Adaptive Backstepping Cancelation of Unmatched Unknown Sinusoidal Disturbances for Unknown LTI Systems by State Derivative Feedback

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Abstract—Solutions already exist for the problem of canceling sinusoidal disturbances by measurement of the state or by measurement of an output for unknown linear and nonlinear systems. In this paper, we design an adaptive backstepping controller to cancel sinusoidal disturbances forcing an unknown linear time-invariant system in controllable canonical form which is augmented by a linear input subsystem with unknown system parameters by using only measurement of state-derivatives of the original subsystem and state of the input subsystem. Our design is based on four steps, 1) parametrization of the sinusoidal disturbance as the output of a known feedback system with an unknown output vector that depends on both unknown disturbance parameters and unknown plant parameters, 2) design of an adaptive disturbance observer for both disturbance and its derivative, 3) design of an adaptive controller for virtual control input, and 4) design final adaptive controller by using backstepping procedure. We prove that the equilibrium of the closed-loop adaptive system is stable and step of the considered original subsystem goes to zero as $t \to \infty$ with perfect disturbance estimation. The effectiveness of the controller is illustrated with a simulation example of a third order system.

I. INTRODUCTION

The problem of canceling sinusoidal disturbances in dynamical systems is a fundamental control problem, with many applications such as vibrating structures [1], active noise control [2] and rotating mechanisms control [3]. The common method to approach this problem is the internal model principle for which a general solution is given in [19], [20] in the case of linear systems. In the internal model approach, the disturbance is modeled as the output of a linear dynamic system which is called an exosystem. Then the effect of the disturbance on the plant response can be completely compensated by adding a replica of the exosystem model in the feedback loop.

The output regulation problem for minimum phase, uncertain nonlinear systems is solved in [4], [7], and extended for non-minimum phase plants in [6]. The regulation of a linear time-varying system is considered in [12], and the regulation problem for time-varying known exosystem is studied in [8]. On the other hand, disturbance cancelation designs also exist for continuous-time linear systems [5], [11], [13], [17] and discrete-time linear systems [14]. Moreover, designs for nonlinear systems are proposed in [9], [10], [16], [18]. In all of these references, the controllers are designed by using measurement of state or an output.

In the last decade, the state derivative feedback control has drawn the attention of many researchers [21]—[24] due to its various advantages in applications. In most practical problems, especially disturbance cancelation problems, using accelerometers as sensors is easier, cheaper and more reliable than using position sensors. In this case, from the signals of the accelerometers it is possible to establish the velocities with a sufficient precision but not the displacements. Then, the system can be modeled by considering position and velocity as states and the state-derivatives are available for control design. A control design by state-derivative feedback for known and unknown linear time-invariant systems with matched unknown sinusoidal disturbances is proposed in [25], [26].

We extend the result in [26] by relaxing the matched disturbance condition. Employing an approach inspired by [15], we design an adaptive backstepping controller by state derivative feedback to cancel unmatched unknown sinusoidal disturbances forcing linear time-invariant systems with unknown system parameters. We prove that the equilibrium of the closed loop error system is stable and the state of the considered error system goes to zero as $t \to \infty$ with perfect disturbance estimation.

In Section II, we introduce the problem and state our main stability theorem. In Section III, we prove the theorem. A simulation example is presented in Section IV.

II. PROBLEM STATEMENT AND ADAPTIVE CONTROLLER DESIGN

We consider the single-input LTI system

\[
\dot{x} = Ax + Bu + \nu + p, \quad (1)
\]

\[
\dot{p} = \overline{\nu} + \overline{\overline{u}}, \quad (2)
\]

where

\[
A = \begin{bmatrix}
0_{n-1} & 1 \\
0 & 0_{n-1}
\end{bmatrix} \quad (3)
\]

\[
B = \begin{bmatrix}
0_{n-1} \\
1
\end{bmatrix} \quad (4)
\]

\[
\gamma_1 = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]^T \quad (5)
\]

\[
\gamma_2 = [a_{21}, a_{22}, a_{23}, \ldots, a_{2n}]^T \quad (6)
\]

with $0_{n-1} = [0, \ldots, 0]^T \in \mathbb{R}^{n-1}$, the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$, and sinusoidal disturbance $\nu \in \mathbb{R}$ given by

\[
\nu(t) = \sum_{i=1}^{q} g_i \sin(\omega_i t + \phi_i), \quad (7)
\]

where $i \neq j \Rightarrow \omega_i \neq \omega_j$, $\omega_i \in \mathbb{Q}$, $g_i, \phi_i \in \mathbb{R}$. This work was supported by ONR

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The sinusoidal disturbance $\nu$ can be represented as the output of a linear exosystem,
\[ \dot{w} = Sw \quad (8) \]
\[ \nu = h^T w \quad (9) \]
where $w \in \mathbb{R}^{2q}$ and the choice of $S \in \mathbb{R}^{2q \times 2q}$ and $h \in \mathbb{R}^{2q}$ is not unique.

We make the following assumptions regarding the plant (1)-(2) and the exosystem (8)-(9):
\begin{enumerate}
  \item \textbf{Assumption 1:} $\beta_1, \beta_2, \gamma_1$ and $\gamma_2$ are unknown.
  \item \textbf{Assumption 2:} The sign of $a_{11}$ and $b_2$ is known and the lower bound of $|a_{11}|$ is known, $|a_{11}| \geq \varsigma > 0$.
  \item \textbf{Assumption 3:} $x$ and $\nu$ are not measured but $\dot{x}$ and $p$ are measured.
  \item \textbf{Assumption 4:} The pair $(h^T, S)$ is observable.
  \item \textbf{Assumption 5:} The eigenvalues of $S$ are imaginary, distinct and rational.
  \item \textbf{Assumption 6:} $q$ is known.
  \item \textbf{Assumption 7:} $S$ and $h$ are unknown.
  \item \textbf{Assumption 8:} $g_i \neq 0$ for all $i \in \{1, \ldots, q\}$
\end{enumerate}

The adaptive controller for system (1), (2), (8), (9) is given by
\[ u = \frac{\hat{b}_2}{a_{11} k_n} \pi, \quad (10) \]
where $k_n$ is the $n^{th}$ element of the control gain $K \in \mathbb{R}^{1 \times n}$ is chosen so that $(A_0^T + BK)$ is Hurwitz and the positive definite matrix $P$ is a solution of the matrix equation
\[ (A_0^T + BK)^T P + P(A_0^T + BK) = -4I, \quad (11) \]
where
\[ B = \begin{bmatrix} 1, & 0_{n-1} \end{bmatrix}^T, \quad (12) \]
and
\[ \pi = \begin{pmatrix} \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 & \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 \\ \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 & \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 \\ \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 & \hat{\theta}_1 \theta_1 + \hat{\theta}_2 \theta_2 + (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 \end{bmatrix} \]
\[ (13) \]

The update laws are given by
\[ \dot{\gamma}_1 = -\kappa_{\gamma_1} \text{sgn}(a_{11}) \hat{\theta}_1 \theta_1 P \hat{x}, \quad \kappa_{\gamma_1} > 0, \quad (15) \]
\[ \dot{\gamma}_1 = -\kappa_{\gamma_1} \theta_1 (1 - \hat{\theta}_1 \theta_1) \hat{\theta}_1 \theta_1 \hat{e}, \quad \kappa_{\gamma_1} > 0, \quad (16) \]
\[ \dot{\gamma}_2 = \kappa_{\gamma_2} \hat{\theta}_1 \theta_1 \hat{e}, \quad \kappa_{\gamma_2} > 0, \quad (17) \]
\[ \dot{\alpha}_{11} = \left\{ \begin{array}{ll} \tau_{a_{11}}, & \hat{a}_1 \text{sgn}(a_{11}) > \varsigma \text{ or } \tau_{a_{11}} \text{sgn}(a_{11}) \geq 0 \\ 0, & \hat{a}_1 \text{sgn}(a_{11}) \leq \varsigma \text{ and } \tau_{a_{11}} \text{sgn}(a_{11}) < 0 \end{array} \right\} \quad (18) \]
\[ \dot{\alpha}_{11} = -\kappa_{11} \text{sgn}(a_{11}) K \hat{x} \hat{B}^T P \hat{x}, \quad \kappa_{a_{11}} > 0. \quad (21) \]

The disturbance observer is given by
\[ \dot{\eta}_i = G \eta_i - NB \hat{x} \hat{e}_i, \quad 1 \leq i \leq n, \quad (29) \]
\[ \dot{\eta}_0 = G (\eta_0 + N(\hat{x} - Bp)), \quad (30) \]
\[ \xi = \eta_0 + N(\hat{x} - Bp), \quad (31) \]
where $G$ is a $2q \times 2q$ Hurwitz matrix with distinct poles and constitutes a controllable pair with a chosen vector $l \in \mathbb{R}^{2q}$ and $N$ is a $2q \times n$ matrix which is given by
\[ N = \frac{1}{B^T B} l B^T, \quad (32) \]
where the given $N$ is one of the many solutions of the following equation
\[ NB = l. \quad (33) \]

Since the matrices $G$ and $S$ have disjoint spectra, the pair $(h^T, S)$ is observable, and the pair $(G, l)$ is controllable, the Sylvester equation
\[ MS - GM = lh^T, \quad (34) \]
has a unique solution [27]. This fact is exploited in the proof of our stability result (Lemma 1).

We first define the signals needed in the analysis and state a theorem describing our main stability result. Then we prove the theorem using a series of technical lemmas in Section III.
Estimation errors of the unknown parameters are denoted by
\[
\tilde{a}_{11}(t) = a_{11} - \hat{a}_{11}(t), \quad (35)
\]
\[
\tilde{\sigma}_{11}(t) = \frac{1}{a_{11}} - \hat{\sigma}_{11}(t), \quad (36)
\]
\[
\tilde{b}_1(t) = \left( \frac{b_1}{1 - a_{21}} \right) - \tilde{b}_1(t), \quad (37)
\]
\[
\tilde{b}_2(t) = \frac{1}{b_2} - \tilde{b}_2(t), \quad (38)
\]
\[
\tilde{c}_1(t) = [a_{12}, a_{13}, a_{14}, \ldots, a_{1n}, 0]^T - \tilde{c}_1(t), \quad (39)
\]
\[
\tilde{c}_2(t) = \left( \frac{a_{21}}{a_{11}} + a_{22}, -a_{21}a_{13} \right) + a_{23}, \quad (40)
\]
\[
\tilde{c}_2(t) = \left( \frac{a_{21}a_{12}}{a_{11}} + a_{22}, -a_{21}a_{13} \right) + a_{23} + a_{24}, \quad \ldots, \quad (41)
\]
\[
\tilde{d}_1(t) = (\tilde{MS})^{-T} h - \tilde{d}_1(t), \quad (42)
\]
\[
\tilde{d}_2(t) = \left( \frac{a_{21}(MS)^{-T} h - \tilde{d}_2(t)}{a_{11}} \right) \quad (43)
\]
\[
\tilde{d}_3(t) = a_{11}(MS)^{-T} h - \tilde{d}_3(t), \quad (44)
\]
\[
\tilde{d}_4(t) = a_{11}(MS)^{-T} h - \tilde{d}_4(t), \quad (45)
\]
\[
\tilde{d}_5(t) = a_{11}(MS)^{-T} h - \tilde{d}_5(t), \quad (46)
\]
\[
\tilde{d}_6(t) = a_{11}(MS)^{-T} h - \tilde{d}_6(t), \quad (47)
\]
and \(\delta(t)\) and \(\tilde{\eta}_0(t)\) denotes the signals,
\[
\delta(t) = GMw(t) - \eta_0(t) - t\gamma_2^T x(t) - \sum_{i=1}^n a_{1i} \eta_i(t), \quad (48)
\]
\[
\tilde{\eta}_0(t) = \eta_0(t) - \tilde{\eta}_0(t), \quad (49)
\]
where
\[
\tilde{\eta}_0(t) = \int_0^t e^{G(t-\tau)} G\nu d\tau. \quad (50)
\]

**Theorem 1:** Consider the closed-loop system consisting of the plant (1), (2) forced by the unknown sinusoidal disturbance (9), the disturbance observer (29), (30), (31) and the adaptive controller (10), (15)–(27). Under Assumptions 1–7, the following holds:

(a) The equilibrium \(x = \gamma_1 = \gamma_2 = 0, \eta_1 = \ldots = \eta_n = 0, \delta = \tilde{\eta}_0 = \tilde{\eta}_1 = \tilde{\eta}_2 = \tilde{\eta}_11 = \ldots = \tilde{\eta}_n = 0, \tilde{a}_{11} = \tilde{a}_{11} = \tilde{b}_1 = \tilde{b}_2 = e = 0\) is stable.

(b) For all initial condition \(\tilde{a}_{11}(0) \), \(\tilde{\sigma}_{11}(0) \), \(\tilde{b}_1(0) \), \(\tilde{b}_2(0) \), \(\tilde{c}_1(0) \), \(\tilde{c}_2(0) \), \(\tilde{d}_1(0) \), \(\tilde{d}_2(0) \), \(\tilde{d}_3(0) \), \(\tilde{d}_4(0) \), \(\tilde{d}_5(0) \), \(\tilde{d}_6(0) \) and \(\nu(0) \), \(\hat{\nu}(0) \), such that Assumption 8 holds, the signals \(e(t), z(t), \eta(t), \ldots, \eta_i(t), \hat{\theta}_1(t), \nu(t) - \hat{\theta}_1^T \xi(t), \delta(t), \tilde{\eta}_0(t)\) converge to zero as \(t \to \infty\).

### III. Stability Proof

The following lemma enables us to represent the unknown sinusoidal disturbance as the output of a linear system whose input is the disturbance itself, whose state and input matrices are known, and whose output matrix is unknown.

**Lemma 1:** Let \(G \in \mathbb{R}^{2q \times 2q}\) be a Hurwitz matrix with distinct eigenvalues and let \((G, \nu)\) be a controllable pair. Then, \(\nu\) can be represented as the output of the model
\[
\dot{z} = G z + \nu \quad (51)
\]
\[
\nu = \theta_1^T \xi \quad (52)
\]
\[
\theta_1^T = h^T (MS)^{-1} \quad (53)
\]

**Proof:** This result and its proof are inspired by [15]. To establish (51) from (8), consider
\[
\dot{z} = M \nu \quad (54)
\]
Differentiating (54), using (34), we have
\[
\dot{z} = MS \nu = GMw + lh^T w \quad (55)
\]
Substituting (9) and (54) into (55) yields (51). Substituting \(w = (MS)^{-1} \dot{z}\) into (9), we obtain (52) and (53).

The previous lemma enables us to write the unknown external disturbance \(\nu\) as the product of an unknown constant \(\theta\) and the vector \(\dot{z}\). However, \(\dot{z}\) is not accessible, since the signal \(\nu\) cannot be measured. To overcome this problem, we design the observer (29)–(31).

The following lemma establishes the properties of the observer.

**Lemma 2:** The inaccessible disturbance \(\nu\) and its derivative \(\dot{\nu}\) can be represented in the form
\[
\nu = \theta_1^T \xi + \sum_{i=1}^n a_{1i} \theta_i^T \eta_i + \theta_i^T \delta, \quad (56)
\]
\[
\dot{\nu} = \frac{1}{1 - \theta_1^T l} \left( \theta_1^T G \xi + \sum_{i=1}^n a_{1i} \theta_i^T G \eta_i + \theta_i^T G \delta \right) \quad (57)
\]
and \(\delta \in \mathbb{R}^q\) obeys the equation
\[
\delta = G \delta. \quad (58)
\]

**Proof:** By adding bounded signal \(\pm l\nu\) to (48), using (54), (33), (51), the fact that \(B(\gamma^T_1 x + \nu) = \dot{x} - Bp - A_0 x\) and by noting that \(NA_0 x = 0\), we obtain
\[
\delta = \dot{z} - \left( \eta_0 + N(\dot{x} - Bp) + \sum_{i=1}^n a_{1i} \eta_i \right) \quad (59)
\]
Differentiating \(\delta\) with respect to time and in view of (1), (29), (30), (51) and using (33), (59) we get (58). Using (31), (52) and (59), we obtain (56). Differentiating (51), (52) and using (31), (59), we obtain (57).

The representation (56) and (57) established with Lemmas 1 and 2, allows us to represent a time-varying unknown sinusoidal disturbance \(\nu(t)\) and its derivative \(\dot{\nu}(t)\) as a constant unknown vector multiplied by a known regressor, plus an unknown exponentially decaying disturbances \(\theta_1^T \delta(t)\) and \(\frac{1}{1 - \theta_1^T l} \theta_1^T G \delta\). We need \(\dot{\nu}(t)\) while taking the time derivative of \(\xi\) due to backstepping procedure. Thus, we convert the
problem from cancelation of an unknown sinusoidal disturbance with unknown system parameters to an adaptive control problem.

The following lemma is used in the proof of the theorem. **Lemma 3:** There exists $\rho > 0$ such that for all $t_0 \geq 0$, the following holds
\[
Q_p(\rho, t_0) = \int_{t_0}^{t_0 + \rho} \xi(t) \dot{\xi}(t) \, dt - \frac{1}{\rho} \int_{t_0}^{t_0 + \rho} \xi(t) \, dt \int_{t_0}^{t_0 + \rho} \dot{\xi}(t) \, dt > 0,
\]
where
\[
\xi = \xi + \sum_{i=1}^{n} a_i \eta_i.
\]
The proof of the lemma is given in [26].

**Proof of Theorem 1:** The stability of the equilibrium of the closed-loop system is established by use of Lyapunov function
\[
V = \frac{1}{2} x^T P x + \frac{1}{2 |a_{11}|} \left( \frac{1}{\kappa_{\alpha_1}} \tilde{\alpha}_{11}^T + \frac{1}{\kappa_{\gamma_1}} \tilde{\gamma}_{11} T \tilde{\gamma}_1 + \frac{1}{\kappa_{\theta_1}} \tilde{\theta}_1^T \tilde{\theta}_1 
+ \sum_{i=1}^{n} \frac{1}{\kappa_{\beta_{1i}}} \tilde{\beta}_{1i}^T \tilde{\beta}_{1i} + \frac{1}{2} \left( \frac{1}{\kappa_{\tau_1}} \tilde{\tau}_{11}^T + \frac{1}{2} \frac{\tilde{\tau}_{11}^T \tilde{\tau}_{11}}{\kappa_{\tau_2}} \right)
+ \frac{1}{\kappa_{\eta_1}} \tilde{\eta}_1^T \tilde{\eta}_1 \right) + \frac{1}{2 |b_2|} \left( \frac{1}{\kappa_{\beta_{2i}}} \tilde{\beta}_{2i}^T \tilde{\beta}_{2i} + \frac{1}{2} \frac{\tilde{\beta}_{2i}^T \tilde{\beta}_{2i}}{\kappa_{\beta_{22}}} \right)
\]
and using (62) and (67), we get
\[
|\Theta(t)|^2 \leq M_1 |\Theta(0)|^2,
\]
for some $M_1 > 0$. Taking the derivative of (49) and using (30), (50) and the fact that $\dot{x} - Bp = (A_0 + B\sigma T_1^T) x + B\nu$, we get
\[
\dot{\eta}_0(t) = G\tilde{\eta}_0 + GN (A_0 + B\sigma T_1^T) x.
\]
Since $G$ is Hurwitz, using (70), we have
\[
|\tilde{\eta}_0(t)| \leq M_2 e^{-\alpha_1 t} |\tilde{\eta}_0(0)| + M_3 \sup_{\tau \in [0,t]} |x(\tau)|
\]
for some $M_2$, $M_3$, $\alpha_1 > 0$. By using (69) and (71), we obtain
\[
|\tilde{\eta}_0(t)| \leq M_2 |\tilde{\eta}_0(0)| + M_3 \sqrt{M_1 |\Theta(0)|}.
\]
By using (69) and (72), we obtain
\[
|\Xi(t)| \leq M_4 |\Xi(0)|,
\]
where
\[
\Xi(t) = \left[ \frac{\Theta(t)}{\tilde{\eta}_0(t)} \right],
\]
for some $M_4 > 0$. This proves part (a) of Theorem 1.

We write the closed-loop system in the following form
\[
\dot{x} = \bar{A}^{-1} x + \bar{A}^{-1} B \left( \frac{1}{a_{11}} \left( \tilde{\theta}_1^T \xi + \tilde{\theta}_1^T \delta + \sum_{i=1}^{n} \tilde{\beta}_{1i}^T \eta_i + e \right) \right),
\]
where
\[
\bar{A} = A_0^T + B \left( \frac{1}{a_{11}} \left( -\tilde{\gamma}_1^T + K \tilde{a}_1 \right) \right).
\]
By using (1), (31), (34), (49) and (56), we get
\[
\xi = MS(GM)^{-1} \left( \tilde{\eta}_0 + \tilde{\eta}_0 + l (\tilde{\gamma}_1^T x + \tilde{\theta}_1^T \delta + \sum_{i=1}^{n} \tilde{\beta}_{1i}^T \eta_i) \right).
\]
Substituting (77) into (75), we obtain
\[
\dot{x} = \bar{A}^{-1} x + \bar{A}^{-1} B \left( \frac{1}{a_{11}} \left( e + \tilde{\theta}_1^T \delta + \sum_{i=1}^{n} \tilde{\beta}_{1i}^T \eta_i + \tilde{\theta}_1^T MS \times (GM)^{-1} \left( \tilde{\eta}_0 + \tilde{\eta}_0 + l \left( \tilde{\gamma}_1^T x + \tilde{\theta}_1^T \delta + \sum_{i=1}^{n} \tilde{\beta}_{1i}^T \eta_i \right) \right) \right) \right).
\]
From (3), (12), (39) and (76), we get
\[
\det(\bar{A}) = (-1)^{n+1} \left( \frac{\tilde{a}_{11} k_n}{a_{11}} \right).
\]
From (79) it follows that, for all $\Xi$ such that $\tilde{a}_{11} \text{sgn}(a_{11}) \geq \zeta > 0$, the right-hand side of (78) is continuous in $\Xi$ and $t$, which implies that the right-hand side of (66) is continuous in $\Xi$ and $t$. Furthermore, the right-hand side of (66) is zero at $\Xi = 0$. By the LaSalle-Yoshizawa theorem, (66) ensures that $\dot{x}, e, \eta_1, \ldots, \eta_n$ and $\delta$ converge to zero as $t \to \infty$.  

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By adding \( \pm \kappa_\theta \text{sgn}(a_{11}) \left( \sum_{i=1}^{n} a_{i1} \eta_i \right) B^T P \dot{x} \) and \( \pm \frac{1}{a_{11}} B \tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i \) to (22) and (75) respectively, we represent the close-loop system as a linear-time varying (LTV) system which is given by
\[
\dot{\zeta} = E(t)\zeta + F(t)d,
\] where
\[
E(t) = \begin{bmatrix} A_{cl} & \frac{1}{a_{11}} B \tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i \end{bmatrix} P A_{cl} + \frac{1}{a_{11}} B \tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i P
\]
\[
F(t) = \begin{bmatrix} \frac{1}{a_{11}} B (\tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i) P A_{cl} + \frac{1}{a_{11}} B \tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i P \end{bmatrix}
\]
(80)
\[
\zeta = \begin{bmatrix} \alpha_1 \text{sgn}(a_{11}) \left( \sum_{i=1}^{n} a_{i1} \eta_i \right) \end{bmatrix} (\sum_{i=1}^{n} a_{i1} \eta_i) P A_{cl} + \frac{1}{a_{11}} B \tilde{\eta}_1^T \sum_{i=1}^{n} a_{i1} \eta_i P
\]
\[
d = \begin{bmatrix} x^T, \eta_1^T, \ldots, \eta_n^T, \delta^T, e \end{bmatrix}^T
\]
(83)
\[
\zeta = \begin{bmatrix} x^T, \tilde{\eta}_1^T \end{bmatrix}^T
\]
(84)
with
\[
A_{cl} = (A_{cl}^T + B K)^{-1}
\]
\[
B = A_{cl} B
\]
(85)
\( (86) \)
We first show that the equilibrium \( \zeta = 0 \) of the homogenous part of the LTV system (80) is exponentially stable. Towards that end, we choose the following Lyapunov function
\[
V_c = \frac{1}{2} \zeta^T P_c \zeta,
\]
(87)
where
\[
P_c = \text{diag}\{P_1, \frac{1}{a_{11}} |\theta_1| J_{2q \times 2q} \}
\]
(88)
Taking the time derivative of \( V_c \) and using (11), we get
\[
\dot{V}_c = -2 \zeta^T C(t)C(t)^T \zeta,
\]
(89)
where
\[
C(t)^T = A_{cl} \frac{1}{a_{11}} B \tilde{\eta}_1^T
\]
(90)
Therefore, it follows that \( P_c \), as defined in (88), satisfies the following inequality
\[
E^T(t) P_c + P_c E(t) + \alpha C^T(t) C(t) \leq 0
\]
(91)
for some \( \alpha > 0 \).

The equilibrium \( \zeta = 0 \) of the homogenous part of (80) is exponentially stable if \( (C(t), E(t)) \) is a uniformly completely observable (UCO) pair [29]. For a bounded \( H(t) \), the pairs \( (C(t), E(t)) \) and \( (C(t), E(t) + H(t)C(t)^T) \) have the same UCO property [29]. Choosing \( H(t) = \)
\[
[ -I, -\kappa_\theta \text{sgn}(a_{11}) P B \tilde{\eta}_1^T ]^T,
\]
we write the system corresponding to the pair \( (C, E + HC^T) \) as
\[
\dot{Y} = 0
\]
(92)
\[
y = C^T(t)Y.
\]
(93)
The state transition matrix of (92) is \( \Phi = I \). Therefore, \( (C, E + HC^T) \) is a UCO pair if there exist positive constants \( \alpha_1, \alpha_2, \rho \) such that the observability Gramian satisfies
\[
\alpha_2 I \geq \int_{t_0}^{t_0 + \rho} C(t)C^T(t) dt \geq \alpha_1 I,
\]
(94)
for all \( t_0 \geq 0 \). Since \( \zeta \) is bounded, recalling (90), the upper bound of (94) is satisfied. We now prove the lower bound in (94). Let \( S_h \) be the Schur complement of \( A_{cl}^T A_{cl} \rho \) in \( \int_{t_0}^{t_0 + \rho} C(t)C^T(t) dt \), where
\[
S_h = \frac{B^T B}{a_{11}} \left( \int_{t_0}^{t_0 + \rho} \tilde{\eta}_1^T \eta_1^T dt - \frac{1}{\rho} \int_{t_0}^{t_0 + \rho} \tilde{\eta}_1^T \xi dt \int_{t_0}^{t_0 + \rho} \xi^T dt \right)
\]
(95)
Since \( A_{cl}^T A_{cl} \rho \) is positive definite, \( X \) is positive definite if and only if \( S_h \) is positive definite. Since \( B^T B \) is a positive scalar, according to Lemma 3 there exists a positive \( \rho \) such that for all \( t_0 \geq 0 \), \( S_h > 0 \). Hence, \( (C, E + HC^T) \) is UCO, which implies that \( (C, E) \) is UCO. Therefore, the state transition matrix \( \Phi(t, t_0) \) corresponding to \( E(t) \) in (80) satisfies
\[
\| \Phi(t, t_0) \| \leq \kappa_0 e^{-\gamma_0 (t-t_0)}
\]
(96)
for some positive constants \( \kappa_0, \gamma_0 \). From the boundedness of \( \Xi(t) \) and \( \tilde{\eta}_0 \), it follows from (78) that \( \dot{x} \) is bounded. From (83), \( d(t) \) is bounded and, from (82), \( F(t) \) is bounded. Recalling that it has already been established that \( d(t) \) goes to zero, from (80) and (96) it follows that \( \zeta \) is bounded and \( \zeta = [x^T, \tilde{\eta}_1^T]^T \to 0 \) as \( t \to \infty \). By using (70) and the fact that \( G \) is Hurwitz and \( x(t) \) converges to zero, we conclude that \( \tilde{\eta}_0(t) \) converges to zero as \( t \to \infty \). Furthermore, thanks to Lemma 2, \( \tilde{\eta}_1^T \dot{(t)}(t) - \nu(t) \to 0 \) as \( t \to \infty \). This proves part (b) of Theorem 1.
IV. SIMULATION RESULTS

We illustrate the performance of our controller with a third-order system with \( \gamma_1^T = [1, 3], \gamma_2^T = [1, 2], b_1 = 2, b_2 = 1, \) the unknown disturbance \( \nu(t) = 1.2 \sin(0.8t + \pi/4) - 0.5 \sin(t + \pi/2) \) and initial conditions \( x(0) = [1, -1.5]^T, p(0) = 0.5. \) The control gain \( K \) is chosen such that the eigenvalues of \( A_{cl} \) are \(-3, -2\) and \( c = 1. \) We set all update gains to 1. Finally, the controllable pair \((G, l)\) is chosen as \( G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4.37 & -12.12 & -12.60 & -5.80 \end{bmatrix}^T, \ l = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T. \) From Figures 1 and 2, one can observe that \( x(t) \) converges to zero and the unknown disturbance is perfectly estimated, as Theorem 1 predicts.

V. CONCLUSIONS

In the present work we design an adaptive backstep-ping controller by state derivative feedback to cancel unmatched unknown sinusoidal disturbances for a linear time-invariant systems with unknown system parameters. We prove that the equilibrium of the closed loop system is stable and the state of the considered error system \((\bar{x}, e)\) goes to zero as \( t \to \infty \) with perfect disturbance estimation. The effectiveness of our controller is demonstrated with a numerical example.

VI. ACKNOWLEDGMENTS

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