[K]\bot \Leftrightarrow \top$, or
(b) has a definable member, namely $\mathsf{max}\, K$, and in this case $\langle K \rangle \top \Leftrightarrow \top$. □

What can we say about *where* the function attains its bounds?

Proposition 12.13 Any function $f : K \to \mathbb{R}$ on an occupied compact overt space is bounded, and attains its bounds on an occupied compact subspace $Z \subset K$. However, Z need not be overt (Example 16.16).

Proof The image $fK \subset \mathbb{R}$ is an occupied compact overt subspace of $\mathbb{R}$ (Proposition 11.12), and so has a maximum $b \in \mathbb{R}$. The inverse image $Z \equiv \{x : K \mid fx = b\} \subset K$ of b is closed, and so compact (Theorem 8.14), with necessity operator

$$[Z]\phi \;\equiv\; \forall x{:}K.\; \phi x \vee fx \neq b.$$

The subspace Z is occupied because

$$[Z]\bot \;\equiv\; \forall x{:}K.\; fx \neq b \;\equiv\; [fK](\lambda y.\, y \neq b) \;\Rightarrow\; (\lambda y.\, y \neq b)b \;\Longleftrightarrow\; \bot. \qquad \square$$

Remark 12.14 We shall see in Section 14 that, given a polynomial equation (say $x^3 - x = 0$) with distinct roots ($\{-1, 0, +1\}$), the argument above yields the greatest root ($+1$) as the supremum.

In the singular case, $\Box$ and $\Diamond$ only satisfy one of the mixed modal laws, namely the one that was used to prove *disjointness* in Lemma 12.5, whilst locatedness fails. The pseudo-cut (δ, υ) nevertheless defines a compact interval $[\delta, \upsilon]$ in the sense of Proposition 9.11, whose endpoints are the ascending real δ and the descending one υ, which need not be Euclidean.

Concretely, consider the polynomial equation $x^3 + x^2 = 0$. This has a stable zero at -1 and a double (unstable) one at 0, *i.e.* $\{-1\} \equiv S \subset Z \equiv \{-1, 0\}$. Then $\delta = \mathsf{sup}\, S = -1$ and $\upsilon = \mathsf{sup}\, Z = 0$, so the interval-valued supremum is $[-1, 0]$.

13 Connectedness

Connectedness is the abstraction of the intermediate value theorem from $\mathbb{R}$ to other topological spaces, so the classical proof that $\mathbb{I}$ is connected is therefore essentially that of Theorem 1.1(a). This definition denies the existence of a very strong kind of separation, which is weakened in the constructive versions — the latter are therefore *stronger* definitions of connectedess, and *fewer* spaces are constructively connected. Indeed Example 16.5, which deletes “part of” a point from the line, is not connected constructively, whereas the classical definition says that it is. But, as usual, the difference between the classical and constructive results is subtle, so this does not affect the status of the traditional examples of unintuitively connected spaces [SS70]

For their constructive definition and proof that $\mathbb{I}$ is connected, the followers of Brouwer and Bishop make use of an interval halving argument that also needs countable choice, *cf.* Theorems 1.1(b) and 1.7; in [Man83] this is attributed to Ray Mines and Fred Richman. It leads directly to a useful *approximate* form of the intermediate value theorem: see [TvD88, Proposition 6.1.4] for the Brouwer school and [BB85, Theorem 2.4.8] for Bishop’s.

In these definitions, tests of “separation” may be expressed by saying that certain open subspaces cover or are inhabited. The latter ideas are naturally expressed using the modal operators

$\Box$ and $\Diamond$ that we have developed, and these allow us to say respectively whether *compact* and *overt* subspaces are connected.

We therefore have two versions of the definition, leading to *two* approximate intermediate value theorems. This is an example of the way in which ASD elucidates the open–closed duality in topology. It turns out that the Brouwer–Bishop definition agrees with ours for *overt* subspaces.

A consequence of this duality in ASD (in particular Axiom 5.5) is that we may use the *classical* proof that $\mathbb{I}$ is connected, which it is in all of the senses that we consider.

Our dual results for compact spaces do not seem to occur in the constructive literature. This is probably because, for Bishop, a "compact" subspace is (totally bounded and therefore) overt, but need not satisfy the Heine–Borel property. Our results about connectedness of compact spaces that are also overt would, for Bishop, merely be $\neg\neg$ versions of the overt ones that also hold. But we shall also find that the proof that an occupied compact interval (in the sense Proposition 9.11) that need not be overt is connected is rather tricky, and uses Heine–Borel (Theorem 15.14).

Definition 13.1 An open or closed subspace $I \subset H$ of a Hausdorff space is said to be
(a) ***overt connected*** if it is given by $\Diamond : \Sigma^{\Sigma^X}$ (Definition 11.1) for which

$$\Diamond\top \;\Leftrightarrow\; \top \quad \text{and} \quad \ldots,\ \phi,\psi : \Sigma^X,\ \phi\vee\psi = \top_I \;\vdash\; \Diamond\phi\wedge\Diamond\psi \;\Rightarrow\; \Diamond(\phi\wedge\psi);$$

so whenever $I \subset U\cup V$ is covered by open inhabited subspaces, they must intersect;
(b) ***compact connected*** if it is given by $\Box : \Sigma^{\Sigma^X}$ (Definition 8.1) for which

$$\Box\bot \;\Leftrightarrow\; \bot \quad \text{and} \quad \ldots,\ \phi,\psi : \Sigma^X,\ \phi\wedge\psi = \bot_I \;\vdash\; \Box\phi\vee\Box\psi \;\Leftarrow\; \Box(\phi\vee\psi);$$

so whenever $I \subset U + V$ is covered by disjoint open subspaces then one of them is enough.
Recall from Remark 7.13 that when $I \subset H$ is an open subspace classified by $\alpha : \Sigma^H$, the equation $\phi = \top_I$ means $\phi \geqslant \alpha : \Sigma^H$ and $\phi = \bot_I$ means $\phi\wedge\alpha = \bot : \Sigma^H$. On the other hand, if it is a closed subspace co-classified by $\omega : \Sigma^H$, then $\phi = \top_I$ means $\phi\vee\omega = \top : \Sigma^H$ and $\phi = \bot_I$ means $\phi \leqslant \omega : \Sigma^H$.)

We shall show that $\mathbb{R}$ is overt connected and that $\mathbb{I}$ has both properties, but first we see how the two definitions yield the approximate versions of the intermediate value theorem.

Proposition 13.2 Any function $f : X \to \mathbb{R}$ on an overt connected space X that takes values both above $-\epsilon$ and below $+\epsilon$ also takes values within ϵ of zero:

$$\exists xz{:}X.\ (-\epsilon < fx)\wedge(fz < +\epsilon) \;\Rightarrow\; \exists y{:}X.\ (-\epsilon < fy < +\epsilon),$$

so the open, overt subspace $\{x : X \mid |fx| < \epsilon\}$ is inhabited.

Proof Let $\phi x \equiv (-\epsilon < fx)$ and $\psi x \equiv (fx < +\epsilon)$, so $\phi\vee\psi = \top$ by locatedness for $\mathbb{R}$ (Axiom 4.9). Then overt connectedness says that

$$\exists xz.\ (-\epsilon < fx)\wedge(fz < +\epsilon) \;\equiv\; \Diamond\phi\wedge\Diamond\psi \quad\Longrightarrow\quad \Diamond(\phi\wedge\psi) \;\equiv\; \exists y.\ (-\epsilon < fy < +\epsilon). \qquad \Box$$

The second approximate intermediate value theorem follows from compact connectedness.

Proposition 13.3 Let $f : K \to \mathbb{R}$ be a function on a compact connected space such that both of the closed, compact subspaces $\{x : K \mid fx \geq 0\}$ and $\{x : K \mid fx \leq 0\}$ are occupied. Then so is $Z \equiv \{x : K \mid fx = 0\}$.

Proof Let $\phi x \equiv (0 < fx)$ and $\psi x \equiv (fx < 0)$, so $\phi\wedge\psi = \bot$. Then compact connectedness says that

$$(\forall x.\ 0 < fx)\vee(\forall x.\ fx < 0) \;\equiv\; \Box\phi\vee\Box\psi \quad\Longleftarrow\quad \Box(\phi\vee\psi) \;\equiv\; (\forall x.\ fx \neq 0). \qquad \Box$$

Example 13.4 The zero-set in Example 1.2 is occupied. $\Box$

Now that we have (at least) two definitions, we had better show that they agree.

Proposition 13.5 Let $I \subset H$ be a closed, compact, overt subspace, so it has $(\Box, \Diamond)$ satisfying both mixed modal laws (Proposition 11.5). Then the following are equivalent:
(a) $\Diamond$ defines an overt connected subspace;
(b) $\Box$ defines a compact connected subspace;
(c) I is occupied, and any map $f : I \to \mathbf{2}$ is constant; and
(d) I is occupied, and if $K \subset I$ is clopen (both open and closed) then either $K = \emptyset$ or $K = I$.

Proof It is easy to see that $[\mathrm{a} \vdash \mathrm{c} \dashv\vdash \mathrm{d} \dashv \mathrm{b}]$. Examples 16.5 and 16.8 are classically connected but the former fails (a) and compactness, whilst the latter fails (b) and overtness.

When we have both modal operators available (Definitions 8.1 and 11.1), we may use them to turn the equational hypotheses into propositions:

$$\phi \wedge \psi = \bot_I \quad \dashv\vdash \quad \phi \wedge \psi \leqslant \omega \quad \dashv\vdash \quad \Diamond(\phi \wedge \psi) \Leftrightarrow \bot$$

and $$\phi \vee \psi = \top_I \quad \dashv\vdash \quad \phi \vee \psi \vee \omega = \top_X \quad \dashv\vdash \quad \Box(\phi \vee \psi) \Leftrightarrow \top.$$

These may then be transferred across the $\vdash$ in (a,b) using Axiom 5.5, to give

$$\Box(\phi \vee \psi) \wedge \Diamond\,\phi \wedge \Diamond\,\psi \quad \Rightarrow \quad \Diamond(\phi \wedge \psi) \qquad \text{(overt)}$$

and $$\Box(\phi \vee \psi) \quad \Rightarrow \quad \Box\,\phi \vee \Box\,\psi \vee \Diamond(\phi \wedge \psi). \qquad \text{(compact)}$$

These may be shown to be equivalent using the mixed modal laws

$$\Box(\phi \vee \psi) \quad \Rightarrow \quad \Box\,\phi \vee \Diamond\,\psi \qquad \text{and} \qquad \Box\,\phi \wedge \Diamond\,\psi \quad \Rightarrow \quad \Diamond(\phi \wedge \psi)$$

and their analogues with ϕ and ψ interchanged.

Alternatively, we may eliminate the modal operators from the hypotheses:

$$\phi \vee \psi = \top_I, \quad \phi \vee \psi = \bot_I \quad \vdash \quad \Box\,\phi \vee \Box\,\psi \Leftrightarrow \top, \quad \Diamond\,\phi \wedge \Diamond\,\psi \Leftrightarrow \bot,$$

which is how we formulate the classical definitions (c,d) of connectedness. $\Box$

Lemma 13.6 Any clopen subspace of a compact overt space is compact overt.

Proof Let the given compact overt (closed sub)space $I \subset H$ be defined by $(\Box, \Diamond, \omega)$, and its clopen subspace $K \subset I$ by predicates ϕ, ψ that cover but are disjoint, so

$$\Box(\phi \vee \psi) \Leftrightarrow \top \quad \text{and} \quad \Diamond(\phi \wedge \psi) \Leftrightarrow \bot,$$

and therefore $$\phi \vee \omega \vee \psi = \top \quad \text{and} \quad \phi \wedge \psi \leqslant \omega.$$

Then I claim that $\varpi x \equiv \omega x \vee \psi x$, $\blacksquare\,\theta \equiv \Box(\theta \vee \psi)$ and $\blacklozenge\,\theta \equiv \Diamond(\theta \wedge \phi)$ define another closed compact overt subspace in the sense of Proposition 11.5.

The relationship between ϖ and $\blacksquare$ is an example of Theorem 8.14, so we must show that $\blacklozenge\,\theta \equiv \Diamond(\theta \wedge \phi) \Leftrightarrow \bot \dashv\vdash \theta \wedge \phi \leqslant \omega \dashv\vdash \theta \leqslant \varpi \equiv \omega \vee \psi$.

$[\vdash]$ $\top \wedge \theta \;\leqslant\; (\phi \vee \omega \vee \psi) \wedge (\theta \vee \omega \vee \psi) \;=\; (\phi \wedge \theta) \vee \omega \vee \psi \;\leqslant\; \omega \vee \psi.$

$[\dashv]$ $\phi \wedge \theta \;\leqslant\; \phi \wedge (\omega \vee \psi) \;=\; (\phi \wedge \omega) \vee (\phi \wedge \psi) \;\leqslant\; (\phi \wedge \omega) \vee \omega \;=\; \omega.$ $\Box$

Now we specialise to $\mathbb{I} \equiv [d, u]$, which is the closed compact overt subspace of $\mathbb{R}$ defined by

$$\omega x \;\equiv\; x < d \vee u < x, \qquad \Box\,\phi \;\equiv\; \forall x{:}[d,u].\ \phi x \quad \text{and} \quad \Diamond\,\phi \;\equiv\; \exists x{:}[d,u].\ \phi x.$$

We find that the classical proof is valid in ASD:

Lemma 13.7 Any non-empty clopen subspace $K \subset \mathbb{I} \equiv [d, u]$ contains the endpoints.

$$\Box\,\bot \vee \Diamond\,\top \Leftrightarrow \bot, \quad \Box\,\bot \wedge \Diamond\,\top \Leftrightarrow \top, \quad \Diamond\,\top \Leftrightarrow \top \quad \vdash \quad u \in K.$$

Proof By Theorem 12.9, K has a maximum element, a. Then by Example 4.6 it is enough to show that $(a < u) \Leftrightarrow \bot$, *cf.* the proof of Theorem 1.1(a).

Since $a \in K \subset \mathbb{I}$, it satisfies $\phi a \Leftrightarrow \top$, where ϕ and ψ define the clopen subspace K as in Lemma 13.6. But by Theorem 10.2, a lies in the interior of ϕ, so $a \in (e,t) \subset [e,t] \subset \phi$. If $u \neq a$ then, without loss of generality, $a < t < u$, so

$$\begin{array}{rcll} \phi a \wedge (u \neq a) & \Leftrightarrow & \exists et.\, (e < a < t < u) \wedge \forall x{:}[e,t].\ \phi x & \\ & \Rightarrow & \exists t{:}\mathbb{I}.\ \phi t \wedge (\mathsf{max}\, K < t) & a \equiv \mathsf{max}\, K,\ x \equiv t \\ & \Leftrightarrow & \exists t{:}\mathbb{I}.\ \phi t \wedge \forall x{:}K.\ (x < t) & \text{Theorem 12.9} \\ & \Leftrightarrow & \exists t{:}\mathbb{I}.\ \phi t \wedge \forall k{:}\mathbb{I}.\ \psi k \vee (k < t) & \text{def } \blacksquare \equiv \forall x : K \\ & \Rightarrow & \exists t{:}\mathbb{I}.\ \phi t \wedge \forall k.\ \psi k \vee (k \neq t) & \text{Axiom 4.9} \\ & \Rightarrow & \exists t{:}\mathbb{I}.\ \phi t \wedge \psi t \ \Leftrightarrow\ \bot & k \equiv t \end{array}$$

We have proved the statement that $u \neq a \Rightarrow \bot$, which is $u = a \in K$ by Definition 4.8.

Theorem 13.8 The interval $\mathbb{I} \equiv [d,u]$ is connected in all of the above senses.

$$\frac{\ldots,\ x : [d,u] \ \vdash\ (\phi x \vee \psi x) \ \Leftrightarrow\ \top \qquad\qquad (\phi x \wedge \psi x) \ \Leftrightarrow\ \bot}{\ldots\ \vdash\ (\forall x{:}[d,u].\ \phi x)\ \vee\ (\forall y{:}[d,u].\ \psi y)\ \Leftrightarrow\ \top}$$

$$\Box(\phi \vee \psi) \ \Rightarrow\ \Box\,\phi \vee \Box\,\psi \vee \Diamond(\phi \vee \psi)$$

Proof By the previous result for K and its complement K' (obtained by interchanging ϕ and ψ in Lemma 13.6), we have four cases:

$$\begin{array}{llcll} K = \emptyset & K' = \emptyset & \qquad & K = \emptyset & u \in K' \\ u \in K & K' = \emptyset & & u \in K & u \in K'. \end{array}$$

In the first, $\phi u \vee \psi u \Leftrightarrow \bot$, so they don't cover, whilst in the last $\phi u \wedge \psi u \Leftrightarrow \top$, so they're not disjoint. If $K = \emptyset$ then $\blacksquare\,\bot \Leftrightarrow \top$ and $\blacklozenge\,\top \Leftrightarrow \bot$ by Theorem 11.9, so $\forall x{:}\mathbb{I}.\ \psi x \Leftrightarrow \top$ and $\exists x{:}\mathbb{I}.\ \phi x \Leftrightarrow \bot$ by Lemma 13.6. Similarly, if $K' = \emptyset$ then ϕ covers instead. $\Box$

Theorem 13.9 $\mathbb{R}$ is overt connected.

Proof Let $\phi, \psi : \Sigma^{\mathbb{R}}$ with $\phi \vee \psi = \top$, $\Diamond\,\phi \equiv \exists x{:}\mathbb{R}.\ \phi x \Leftrightarrow \top$ and $\Diamond\,\psi \equiv \exists z{:}\mathbb{R}.\ \phi z \Leftrightarrow \top$. Then the restrictions of ϕ and ψ to $[x,z]$ or $[z,x]$ satisfy the hypotheses of the rule, so $\Diamond(\phi \wedge \psi)$ on this interval, whence $\exists y{:}R.\ \phi y \wedge \psi y \Leftrightarrow \top$. $\Box$

The approximate intermediate value theorems for $\mathbb{I} \to \mathbb{R}$ and $\mathbb{R} \to \mathbb{R}$ now follow. We shall characterise the *open* connected subspaces of $\mathbb{R}$ as open intervals in Proposition 15.1, and the *compact* connected subspaces as the compact intervals $[\delta, \upsilon]$ in Theorem 15.14.

Lemma 13.10 $[\delta, \upsilon]$ is occupied iff δ and υ are disjoint, and empty iff they overlap.

Proof By Proposition 9.11,

$$\Box\,\bot \ \equiv\ \exists d < u.\, \delta d \wedge \upsilon u \wedge \forall x{:}[d,u].\ \delta x \vee \upsilon x,$$

and by Theorem 13.8,

$$\forall x{:}[d,u].\ \delta x \vee \upsilon x \ \Rightarrow\ \big(\forall x{:}[d,u].\ \delta x\big) \vee \big(\forall x{:}[d,u].\ \upsilon x\big) \vee \big(\exists x{:}[d,u].\ \delta x \wedge \upsilon x\big),$$

so $$\Box\,\bot \ \equiv\ \exists x.\, \delta x \wedge \upsilon x.$$ $\Box$

The next result apparently solves a problem that we might at first think is impossible. Imagine arriving from a hike at an isolated bus stop to find the timetable obliterated. The one daily bus is not there now (ωx), but how can we decide whether we should wait for it (δx) or if it's already gone (υx)? In fact, this just illustrates that the meaning of $\vee$ is weaker in ASD than in some other constructive logics: our argument strengthens the present observation to a temporal one, saying that either the bus hasn't come yet or it will never come again, but it doesn't say which is the case.

Theorem 13.11 Any compact connected subspace $K \equiv (\square, \omega) \subset \mathbb{R}$ is an occupied compact interval $[\delta, \upsilon]$. In particular, any compact *overt* connected subspace is an interval $[d, u]$.

Proof For $x : \mathbb{R}$, let $\phi_x y \equiv (y < x)$ and $\psi_x y \equiv (x < y)$, so $\phi_x \wedge \psi_x = \bot$. By compact connectedness,

$$\omega x \;\equiv\; \square(\lambda y.\, x \neq y) \;\equiv\; \square(\phi_x \vee \psi_x) \;\Leftrightarrow\; \square\,\phi_x \vee \square\,\psi_x,$$

so $\delta d \equiv \square\,\phi_d$ and $\upsilon u \equiv \square\,\psi_u$ provide the pseudo-cut. It is rounded and bounded by Proposition 9.12. Since a compact connected subspace is by definition inhabited, so $\square\,\bot \Leftrightarrow \bot$,

$$\delta x \wedge \upsilon x \;\equiv\; \square(\lambda y.\, y < x) \wedge \square(\lambda y.\, x < y) \;\Leftrightarrow\; \square(\lambda y.\, y < x \wedge x < y) \;\Leftrightarrow\; \square\bot \;\Leftrightarrow\; \bot,$$

so δ and υ are disjoint. If K is also overt, Theorem 12.9 says that (either $K \cong \emptyset$, which is forbidden by either definition of connectedness or) it has $u \equiv \mathsf{max}\, K$ and similarly $d \equiv \mathsf{min}\, K$. $\square$

We conclude with the main result about maps *out* of a connected (sub)space. This can be used to develop the notion of *path*-connectedness.

Proposition 13.12 The direct image under $f : X \to Y$ of a compact or overt connected subspace $I \subset X$, in the senses of Theorem 8.8 and Notation 11.11 respectively, is also connected as a subspace of Y.

Proof Let $I \subset X \xrightarrow{f} Y \supset fI$ with I compact connected, and $\phi, \psi : \Sigma^Y$ such that $\phi \wedge \psi = \bot_{fI}$, so $\phi \cdot f \wedge \psi \cdot f = \bot_I$. Then

$$[fI](\phi \vee \psi) \;\equiv\; [I](\phi \cdot f \vee \psi \cdot f) \;\Rightarrow\; [I](\phi \cdot f) \vee [I](\psi \cdot f) \;\equiv\; [fI]\phi \vee [fI]\psi,$$

and the overt result is similar. $\square$

The next two sections bring our investigation of the connectedness of the real line to a conclusion by proving the intermediate value theorem and the characterisation of open subspaces.

14 The intermediate value theorem

We are now able to prove the results of Section 2 within ASD. It turns out that, in the non-singular case on a closed bounded interval, the approximate forms of the intermediate value theorem in the previous section are enough to ensure that the solution-set $S = Z$ is compact, overt and non-empty, and therefore has a maximum element. In the singular case, on the other hand, the *useful* zeroes are the stable ones, which we access *via* $\lozenge$. However, to prove the crucial property that this takes unions to disjunctions, we need to use connectedness again. This therefore relies on the whole of the main line of development of this paper: connectedness of $\mathbb{R}$, maxima of non-trivial compact overt subspaces and compactness of the closed interval.

We begin by formulating the relevant definitions from Sections 1 and 2 in ASD.

Definition 14.1 A function $\ldots, x : \mathbb{R} \vdash fx : \mathbb{R}$ (Definition 5.2)
(a) ***doesn't hover*** (*cf.* Definition 1.4) if, for $d, u : \mathbb{R}$,

$$d < u \;\Rightarrow\; \exists x.\, (d < x < u) \;\wedge\; (fx \neq 0);$$

(b) ***doesn't touch from below without crossing*** if

$$(d < u) \;\wedge\; (fd < 0) \;\wedge\; (fu < 0) \;\Rightarrow\; \big(\forall x{:}[d, u].\, fx < 0\big) \;\vee\; \big(\exists x{:}[d, u].\, fx > 0\big);$$

(c) ***doesn't touch from above without crossing*** if

$$(d < u) \ \wedge \ (fd > 0) \ \wedge \ (fu > 0) \ \Rightarrow \ \big(\forall x{:}[d,u].\ fx > 0\big) \ \vee \ \big(\exists x{:}[d,u].\ fx < 0\big);$$

(d) a number $a : \mathbb{R}$ is a ***stable zero*** of f (*cf.* Definition 1.8) if

$$(d < a < u) \ \Rightarrow \ \exists et.\, (d < e < a < t < u) \ \wedge \ (fe < 0 < ft \ \vee fe > 0 > ft);$$

(e) the ***possibility modal operator*** $\Diamond : \Sigma^{\Sigma^{\mathbb{R}}}$ (*cf.* Proposition 2.1) is, for $\phi : \Sigma^{\mathbb{R}}$,

$$\Diamond\, \phi \ \equiv \ \exists d < u.\ \big(\forall x{:}[d,u].\ \phi x\big) \ \wedge \ (fd < 0 < fu \ \vee fd > 0 > fu);$$

(f) and the ***solution co-classifier*** (or ***non-solution classifier***) is $\omega x \ \equiv \ (fx \neq 0)$.

(g) Restricting to an interval $\mathbb{I} \equiv [d,u]$, the ***necessity modal operator*** $\Box : \Sigma^{\Sigma^{\mathbb{I}}}$ is

$$\Box\, \phi \ \equiv \ \forall x{:}\mathbb{I}.\ (fx \neq 0) \vee \phi x.$$

Lemma 14.2
(a) ω defines a compact closed subspace (Z), as in Proposition 2.13;
(b) $\Diamond \bot \Leftrightarrow \bot$ trivially;
(c) $fd < 0 < fu \ \Rightarrow \ \Diamond \top$;
(d) $\Diamond\, \omega \Leftrightarrow \bot$ by Proposition 13.3;
(e) if $a \in \Diamond$ then $a \in \{x \mid \neg \omega x\}$ (so $S \subset Z$ in the notation of Section 2);
(f) restricting to a bounded closed interval, $\Box$ defines a compact closed subspace by Theorem 8.14. $\Box$

First we prove the analogue of Proposition 2.1.

Proposition 14.3 If $a : \mathbb{R}$ is a stable zero then it is an accumulation point of $\Diamond$ in the sense of Definition 11.7, *i.e.* $\phi a \Rightarrow \Diamond\, \phi$. The converse holds if f doesn't hover.

Proof If a is a stable zero then, for $c < d < a < u < v$,

$$\begin{array}{rll} \phi a & \Rightarrow \ \exists cv.\, (c < a < v) \wedge \forall x{:}[c,v].\ \phi x & \text{Theorem 10.2} \\ & \Rightarrow \ \exists du.\, (d < a < u) \wedge \forall x{:}[d,u].\ \phi x & \text{stability in } (c,v) \\ & \qquad \wedge (fd < 0 < fu \vee fd > 0 > fu) & \\ & \equiv \ \Diamond\, \phi & \end{array}$$

Conversely, suppose that $\phi a \Rightarrow \Diamond\, \phi$ and let $c < a < v$. With $\phi x \equiv (c < x < v)$,

$$\top \ \Leftrightarrow \ \phi a \ \Rightarrow \ \Diamond\, \phi \ \Rightarrow \ \exists de.\, (c < d < e < v) \wedge (fd < 0 < fe \vee fd > 0 > fe).$$

If d and e lie on opposite sides of a then it is a stable zero. Otherwise, say $d < e < a$, we use the hypothesis that f doesn't hover, so there is some $a < u < v$ with $fu \neq 0$, which replaces either d or e in what is required. $\Box$

Next we characterise the non-singular case.

Proposition 14.4 If all zeroes are stable then $(\Diamond, \omega)$ define a closed overt subspace (Definition 11.1). In a bounded interval this is also compact, and $\Diamond$ and $\Box$ satisfy all of the modal laws, in particular

$$\Diamond(\phi \vee \psi) \ \Rightarrow \ \Diamond\, \phi \vee \Diamond\, \psi \qquad \Box\, \phi \wedge \Diamond\, \psi \ \Rightarrow \ \Diamond(\phi \wedge \psi) \quad \text{and} \quad \Box(\phi \vee \psi) \ \Rightarrow \ \Box\, \phi \vee \Diamond\, \psi.$$

Proof $\omega x \Leftrightarrow \bot \dashv\vdash fx = 0 \dashv\vdash \lambda\phi.\, \phi x \leqslant \Diamond$, so $\phi x \Rightarrow \Diamond\, \phi \vee \omega x$ and $\Diamond\, \phi \Leftrightarrow \bot \dashv\vdash \phi \leqslant \omega$ by Axiom 5.5. Propositions 11.2 and 11.5 derived the modal laws from this. $\Box$

So when are all zeroes stable?

Lemma 14.5 If f doesn't hover or touch without crossing then all zeroes are stable.

Proof Let $a:\mathbb{R}$ with $fa=0$.

Given $d<a<u$, since f doesn't hover, there are $d<e<a<t<u$ with $fe\neq 0$ and $ft\neq 0$. If $fe<0<ft$ or $fe>0>fu$ then a is stable. Suppose $fe,ft<0$, whilst f doesn't touch from below without crossing, so perhaps $\forall x{:}[e,t].\ (fx<0)$, but $x\equiv a$ falsifies this. Hence there is $e<x<t$ with $fx>0$, so $x\neq a$. Then either $d<e<a<x<t<u$ with $fe<0<fx$ or $d<e<x<a<t<u$ with $ft<0<fx$ so a is stable. Similarly if $fe,ft>0$, since f doesn't touch from above without crossing. □

The converse is also true:

Lemma 14.6 If all zeroes are stable then f doesn't hover.

Proof Let $d<u$ and suppose that $\big(\exists x.\,(d<x<u)\wedge(fx\neq 0)\big)\Leftrightarrow\bot$. By Axiom 5.9,

$$d<x<u\ \vdash\ (fx\neq 0)\Leftrightarrow\bot\ \ \dashv\vdash\ \ fx=0:\mathbb{R},$$

so x is a zero, which should be stable. But then there would be $d<e<x<t<u$ with $(fe<0<ft)$ or $(fe>0>ft)$, but there aren't. Hence $\exists x.\,(d<x<u)\wedge(fx\neq 0)$ by Axiom 5.5. □

Lemma 14.7 If all zeroes are stable then f doesn't touch from below without crossing.

Proof Let $d<u$ with $fd,fu<0$ and suppose that $\big(\exists x{:}[d,u].\ fx>0\big)\Leftrightarrow\bot$. By Axiom 5.9,

$$d\leq x\leq u\ \vdash\ fx\leq 0.$$

If any such x has $fx=0$, *i.e.* it's a zero, then this is stable by hypothesis, so there are $d<e<x<t<u$ for which $(fe<0<ft)$ or $(fe>0>ft)$, but this is impossible since $fe,ft\leq 0$. Hence $\forall x{:}[d,u].\ (fx<0)$, and the result follows by Axiom 5.5. □

Remark 14.8 Now we turn to the singular case, *i.e.* functions that don't hover but may touch without crossing, such as polynomials with double zeroes. Classically, $S\neq Z$, but as *all* accumulation points are stable, S is also a closed subspace. Remark 16.14 suggests that this is not the case in ASD. We then have an operator $\lozenge$ that is related to neither ω nor $\Box$, so the *topological* results of Section 11 do not apply.

The *computational* results (Theorems 2.4 and 2.6), on the other hand, still work, so long as $\lozenge$ takes unions to disjunctions.

Theorem 14.9 If f doesn't hover then $\lozenge$ preserves joins, $\lozenge(\exists j.\,\theta_j)\ \Rightarrow\ \exists j.\ \lozenge\,\theta_j$.

Proof Define $\qquad \phi x\ \equiv\ \exists j.\,\exists y.\,(x<y)\wedge(fy<0)\wedge\forall z{:}[x,y].\ \theta_j z$

and $\qquad \psi x\ \equiv\ \exists j.\,\exists y.\,(x<y)\wedge(fy>0)\wedge\forall z{:}[x,y].\ \theta_j z,$

which say that x is connected *via* a *component* of one of the θ_j to some negative or positive value respectively. Then

$$\begin{array}{rcll}\phi x\vee\psi x & \Rightarrow & \exists j.\,\exists y.\,(x<y)\wedge\forall z{:}[x,y].\ \theta_j z & \text{def. }\phi,\psi\\ & \Rightarrow & \exists j.\,\theta_j x & z\equiv x\\ & \Rightarrow & \exists j.\,\exists du.\,(d<x<u)\wedge\forall z{:}[d,u].\ \theta_j z & \text{Theorem 10.2}\\ & \Rightarrow & \exists j.\,\exists y.\,(x<y)\wedge(fy\neq 0)\wedge\forall z{:}[x,y].\ \theta_j z & \text{non-hovering}\\ & \equiv & \phi x\vee\psi x, & \text{def. }\phi,\psi\end{array}$$

since f doesn't hover in the interval (x,u). By a similar argument,

$$(fx<0)\ \wedge\ \exists j.\,\theta_j x\ \Rightarrow\ \phi x\qquad\text{and}\qquad(fx>0)\ \wedge\ \exists j.\,\theta_j x\ \Rightarrow\ \psi x.$$

Putting these two things together with the rule $\blacksquare(\phi \vee \psi) \wedge \blacklozenge\phi \wedge \blacklozenge\psi \Rightarrow \blacklozenge(\phi \wedge \psi)$ for connectedness of the interval $[d, u]$, we have

$$\begin{array}{rcll} \Diamond(\exists j.\, \theta_j) & \equiv & \exists d < u.\, (fd < 0 < fu \vee fd > 0 > fu) \wedge \forall z{:}[d,u].\ \exists j.\, \theta_j z & \text{def } \Diamond \\ & \Rightarrow & \exists d < u.\, \big((\phi d \wedge \psi u) \vee (\psi d \wedge \phi u)\big) \wedge \forall z{:}[d,u].\ (\phi z \vee \psi z) & \text{above} \\ & \Rightarrow & \exists d < u.\, \blacklozenge\phi \wedge \blacklozenge\psi \wedge \blacksquare(\phi \vee \psi) & \phi d \Rightarrow \blacklozenge\phi \\ & \Rightarrow & \exists d < u.\, \blacklozenge(\phi \wedge \psi) \ \equiv\ \exists d < u.\, \exists z{:}[d,u].\ \phi z \wedge \psi z & \text{connectedness} \\ & \Rightarrow & \exists e < t.\, \forall z{:}[e,t].\ \phi z \wedge \psi z & \text{Theorem 10.2} \\ & \Rightarrow & \exists x.\, (fx \neq 0) \wedge \phi x \wedge \psi x & \text{non-hovering} \end{array}$$

Finally, by the definition of ψ,

$$\exists x.\, (fx < 0) \wedge \psi x \quad \Rightarrow \quad \exists j.\, \exists x < y.\, (fx < 0 < fy) \wedge \forall z{:}[x,y].\ \theta_j z \quad \Rightarrow \quad \exists j.\ \Diamond\, \theta_j,$$

and similarly $\exists x.\, (fx > 0) \wedge \phi x \ \Rightarrow\ \exists j.\ \Diamond\, \theta_j$, so $\Diamond(\exists j.\, \theta_j) \Rightarrow \exists j.\ \Diamond\, \theta_j$. □

Proposition 14.10 $\Box\, \phi \wedge \Diamond\, \psi \ \Rightarrow\ \Diamond(\phi \wedge \psi)$, *cf.* Proposition 2.13.

Proof $\Diamond\, \psi$ says that ψ contains a straddling interval $[e, t]$, whilst $\Box\, \phi$ says that $\phi \vee \omega$ cover the whole interval $[d, u]$, and in particular $[e, t]$. Hence $\psi \wedge (\phi \vee \omega)$ contains a straddling interval, so since $\Diamond\, \omega \Leftrightarrow \bot$ by Proposition 13.2,

$$\Box\, \phi \wedge \Diamond\, \psi \ \Rightarrow\ \Diamond\, \big((\phi \vee \omega) \wedge \psi\big) \ \Rightarrow\ \Diamond(\phi \wedge \psi) \vee \Diamond\, \omega \ \Rightarrow\ \Diamond(\phi \wedge \psi).$$ □

Theorem 14.11 If all zeroes are stable, they form a closed overt subspace. In any bounded interval this is compact and has a maximum. So, in this case, the "closed formula" in Theorem 1.1(a) is not only valid but also computationally meaningful.

If f doesn't hover but may touch without crossing, $\Diamond\, \top$ is a program that finds a zero, non-deterministically (Proposition 2.6). □

Proof The compact overt subspace is either empty or has a maximum (Theorem 12.9), but it can't be empty by Propositions 13.2f. □

15 Local connectedness

This section completes the classification of open and connected subspaces of $\mathbb{R}$ that we began in Remark 10.13. The classical result says that every open subspace is a disjoint union of at most countably many open intervals, but some of these words need constructive qualification. We shall also see that the main theorem depends on Heine–Borel, so Bishop will leave us before the end of the journey. We present the characterisation in several idioms, using both open equivalence relations and categorical universal properties.

We begin with open intervals, whose endpoints are respectively descending and ascending but not necessarily Euclidean real numbers (*cf.* Definition 3.3 and Example 16.6). The formulations of convexity come from [Man83]; they are a one-dimensional form of path connectedness.

Proposition 15.1 The following are equivalent for the open subspace classified by $\alpha : \Sigma^{\mathbb{R}}$:
(a) it is an overt connected subspace;
(b) it is inhabited and ***paraconvex***: $\alpha x \wedge (x < y < z) \wedge \alpha z \ \Rightarrow\ \alpha y$;
(c) it is inhabited and ***convex***: $\alpha x \wedge (x < z) \wedge \alpha z \ \Rightarrow\ \forall y{:}[x,z].\ \alpha y$;
(d) $\alpha = \delta \wedge \upsilon$, where (δ, υ) is a (rounded and) overlapping pseudo-cut.
Even though its lower endpoint υ is a descending real, and the upper one δ is ascending, *i.e.* they need not be located, we understand the term ***open interval*** to refer to such a subspace, for which we write (υ, δ).

Proof [b⊣⊢c⊣⊢d]: Recall from Exercise 11.20 that any open subspace is overt, with $\Diamond\phi \equiv \exists x{:}\mathbb{R}.\ \alpha x \wedge \phi x$. Put $\delta d \equiv \exists x.\ d < x \wedge \alpha x$ and $\upsilon u \equiv \exists x.\ \alpha x \wedge x < u$, *cf.* Notation 12.2. These are rounded lower/upper and overlap ($\exists x.\ \delta x \wedge \upsilon x$). They satisfy $\alpha \leqslant \delta \wedge \upsilon$ by Theorem 10.2, whilst (b) says that $\delta \wedge \upsilon \leqslant \alpha$.

[d⊢a]: we modify the proof of Theorem 13.9: $\phi \vee \psi \geqslant \alpha$ and $\delta x \wedge \upsilon x \wedge \delta z \wedge \upsilon z \Leftrightarrow \top$ give $\forall y{:}[x,z].\ \delta y \wedge \upsilon y$.

[a⊢b]: Any overt connected subspace is inhabited, by definition. For $x : \mathbb{R}$, consider $\phi x \equiv \alpha x \wedge (x < y)$ and $\psi x \equiv \alpha x \wedge (x > y)$, so $\Diamond(\phi \wedge \psi) \Leftrightarrow \bot$. Then, by Lemma 5.8,

$$\alpha x \ \Rightarrow \alpha x \wedge (\alpha y \vee x \neq y) \ \Rightarrow\ \alpha y \vee \phi x \vee \psi x.$$

If $\alpha y \Leftrightarrow \bot$, this says that $\alpha \leqslant \phi \vee \psi$, so we may apply overt connectedness:

$$\ldots,\ \alpha y \Leftrightarrow \bot\ \vdash\ \alpha x \wedge (x < y < z) \wedge \alpha z\ \Rightarrow\ \phi x \wedge \psi z\ \Rightarrow\ \Diamond\phi \wedge \Diamond\psi\ \Longrightarrow\ \Diamond(\phi \wedge \psi)\ \Leftrightarrow\ \bot,$$

from which paraconvexity follows by Axiom 5.5. □

Remark 15.2 The characterisation of open subsets of the line breaks into two parts, an infinitary one that uses compactness to reduce to a finite open sub-cover, and a finitary, combinatorial one that generalises the definition of overt connectedness from two parts to many. In order to keep Bishop's company for a little longer, we present the finitary part first.

We shall build up a permutation of $\mathbf{m} \equiv \{i : \mathbb{N} \mid i < m\}$ as a composite of swaps. This uses either group theory or programming, but both of a very simple kind. Unfortunately, the simplest combinatorial argument becomes extremely complicated if we dwell on the minutiae of formal logical calculi. It is essential to understand how these relate to the usual idioms of mathematics: this is explained in [Tay99, §§1.6 & 6.5]. The main point is that the existence of an element can contribute to that of a list, without using any form of Choice, so long as we do not attempt to *export* a *particular* list from the argument.

Something also needs to be said about the foundations of list manipulation in ASD, since this is pure topology with no underlying set theory. $\mathsf{List}N$ is a *space*, not a set, and it is constructed for *overt discrete* N in [E]. The central idea of that paper is that a list is a compact overt subspace of N, but the methods of parametric list induction that we shall use in this section are also justified there. However, since, as we have seen in this paper, compact overt subspaces of $\mathbb{R}$ look nothing at all like lists, when we want lists of numbers, we have to use rational ones. This does not present any real handicap, because Theorem 10.2 and its corollaries allow us to replace real numbers by rationals in the relevant situations.

Lemma 15.3 Let $\theta_0, \ldots, \theta_{m-1}$ be open subsets of a space X that are each inhabited in, and together cover, an overt connected subspace $S \subset X$ defined by $\Diamond$, so

$$\ldots,\ i : \mathbb{N}\ \vdash\ \theta_i : \Sigma^X,\quad \Diamond\theta_i \Leftrightarrow \top \qquad \text{and} \qquad \ldots\ \vdash\ m : \mathbb{N},\quad (\exists i < m.\ \theta_i) = \top_S.$$

Then the overlaps of the θ_i define a ***connected graph***, in the sense that there is some permutation $p : \mathbf{m} \cong \mathbf{m}$ for which

$$\forall 1 \le i < m.\ \exists 0 \le j < i.\ \Diamond(\theta_{p(i)} \wedge \theta_{p(j)}).$$

Proof It is important for the subsequent infinitary argument that the number m in this one be a *parameter*, so we "fill out" the definition of θ_i for all $i : \mathbb{N}$ by putting $\theta_i \equiv \top$ for $i \ge m$.

We prove by induction on $1 \le k \le m$ that

$$\exists p{:}\mathbf{m} \cong \mathbf{m}.\ \left\{ \begin{array}{ll} & \forall 1 \le i < k.\ \exists 0 \le j < i.\ \Diamond(\theta_{p(i)} \wedge \theta_{p(j)}) \\ \wedge & \forall k \le i < m.\ p(i) = i, \end{array} \right.$$

where the initial case $k \equiv 1$ is satisfied by $p \equiv \mathsf{id}$ and the final one $k \equiv m$ gives the required result. Assume the induction hypothesis for some $1 \le k < m$ and put

$$\phi x\ \equiv\ \exists 0 \le j < k.\ \theta_{p(j)} x \quad \text{and} \quad \psi x\ \equiv\ \exists k \le i < m.\ \theta_i x.$$

Then $\phi \vee \psi = \top_S$, whilst $\top \Leftrightarrow \Diamond\,\theta_0 \Rightarrow \Diamond\,\phi$ and $\top \Leftrightarrow \Diamond\,\theta_{m-1} \Rightarrow \Diamond\,\psi$. Since $\Diamond$ is overt connected and preserves joins, we deduce

$$\Diamond(\phi \wedge \psi) \;\equiv\; \exists k \le i < m.\, \exists 0 \le j < k.\; \Diamond(\theta_i \wedge \theta_{p(j)}).$$

Let $s : \mathbf{m} \cong \mathbf{m}$ be the swap that just interchanges k with such an $i \equiv p(i)$. Then the new permutation $p' \equiv s \cdot p$ satisfies the induction hypothesis for $k+1$ in place of k. $\square$

The "infinitary" part of the proof uses the Heine–Borel property, in the form of Corollary 9.5, where the directed relation uses $\mathbb{N}$ rather that $\mathbb{Q}$. So this is where Bishop takes his leave of us.

Lemma 15.4 Let $\sim$ be an open reflexive relation on $\mathbb{I} \equiv [0,1]$, that is,

$$\ldots,\; x, y : \mathbb{R} \;\vdash\; x \sim y : \Sigma \quad \text{such that} \quad \forall x{:}[0,1].\; x \sim x.$$

Then $\sim$ is *represented* by finitely many dyadic rationals, in the sense that

$$\exists n{:}\mathbb{N}.\; \forall x{:}\mathbb{I}.\; \exists k{:}\mathbb{N}.\;\; 0 \le k \le 2^n \;\wedge\; x \sim k \cdot 2^{-n}.$$

Proof

$$\begin{array}{llr} & \forall x{:}[0,1].\; x \sim x & \text{reflexivity} \\ \Rightarrow & \forall x.\, \exists d, u{:}\mathbb{R}.\; d < x < u \;\wedge\; \forall y{:}[d,u].\; x \sim y & \text{Theorem 10.2} \\ \Rightarrow & \forall x.\, \exists n, k{:}\mathbb{N}.\; 0 \le k \le 2^n \;\wedge\; x \sim k \cdot 2^{-n} & \text{Lemma 6.9} \\ \Rightarrow & \exists n.\, \forall x.\, \exists k.\; 0 \le k \le 2^n \;\wedge\; x \sim k \cdot 2^{-n} & n \nearrow \infty \;\square \end{array}$$

Theorem 15.5 Let $\sim$ be an open *equivalence* relation on $\mathbb{I} \equiv [0,1]$, *i.e.* one that satisfies the previous lemma and is also symmetric ($x \sim y \Rightarrow y \sim x$) and transitive ($x \sim y \sim z \Rightarrow x \sim z$). Then it is ***indiscriminate***, *i.e.* $\forall x, y{:}\mathbb{I}.\; x \sim y$, and in particular $0 \sim 1$.

Proof Using n from Lemma 15.4, put $m \equiv 2^n + 1$ and $\theta_i x \equiv (x \sim i \cdot 2^{-n})$ in Lemma 15.3, so $\sim$ is connected in the graph-theoretic sense. As it is also symmetric and transitive, $0 \sim 1$, and more generally $\forall xy{:}[0,1].\; x \sim y$. $\square$

Corollary 15.6 Any open partial equivalence relation $\sim$ on $\mathbb{R}$ satisfies

$$\big(\forall y{:}[x,z].\; y \sim y\big) \;\Rightarrow\; x \sim z. \qquad \square$$

Corollary 15.7 Any function $f : X \to N$ with N discrete (where $X \equiv \mathbb{I}$, $\mathbb{R}$, (d,u) or (υ, δ)) is constant.

Proof The open equivalence relation $(x \sim y) \equiv (fx =_N fy)$ is indiscriminate. $\square$

Bishop can easily prove this when equality on N is decidable, or even if $N \cong N_1 + N_2$ non-trivially, but we do not assume either of these things. He is now on his own travels, visiting the Recursive Analysts in Russia, from where he sends us a postcard, although it was actually written by Andrej Bauer:

Example 15.8 In Recursive Analysis, let $n : \mathbb{N} \vdash \theta_n$ be a singular cover of $\mathbb{I}$, *i.e.* a family of intervals of total length $< \frac{1}{2}$ that cover all definable real numbers, whilst no finite sub-family does so. Define the reflexive relation $\sim$ by

$$(x \sim z) \;\equiv\; \exists n.\, \forall y{:}[x,z].\; \theta_n y.$$

Then its symmetric transitive closure has infinitely many equivalence classes in $\mathbb{I}$. $\square$

We have already used this idea in Theorems 2.4 and 14.9, and we use it again to identify the "countably many" intervals, indexing them by the subquotient of $\mathbb{Q}$ by an open partial equivalence relation. However, this relation need not be decidable (Example 16.5), so the indexing set need not be Hausdorff like $\mathbb{N}$ or $\mathbb{Q}$.

Notation 15.9 Given any $\phi : \Sigma^{\mathbb{R}}$, define

$$\begin{aligned}(x \approx_\phi y) \quad &\equiv \quad \big([x,y] \subset \phi\big) \wedge \big([y,x] \subset \phi\big) \\ &\equiv \quad \big(x > y \vee \forall z{:}[x,y].\ \phi z\big) \ \wedge\ \big(x < y \vee \forall z{:}[y,x].\ \phi z\big),\end{aligned}$$

which we explicitly make symmetric because the formulae that we gave in [I, Section 10] for the bounded quantifier $\forall z{:}[x,y].\ \phi z$ depended on $x \leq y$.

Proposition 15.10 This an *open partial equivalence relation* on $\mathbb{R}$ that is reflexive on the open subspace classified by ϕ:

$$\phi x \Rightarrow (x \approx x), \qquad x \approx y \Rightarrow y \approx x \quad \text{and} \quad x \approx y \approx z \Rightarrow x \approx z.$$

As with any equivalence relation, the classes are "disjoint" in the sense that if any two overlap, they actually coincide. Each of these classes is open and connected (Proposition 15.1).

It is the *sparsest* such relation: any other one, $\sim$, satisfies $(x \approx y) \Rightarrow (x \sim y)$.

Finally, $(x \approx x) \Rightarrow \phi x$ from the definition. □

Superlatives, such as "sparsest", are known as ***universal properties***, albeit only in the *lattice* of open partial equivalence relations. We shall now express the characterisation as a *categorical* universal property. We present this without parameters, partly because category theory expresses dependent types in a rather clumsy way that involves stability of various properties under pullback [Tay99, Chapters VIII & IX]. It is also because these pullbacks do not exist amongst (non-Hausdorff) locally compact spaces, and so the relevant quotient was constructed without parameters in [C, §10].

The next result makes heavy use of the properties of (pre-)open maps in [C].

Proposition 15.11 From any open subspace $U \subset \mathbb{R}$ there is an open surjection $p : U \twoheadrightarrow N/{\approx}$ with open connected fibres onto an overt discrete space.

Proof U is classified by $\phi : \Sigma^{\mathbb{R}}$ without parameters. Let $N \equiv \{a : \mathbb{Q} \mid \phi a\} \subset \mathbb{Q}$, which is an open subspace of an overt discrete space, and therefore itself overt discrete. Then $\approx$ restricts to a (total) open equivalence relation on N, so the quotient $N/{\approx}$ is also an overt discrete space, and the map $q : N \to N/{\approx}$ is an open surjection.

By Theorem 10.2, the partial equivalence relation $\approx$ satisfies

$$\phi x \ \Rightarrow\ \exists a.\, x \approx a \qquad \text{and} \qquad x \approx a \ \wedge\ x \approx b \ \Rightarrow\ a \approx b,$$

for $a, b : \mathbb{Q}$. Hence, for $x \in U$, $x \approx a$ is a description that defines $[a] : N/{\approx}$, so there is a function $p : U \to N/{\approx}$ that makes the triangle commute:

$$\begin{tikzcd} \mathbb{Q} \arrow[d] & N \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, "q"] & \\ \mathbb{R} & U \arrow[l, hook'] \arrow[r, dotted, "p"] & N/{\approx} \end{tikzcd}$$

The "inclusion" $\mathbb{Q} \to \mathbb{R}$ is dense, so $\Sigma^{\mathbb{R}} \to \Sigma^{\mathbb{Q}}$ is mono. Since the square is a pullback and the horizontal maps are open inclusions, $\Sigma^U \to \Sigma^N$ is also mono. Hence its composite with the map $\exists_q : \Sigma^N \to \Sigma^{N/\approx}$ that makes q open is $\exists_p : \Sigma^U \to \Sigma^{N/\approx}$ making p open too. □

Lemma 15.12 The relation $\leq$ on $N/{\approx}$ defined by

$$[x] \leq [y] \ \equiv\ \exists a, b{:}\mathbb{Q}.\ x \approx a \leq b \approx y$$

is a total but not necessarily decidable order, in the sense that

$$[x] \leq [x], \qquad [x] \leq [y] \leq [x] \Rightarrow [x] \leq [z],$$

$$[x] \leq [y] \leq [x] \Rightarrow (x \approx y), \qquad [x] \leq [y] \vee [y] \leq [x]. \qquad \square$$

Theorem 15.13 Every open subspace $U \subset \mathbb{R}$ is ***locally connected***:
(a) there is a map $p : U \twoheadrightarrow N/{\approx}$ with $N/{\approx}$ discrete;
(b) any map $f : U \to M$ to a discrete space factors uniquely as

$$\begin{tikzcd} U \arrow[d, two heads, "p"'] \arrow[dr, "f"] & \\ N/{\approx} \arrow[r, dotted] & M \end{tikzcd}$$

(c) $N/{\approx}$ is overt and p is an open surjection;
(d) this representation is unique up to unique isomorphism.

Proof [b] $(x \sim y) \equiv (fx =_M fy)$ is an open partial equivalence relation on $\mathbb{R}$ with $\phi x \Rightarrow x \sim x$, so $x \approx y \Rightarrow x \sim y$ by Proposition 15.10. This means that f factors uniquely through the quotient [C, Lemma 10.8]. [a,c] are just restatements, and [d] is the standard argument for universal properties: put the alternative candidate in the place of M to obtain the unique isomorphism. $\square$

Finally, we complete the characterisation of compact connected subspaces of $\mathbb{R}$ that we began in Theorem 13.11.

Theorem 15.14 Every compact interval $[\delta, \upsilon]$ with δ and υ disjoint is compact connected.

Proof Proposition 9.11 defines $\Box\,\phi \equiv \forall x{:}[\delta, \upsilon].\ \phi x$ as

$$\Box\,\phi \;\equiv\; \exists d < u.\ \delta d \wedge \upsilon u \wedge \forall x{:}[d, u].\ \delta x \vee \phi x \vee \upsilon x.$$

Compact connectedness (Definition 13.1(b)) says that the compact space must be occupied (as it is, by Lemma 13.10) and, for $\phi, \psi : \Sigma^{\mathbb{R}}$,

$$\phi \wedge \psi \leqslant \delta \vee \upsilon \;\vdash\; \Box(\phi \vee \psi) \;\Rightarrow\; \Box\,\phi \vee \Box\,\psi.$$

By inspection of the definition of $\Box$, it is enough to show that

$$\forall x{:}\mathbb{I}.\ (\delta x \vee \phi x \vee \psi x \vee \upsilon x) \;\Rightarrow\; \forall x{:}\mathbb{I}.\ (\delta x \vee \phi x \vee \upsilon x) \;\vee\; \forall x{:}\mathbb{I}.\ (\delta x \vee \psi x \vee \upsilon x)$$

on the assumption that $\phi \wedge \psi \leqslant \delta \vee \upsilon$, $\delta 0$, $\upsilon 1$, $\neg\upsilon 0$ and $\neg\delta 1$ (using Axiom 5.5).

Let $\approx_\delta$, $\approx_\phi$, $\approx_\psi$ and $\approx_\upsilon$ be the symmetric, transitive relations defined from these four predicates by Notation 15.9. By hypothesis, their union is reflexive:

$$\forall x{:}\mathbb{I}.\ x \approx_\delta x \;\vee\; x \approx_\phi x \;\vee\; x \approx_\psi x \;\vee\; x \approx_\upsilon x.$$

We could use Lemma 15.4 at this point, but it seems to lead to a longer proof.

Instead, we consider the symmetric–transitive closure $\sim$ of this union, which is a total equivalence relation on $\mathbb{I}$. Because of Remark 15.2, the definition of $\sim$ requires some care. We write

$$x \sim z \;\equiv\; \exists m.\, \exists \theta_0, \ldots, \theta_{m-1} \in \{\delta, \phi, \psi, \upsilon\}.\, \exists y_1, \ldots, y_{m-1}{:}\mathbb{Q}.\ \bigwedge_{i=0}^{m-1} (y_i \approx_{\theta_i} y_{i+1}),$$

in which we understand that $y_0 \equiv x$, $y_m \equiv z$ and $0 \leq y_i \leq 1$, whilst the θ_i are *distinguishable labels* for predicates, not the predicates themselves. There is no need for the y_i to be listed in ascending arithmetical order.

It follows from Theorem 15.5 that $0 \sim 1$, whence (making essential use of [E]), there is a *chain* of m *links* θ_i and $m+1$ *nodes* y_i with $y_0 \equiv 0$ and $y_m \equiv 1$. We may also assume that there is no shorter such chain.

If two adjacent links have the same label, $\theta_{i-1} \equiv \theta_i \equiv \theta$, then

$$\forall x{:}[y_{i-1}, y_i].\ \theta x \ \wedge\ \forall x{:}[y_i, y_{i+1}].\ \theta x, \quad \text{so} \quad \forall x{:}[y_{i-1}, y_{i+1}].\ \theta x,$$

whatever the arithmetical order of y_{i-1}, y_i and y_{i+1}. In this case, we may omit the node y_i, to obtain a shorter chain. So in fact successive links have different labels.

If any node satisfies δy_{i+1} then $\forall x{:}[y_0, y_{i+1}].\ \delta x$. If $i > 0$ then this offers us a shorter chain, so $i = 0$ and we may also re-label the first link as $\theta_0 \equiv \delta$. Similarly, the only occurrence of υ as a label is as the final link.

This means that, if both ϕ and ψ occur as labels, then they must do so adjacently. Then $\phi y_i \wedge \psi y_i$ for some i, so $\delta y_i \vee \upsilon y_i$ by hypothesis. But then $\theta_{i-1} = \delta$ or $\theta_i = \upsilon$ by the foregoing argument.

Hence the chain has at most three links, being either $\{\delta, \phi, \upsilon\}$ or $\{\delta, \psi, \upsilon\}$ or a sub-chain of one of these, and the result follows. $\square$

16 Some counterexamples

No account of real analysis would be complete without some pathological examples, but we are saved from a morbid fascination with them by the fact that the ASD λ-calculus only allows us to define continuous functions. We concentrate on demonstrating the necessity of the new concept of overtness, leaving the verification of the defining properties of compact overt subspaces in Definition 8.1 and Proposition 11.5 as exercises for the reader.

Remark 16.1 A word of caution. As these counterexamples rely on recursion-theoretic ideas, the difference between falsity and unprovability is crucial. This is particularly significant when we explore the notion of connectedness, because of the fact that disconnected spaces are of *positive* interest in topology, in a way that is not the case for (sub)spaces that fail to be open, closed or compact.

Recall from Remark 4.5 that statements $\sigma \Leftrightarrow \tau$ are equations between terms of type Σ. In algebra, when we say that $s = 1$ doesn't hold, it doesn't mean that $s = 0$. Similarly, failure of $\sigma \Leftrightarrow \top$ does not signify $\sigma \Leftrightarrow \bot$ or *vice versa*. We shall see why this is the best use of language, in order to take full practical advantage of the new concept of overtness.

We find that a verbatim reading of a classical definition in a constructive context is not the same thing as what a classical mathematician would say. In particular, we can say that Examples 16.5 and 16.8 are classically connected but constructively disconnected. The classical mathematician, on the other hand, despite his religious conviction that all questions are decidable, would actually be *unable to say* which they are!

Notation 16.2 Throughout this section, consider any program P whose termination is undecidable [Tur35], and let

$$g_n \ \equiv\ \left\{ \begin{array}{ll} 0 & \text{if } P \text{ has terminated} \\ 4 & \text{if } P \text{ is still running} \end{array} \right\} \text{ at time } n, \quad \text{and} \quad g \ \equiv\ \sum_{n=0}^{\infty} (-\tfrac{1}{2})^n (4 - g_n).$$

Hence $g \neq 0$ iff $\exists n.\, g_n < 1$ iff P ever terminates. We also assume that it's undecidable whether P terminates (if ever) at an odd or even step, *i.e.* whether $g < 0$ or $g > 0$.

The first two examples are intersections of intervals.

Example 16.3 The compact interval $K \equiv \{g\} \cap \{0\} \subset \mathbb{R}$ (Example 8.7) is co-classified by $\omega \equiv \delta \vee \upsilon$, where

$$\delta d \equiv (d < 0 \vee d < g \vee g \neq 0) \quad \text{and} \quad \upsilon u \equiv (0 < u \vee g < u \vee g \neq 0)$$

define a pseudo-cut that is rounded, bounded and *located.* By Proposition 9.11,

$$\begin{aligned}[K]\phi &\equiv \exists d < u.\, \forall x{:}[d,u].\ \phi x \vee x \neq 0 \vee x \neq g \vee 0 \neq g \\ &\Leftrightarrow \forall x{:}[-1,+1].\ \phi x \vee x \neq 0 \vee x \neq g \vee 0 \neq g \\ &\Leftrightarrow \phi 0 \vee (g \neq 0),\end{aligned}$$

but, since $g \neq 0$ is neither provably $\bot$ nor $\top$, the proposition

$$(K \cong \emptyset) \ \equiv\ [K]\bot \ \equiv\ \exists x.\, \delta x \wedge \upsilon x \ \equiv\ (g \neq 0)$$

is undecidable. Topologically, this means that δ and υ neither overlap nor are disjoint, whilst K is neither empty nor occupied (*cf.* Theorem 11.9). This example therefore fails the nullary condition for compact connectedness, although it obeys the binary one ($\Box(\phi \vee \psi) \Leftrightarrow \Box\,\phi \vee \Box\,\psi$).

Example 16.4 Classically, K is either $\emptyset$ or $\{0\}$, which are both overt, but K itself is not.

If we were talking about an equation between generally defined terms that is valid in both special cases, the classical rule of inference (that separate proofs for an open set and its complementary closed set suffice) would actually be admissible [D].

However, overtness is not a *property* of some terms or of a subspace, but a *structure*, $\Diamond$. This cannot be patched together from its values for $\emptyset$ and $\{0\}$, as the following more complicated examples illustrate. This is in keeping with the idea that overtness is about *computational evidence.*

Example 16.5 Now consider the open overt complement U of K, classified by ω; classically this is either $\mathbb{R}$ or $\mathbb{R} \setminus \{0\}$. Whilst $\omega = \delta \vee \upsilon$, we can neither prove that δ and υ are disjoint, nor that they intersect. Hence U is not connected in either the constructive or overt senses, since

$$\delta \vee \upsilon = \top_U \quad \text{and} \quad \Diamond\,\delta \Leftrightarrow \Diamond\,\upsilon \Leftrightarrow \top \quad \text{do not entail} \quad \Diamond(\delta \wedge \upsilon) \ \equiv\ (g \neq 0).$$

However, for any map $f : U \to \mathbf{2}$, the inverse images of $0, 1 \in \mathbf{2}$ must be *disjoint* open subspaces. Hence f must be constant, *i.e.* U is connected in the classical sense, though not path connected.

Also, the family $N/{\approx}$ in Notation 15.9 is Kuratowski-finite (overt, discrete and compact), but not Hausdorff. $\Box$

Example 16.6 Consider $\upsilon u \equiv \exists n.\, g_n < u$, which is a descending real (*cf.* Definition 3.3). It is $(0, \infty)$ if P terminates and $(4, \infty)$ otherwise, and in particular $\upsilon 1 \Leftrightarrow \upsilon 3 \Leftrightarrow \upsilon 4$.

Although υ is bounded on both sides in the sense that $\upsilon 0 \Leftrightarrow \bot$ and $\upsilon 5 \Leftrightarrow \top$, it has no partner δ. For if (δ, υ) were a Dedekind cut, we would have

$$2 < 1 \ \Leftarrow\ \delta 2 \wedge \upsilon 1 \quad \text{and} \quad 2 < 3 \ \Rightarrow\ \delta 2 \vee \upsilon 3$$

by disjointness and locatedness respectively, so $\delta 2$ would be the complement of $\upsilon 1 \Leftrightarrow \upsilon 3$, solving the halting problem for P.

The convex open overt subspace υ has no left endpoint or closure, whilst its closed complement has no interior, *cf.* Remark 10.13(c) and [I, Warning 10.14]. $\Box$

Example 16.7 Now let $\delta \equiv -\upsilon$, so $\delta d \equiv \exists n.\, d < -g_n$. The compact interval

$$K \ \equiv\ \bigcap_n [-g_n, +g_n] \ \subset\ [-5, +5]$$

co-classified by $\delta \vee \upsilon$ is bounded, and therefore compact. It is the directed intersection of intervals with endpoints, *i.e.* of compact overt subspaces. Classically, $K = \{0\}$ if P terminates and $[-4, +4]$ otherwise. Indeed we can show that $\delta 0 \Leftrightarrow \upsilon 0 \Leftrightarrow \bot$, so $0 \in K$. However, K doesn't have a supremum, so by Theorem 12.9 it is not overt. $\Box$

Example 16.8 Consider the closed subspace $K \subset [0,8] \subset \mathbb{R}$ co-classified by

$$\omega x \;\equiv\; (x<0) \;\vee\; (\exists n.\, g_n < x < 8-g_n) \;\vee\; (8<x).$$

This is an "inside out" version of the previous example: classically it is the doubleton $\{0,8\}$ if P terminates, and the closed interval $[0,8]$ otherwise. It is not overt, because $\Diamond(0,8)$ would solve the halting problem.

It is compact but not compact connected: $\phi x \equiv x < 4$ and $\psi x \equiv 4 < x$ satisfy $\Box\phi \Leftrightarrow \Box\psi \Leftrightarrow \bot$ and $\phi \wedge \psi = \bot$, but $\Box(\phi \vee \psi)$ says that P terminates, which is not provably $\bot$.

It is classically connected: let $f : K \to \mathbf{2}$; then since the halves are connected,

$$\forall x{:}[0,4].\, \omega x \vee (fx = f0) \quad\text{and}\quad \forall x{:}[4,8].\, \omega x \vee (fx = f8),$$

so $\omega 4 \vee (f0 = f8) \Leftrightarrow \top$ and $(f0 \neq f8) \Rightarrow \omega 4$. But $\omega 4$ says that P terminates, which is not decidable, so this can only happen if $(f0 = f8)$ had been given. $\Box$

Example 16.9 Now let us reconsider our introductory Example 1.2 of a function that hovers near zero. The subspace

$$Z \;\equiv\; \{(t,x) \mid f_t(x) = 0\} \;\subset\; [-1,+1]\times[0,3] \;\subset\; \mathbb{R}^{\mathbf{2}}$$

of all (parametric) zeroes is closed, being co-classified by

$$\omega(t,x) \;\equiv\; (f_t x \neq 0) \;\equiv\; (x<1) \;\vee\; (t<0 \wedge x<2) \;\vee\; (t>0 \wedge x>1) \;\vee\; (x>2),$$

so it is compact, with necessity modal operator $[Z]\theta \;\equiv\; \forall t.\, \forall x.\, (f_t \neq 0) \vee \theta(t,x)$.

It is also overt, with possibility operator

$$\langle Z\rangle\theta \;\equiv\; \big(\exists t{:}[0,+1].\, \theta(t,1)\big) \;\vee\; \big(\exists x{:}[1,2].\, \theta(0,x)\big) \;\vee\; \big(\exists t{:}[-1,0].\, \theta(t,2)\big).$$

Remark 16.10 Observe that, in these formulae, the variable t

(a) is an argument of the co-classifier ω of Z considered as a closed subspace, and we obtain the co-classifier ω_t of the intersection

$$Z_t \;\cong\; Z \cap \big(\{t\}\times[0,3]\big) \;\subset\; [-1,+1]\times[0,3]$$

simply by applying ω to t, so $\omega_t(x) \equiv \omega(t,x)$; but it

(b) is *not* an argument of (or free in) either of the modal operators $[Z]$ or $\langle Z\rangle$.

This is the symbolic manifestation of the fact that inverse images preserve open and closed subspaces (Lemma 7.15). In this case the map is *proper*, so the inverse image of any compact subspace is compact, but it is not *open*, so the inverse image of an overt subspace need not be overt.

Remark 16.11 Nevertheless $Z_t \equiv \{x \mid f_t(x) = 0\} \subset [0,3]$ is compact, because we obtain $[Z_t]$ from ω_t in the same way as we obtained $[Z]$ from ω, namely

$$[Z_t]\phi \;\Leftrightarrow\; (\forall x.\, f_t x \neq 0 \vee \phi x) \;\Leftrightarrow\; (t>0 \wedge \phi 1) \;\vee\; (\forall x{:}[1,2].\, \phi x) \;\vee\; (t<0 \wedge \phi 2).$$

In *the three separate cases* where $t<0$, $t \equiv 0$ and $t>0$, Z_t is an overt closed interval with endpoints (max and min):

	Z_t	$\min Z_t$	$\max Z_t$	ω_t	$[Z_t]\phi$	$\langle Z_t\rangle\phi$
$t<0$	$[2,2]$	2	2	$x \neq 2$	$\phi 2$	$\phi 2$
$t \equiv 0$	$[1,2]$	1	2	$x<1 \vee x>2$	$\forall x{:}[1,2].\, \phi x$	$\exists x{:}[1,2].\, \phi x$
$t>0$	$[1,1]$	1	1	$x \neq 1$	$\phi 1$	$\phi 1$

For any value t such that Z_t is overt, and therefore has endpoints, the *observable predicate*

$$\mathsf{min}\, Z_t < \mathsf{max}\, Z_t$$

is equivalent to the *statement* $t = 0$, which is not observable. Such an equivalence cannot be a single assertion — we must distinguish the cases $t = 0$ and $t \neq 0$ before we can make it. Hence there is no single definition of $\langle Z_t \rangle$, $\mathsf{max}\, Z_t$, $\mathsf{min}\, Z_t$ or of a zero of the function that is continuous in the parameter t.

In particular, for the undecidable value g, the subspace of all zeroes,

$$Z_g \;\cong\; Z \cap \big(\{g\} \times [0,3]\big) \;\subset\; [-1,+1] \times [0,3],$$

which is the intersection of two overt subspaces, is not itself overt. If it were, the proposition $\mathsf{min}\, Z_g < \mathsf{max}\, Z_g$ would solve the halting problem for the given program P.

Remark 16.12 We could alternatively try to define $\langle S_t \rangle$ either

(a) naïvely from the *set* of stable zeroes, *cf.* Exercise 2.7, but then $\langle S_0 \rangle \phi \equiv \bot$; or

(b) from the function, as in Definition 14.1(e),

$$\langle S_t \rangle \phi \;\equiv\; \exists xy.\, (\forall z{:}[x,y].\ \phi z) \wedge (f_t x < 0) \wedge (f_t y > 0),$$

but then $\langle S_t \rangle (0,3) \Leftrightarrow \top$, whilst $\langle S_t \rangle (0,2) \Leftrightarrow (t > 0)$ and $\langle S_t \rangle (1,3) \Leftrightarrow (t < 0)$, so this only preserves $\vee$ when $t \neq 0$.

The functions in the remaining examples don't hover, but our search for a non-stable zero is hindered by the failure of *existence* and *uniqueness* respectively.

Example 16.13 Consider the parametric function that ***touches without crossing***,

$$-1 \le t \le +1,\ -1 \le x \le +1 \;\vdash\; f_t(x) \;\equiv\; x^2 - t.$$

The subspaces $Z \subset \mathbb{R}^{\mathbf{2}}$ and $Z_t \subset \mathbb{R}$, which are defined as in the previous example, are both closed and compact. The co-classifier ω_t and necessity operator $[Z_t]$ depend continuously on t, with

$$(Z_t \cong \emptyset) \;\equiv\; [Z_t]\bot \;\equiv\; (t < 0).$$

But, for $t \equiv g$, this is not decidable, so Z_g is not overt, by Theorem 11.9. Indeed,

$$\begin{array}{lcl} b^2 = t \ge 0 & \vdash & \langle Z_t \rangle \phi \;\Leftrightarrow\; \phi(-b) \vee \phi(+b) \\ t < 0 & \vdash & \langle Z_t \rangle \phi \;\Leftrightarrow\; \bot, \end{array}$$

which is not continuous in t.

On the other hand, we may define $\langle S_t \rangle$ either naïvely from the set of stable zeroes (Exercise 2.7), or from the function (Definition 14.1(e)):

$$\begin{array}{rcl} \langle S_t \rangle \phi & \Leftrightarrow & \exists xy.\, (x^2 < t < y^2) \wedge \forall z{:}[x,y].\ \phi z \\ & \Leftrightarrow & \begin{cases} \phi(-b) \vee \phi(+b) & \text{if } b^2 = t > 0 \\ \bot & \text{if } t \le 0 \end{cases} \end{array}$$

Notice the subtle change in the case analysis: the one for $\langle S_t \rangle$ is Scott-continuous because the inverse image of $\bot$ is the *closed* subspace with $t \le 0$, whereas previously we had tried to make it the *open* subspace with $t < 0$.

Remark 16.14 Of course, as t decreases, the square roots merge and disappear, but they vanish from S_t before they go from Z_t. In fact, we have

$$S_t \;=\; Z_t \cap \{x \mid t > 0\},$$

so that S_t is *locally closed*, *i.e.* the intersection of a closed subspace with an open one. In order to define it as an *overt* subspace of $\mathbb{R}$, we therefore have to combine the techniques of Definition 11.1 and Exercise 11.20; these ensure that S_t is still locally compact, and definable within the existing ASD calculus. $\square$

Remark 16.15 We see, therefore, that *the space Z of all real roots of a polynomial need not be overt.* The situation is entirely different in the complex plane, where we can use Cauchy's integral formula,

$$\oint \frac{\mathsf{d}z}{z^n} \;=\; 2\pi i n,$$

to identify not only the presence but also the multiplicity of roots in any open disc.

Example 16.16 Consider the parametric function

$$-1 \leq t \leq +1,\ -\pi \leq x \leq +\pi \ \vdash\ f_t(x) \;\equiv\; t \sin x,$$

but this time we want to find x for which $f_t(x)$ takes its *maximum value* (over all x but particular t), rather than 0. As we know that the maximum *value* is $|t|$, this is the same as solving the equation

$$|t| - t \sin x \;=\; 0.$$

(Using Theorem 6.13 we may define $\sin x$ as a power series, but a polynomial such as $f_t x \equiv tx(1-x^2)$ would do instead.) Then, as before, the parametric space of all solutions,

$$Z \quad\equiv\quad [0,1] \times \left\{\tfrac{+\pi}{2}\right\} \quad\cup\quad \{0\} \times [-\pi,+\pi] \quad\cup\quad [-1,0] \times \left\{\tfrac{-\pi}{2}\right\},$$

is closed, compact and overt, whilst Z_t is closed, compact and occupied, with

$$[Z_t]\phi \;\equiv\; \big(\phi(+\pi/2) \vee t < 0\big) \;\wedge\; \big(\forall x{:}[-\pi,+\pi].\ \phi x \vee t \neq 0\big) \;\wedge\; \big(\phi(-\pi/2) \vee t > 0\big).$$

If Z_t were overt, it would have a maximum and minimum, but this varies discontinuously near $t = 0$. Indeed

$$\begin{array}{rcl} t < 0 & \vdash & \langle Z_t \rangle \phi \;\Leftrightarrow\; \phi(+\pi/2) \\ t \equiv 0 & \vdash & \langle Z_t \rangle \phi \;\Leftrightarrow\; \exists x{:}[-\pi,+\pi].\ \phi x \\ t > 0 & \vdash & \langle Z_t \rangle \phi \;\Leftrightarrow\; \phi(-\pi/2), \end{array}$$

which is not continuous in t. On the other hand, $\langle S_t \rangle = \bot$, and in particular $\langle S_t \rangle = \bot$, whether we define them from the function or from the set of stable zeroes. $\square$

References

[Bai85] Günter Baigger. Die Nichtkonstruktivität des Brouwerschen Fixpunktsatzes. *Archiv für Mathematische Logik und Grundlagenforschung*, 25(3-4):183–188, 1985.

[Bar81] Henk Barendregt. *The Lambda Calculus: its Syntax and Semantics.* Number 103 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1981. Second edition, 1984.

[BB85] Errett Bishop and Douglas Bridges. *Constructive Analysis.* Number 279 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1985.

[BBS04] Andrej Bauer, Lars Birkedal, and Dana Scott. Equilogical spaces. *Th. Comp. Sci.*, 315:35–59, 2004.

[Bis67] Errett Bishop. *Foundations of Constructive Analysis.* Higher Mathematics. McGraw–Hill, 1967.

[BIS02] Douglas Bridges, Hajime Ishihara, and Peter Schuster. Compactness and continuity, constructively revisited. In Julian Bradfield, editor, *Computer Science Logic*, volume 2471 of *Lecture Notes in Computer Science*, pages 89–102. Springer-Verlag, 2002.

[Bou66] Nicolas Bourbaki. *Topologie Générale.* Hermann, 1966. English translation, "General Topology", distrubuted by Springer-Verlag, 1989.

[BR87] Douglas Bridges and Fred Richman. *Varieties of Constructive Mathematics*. Number 97 in LMS Lecture Notes. Cambridge University Press, 1987.

[Bri99] Douglas Bridges. Constructive mathematics: a foundation for computable analysis. *Theoretical Computer Science*, 219:95–109, 1999.

[Bro75] L.E.J. Brouwer. *Collected Works: Philosophy and Foundations of Mathematics*, volume 1. North-Holland, 1975. Edited by Arend Heyting.

[Cle87] John Cleary. Logical arithmetic. *Future Computing Systems*, 2:125–149, 1987.

[Dav05] Brian Davies. Pluralism in mathematics. *Philosophical Transactions of the Royal Society*, A 363:2449–60, 2005.

[Dug66] James Dugundji. *Topology*. Allyn and Bacon, 1966.

[EL04] Abbas Edelat and André Lieutier. Domain theory and differential calculus. *Mathematical Structures in Computer Science*, 14:771–802, 2004.

[Fox45] Ralph Fox. On topologies for function-spaces. *Bulletin of the American Mathematical Society*, 51:429–32, 1945.

[Gen35] Gerhard Gentzen. Untersuchungen über das Logische Schliessen. *Mathematische Zeitschrift*, 39:176–210 and 405–431, 1935. English translation in [Gen69], pages 68–131.

[Gen69] Gerhard Gentzen. *The Collected Papers of Gerhard Gentzen*. Studies in Logic and the Foundations of Mathematics. North-Holland, 1969. Edited by M. E. Szabo.

[GHK+80] Gerhard Gierz, Karl Heinrich Hoffmann, Klaus Keimel, Jimmie Lawson, Michael Mislove, and Dana Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980. Second edition, *Continuous Lattices and Domains*, published by Cambridge University Press, 2003.

[GLT89] Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*. Number 7 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989.

[Göd31] Kurt Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38:173–198, 1931. English translations, "On Formally Undecidable Propositions of 'Principia Mathematica' and Related Systems" published by Oliver and Boyd, 1962 and Dover, 1992; also in "From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931", ed. Jan van Heijenoort, Harvard University Press, 1967.

[Har08] G.H. Hardy. *A Course of Pure Mathematics*. Cambridge University Press, 1908. Tenth edition, 1952, frequently reprinted.

[Hey56] Arend Heyting. *Intuitionism, an Introduction*. Studies in Logic and the Foundations of Mathematics. North-Holland, 1956. Revised edition, 1966.

[HM81] Karl Hofmann and Michael Mislove. Local compactness and continuous lattices. In Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, volume 871 of *Springer Lecture Notes in Mathematics*, pages 209–248, 1981.

[Hyl79] Martin Hyland. Filter spaces and continuous functionals. *Annals of Mathematical Logic*, 16:101–143, 1979.

[Hyl91] Martin Hyland. First steps in synthetic domain theory. In *Category Theory*, number 1488 in Lecture Notes in Mathematics. Springer Verlag, 1991.

[Joh77] Peter Johnstone. *Topos Theory*. Number 10 in London Mathematical Society Monographs. Academic Press, 1977.

[Joh82] Peter Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982.

[Joh84] Peter Johnstone. Open locales and exponentiation. *Contemporary Mathematics*, 30:84–116, 1984.

[JT84] André Joyal and Myles Tierney. *An Extension of the Galois Theory of Grothendieck*, volume 309 of *Memoirs*. American Mathematical Society, 1984.

[Man83] Mark Mandelkern. Constructive continuity. *Memoirs of the American Mathematical Society*, 42(277), 1983.

[Moo66] Ramon Moore. *Interval Analysis*. Automatic Computation. Prentice Hall, 1966.

[MS02] Matías Menni and Alex Simpson. Topological and limit-space subcategories of countably-based equilogical spaces. *Mathematical Structures in Computer Science*, 12:739–770, 2002.

[Pea97] Giuseppe Peano. Studii di logica matematica. *Atti della Reale Accademia di Torino*, 32:565–583, 1897. Reprinted in Peano, *Opere Scelte*, Cremonese, 1953, vol. 2, pp. 201–217, and (in English) in H.C. Kennedy, *Selected Works of Giuseppe Peano*, Toronto University Press, 1973, pp 190–205.

[Plo77] Gordon Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.

[Pre05] Matija Pretnar. *Sintentična topologija.* PhD thesis, Faculty of Mathematics and Physics, University of Ljubljana, 2005.

[Ros00] Giuseppe Rosolini. Equilogical spaces and filter spaces. *Rend. Circ. Mat. Palermo Suppl.*, 64:157–175, 2000.

[Sch03] Peter Schuster. Unique existence, approximate solutions, and countable choice. *Theoretical Computer Science*, 305:433–455, 2003.

[Sch05] Peter Schuster. What is continuity, constructively? *Journal of Universal Computer Science*, 11:2076–85, 2005.

[Sco72] Dana Scott. Continuous lattices. In Bill Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, number 274 in Lecture Notes in Mathematics, pages 97–137. Springer-Verlag, 1972.

[SS70] J. Arthur Seebach and Lynn Arthur Steen. *Counterexamples in Topology.* Holt, Rinehart and Winston, 1970. Republished by Springer-Verlag, 1978 and by Dover, 1995.

[Tay99] Paul Taylor. *Practical Foundations of Mathematics.* Number 59 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[Tur35] Alan Turing. On computable numbers with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society (2)*, 42:230–265, 1935.

[TvD88] Anne Sjerp Troelstra and Dirk van Dalen. *Constructivism in Mathematics, an Introduction.* Number 121 and 123 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1988.

[Ver94] Japie Vermeulen. Proper maps of locales. *Journal of Pure and Applied Algebra*, 92:79–107, 1994.

[Wei00] Klaus Weihrauch. *Computable Analysis.* Springer, Berlin, 2000.

[Wil70] Peter Wilker. Adjoint product and hom functors in general topology. *Pacific Journal of Mathematics*, 34:269–283, 1970.

The papers on abstract Stone duality may be obtained from
`www.PaulTaylor.EU/ASD`

[A] Paul Taylor, Sober spaces and continuations. *Theory and Applications of Categories*, 10(12):248–299, 2002.

[B] Paul Taylor, Subspaces in abstract Stone duality. *Theory and Applications of Categories*, 10(13):300–366, 2002.

[C] Paul Taylor, Geometric and higher order logic using abstract Stone duality. *Theory and Applications of Categories*, 7(15):284–338, 2000.

[D] Paul Taylor, Non-Artin gluing in recursion theory and lifting in abstract Stone duality. 2000.

[E] Paul Taylor, Inside every model of Abstract Stone Duality lies an Arithmetic Universe. *Electronic Notes in Theoretical Computer Science* **122** (2005) 247-296, Elsevier.

[F] Paul Taylor, Scott domains in abstract Stone duality. March 2002.

[G–] Paul Taylor, Local compactness and the Baire category theorem in abstract Stone duality. *Electronic Notes in Theoretical Computer Science* **69**, Elsevier, 2003.

[G] Paul Taylor, Computably based locally compact spaces. *Logical Methods in Computer Science*, **2** (2006) 1–70.

[H-] Paul Taylor, An elementary theory of the category of locally compact locales. APPSEM Workshop, Nottingham, March 2003.

[H] Paul Taylor, An elementary theory of various categories of spaces and locales. November 2004.

[I] Andrej Bauer and Paul Taylor, The Dedekind reals in abstract Stone duality. Computability and Complexity in Analysis, Kyoto, August 2005.

[K] Paul Taylor, Interval analysis without intervals. February 2006.

As I was a reluctant analyst, this work would never have been done without the persistent (but friendly) cajoling of Graham White and Andrej Bauer. I would like to express my warmest appreciation for the ongoing encouragement that they have both given me. In particular, during my visit to Ljubljana in November 2004, Andrej provided the construction of the supremum of a compact overt subspace in Section 12 using Dedekind cuts that is the keystone of this paper. He also pointed out Exercise 1.10 and Examples 2.8.

This work was presented at *Computability and Complexity in Analysis* in Kyoto in August 2005, and I am grateful to Peter Hertling and the CCA programme committee for the indulgence of allowing me to occupy altogether 80 pages of their proceedings.

The paper that you see here is the fruit of illuminating discussions with numerous people since the conference. I would particularly also like to thank Vasco Brattka, Douglas Bridges, Robin Cockett, Thierry Coquand, Nicola Gambino, André Joyal, Achim Jung, Norbert Müller, Petrus Potgieter, Pino Rosolini, Giovanni Sambin, Peter Schuster, Anton Setzer and Bas Spitters for their interest and encouragement in this project.