More economists are interested in analyzing situations in which cooperative behavior within a coalition coexists with strategic behavior across the coalitions. The underlying equilibrium is defined as the hybrid solution with a distribution rule (HSDR), which could contribute to these studies to the same extent that Nash equilibrium does to strategic behavior. The paper also defines the stable coalition structure, which extends the ideas in Thrall and Lucas (1963), Shenoy (1979) and Hart and Kurz (1983) to normal form TU games. These extensions not only make earlier results applicable to oligopoly markets but also provide insights on the refinement of Nash equilibria. Sufficient conditions for the existence of HSDR are provided, and three models of endogenizing coalition structure are constructed (JEL#: C62, C71, C72).

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1. INTRODUCTION

More economists are interested in analyzing situations in which cooperative behavior within a coalition coexists with strategic behavior across the coalitions. This paper introduces the hybrid solution with a distribution rule (HSDR) for these studies, which contributes to the underlying equilibria to the same extent that Nash equilibrium does to strategic behavior (see Remark 3 in Section 2).

Secondly, this paper introduces the stability of coalition structures. The stability of a coalition structure $\Delta$ is, essentially, the stability of a HSDR for $\Delta$ in normal form games, or the stability of a payoff vector associated with $\Delta$ in "partition function form games" and in "coalition structure value form games". These stable concepts provide new insights into many economic problems and into the refinement of Nash equilibria. For example, coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston, 1987) is equivalent to the stability of the finest coalition structure (see Section 3); the cartel stability (d'Aspremont et al., 1983) and the formation of customs unions (Riezman, 1985) are both the stability of some specific coalition structures, though these authors phrased the ideas differently (see Section 2).

Finally, this paper constructs three models of coalition structure formation by using the coalition structure formation game (von Neumann and Morgenstern (1944)) and the cooperation structure formation game (Myerson, 1977). In order to determine the final payoffs in normal form TU games, players need to decide three issues: (1) what coalition structure to form, (2) what strategies to choose, and (3) how to split a coalition's joint payoff. Our models incorporated all three issues. Similarly as in Hart and Kurz (1983), we establish the equivalence between a stable coalition structure and the strong equilibrium in the coalition structure formation game (see Propositions 6 and 7 in Section 5).

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2 See Section 26 of all editions or P. 243-245 of Third edition printed by Wiley & Sons (1953).
There is a large literature on games with coalition structures\(^3\) (see Kurz(1988) and Greenberg(1994) for surveys) since the publications of Thrall and Lucas (1963) and Aumann and Dreze (1974), and the interests in this field have been growing recently. However, only Thrall and Lucas (1963), Shenoy (1979), Hart and Kurz (1983), and Ray & Vohra (1993) studied the stable coalition structures in a general game, others studied related issues like cooperation structure or coalition formation or society or applications.\(^4\) This paper focuses on normal form TU games, which include the quantity oligopoly models.

The paper is organized as follows. The next section introduces the concept of HSDR (Definition 3) and the concept of stable coalition structures (Definition 4). Section 3 discusses the new insights on Nash equilibrium and its refinements. Section 4 provides existence results for HSDR and studies the properties of stable coalition structures. Section 5 discusses the endogenization of coalition structures, Section 6 provides all the proofs, and Section 7 concludes.

### 2. DEFINITIONS

Throughout the paper superscripts in small letters denote individual players, and subscripts in capital letters denote coalitions. An n-person game in normal form is defined as

\[ \Gamma = \{N, x^i, u^i\}, \]

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\(^3\) This literature concentrated on the core and the Shapley value, most subsequent works on the core of games with coalition structure concentrated on the applications to goods and taxation (See Kurz, 1988).

where $N = \{1, 2, ..., n\}$ is the set of players. For each $i \in N$, $X_i$ (a non-empty subset in $\mathbb{R}^{m_i}$) is $i$'s strategy set, $u_i : X_i \rightarrow \mathbb{R}$, is $i$'s payoff function, where $X = \prod_{i=1}^{n} X_i$. When this game is played with side-payments, it is called an n-person TU game in normal form.\(^5\)

TU games are usually given in coalition function form, $\Gamma_{\text{CF}} = \{N, V(\cdot)\}`,\(^6\) which specifies a joint payoff $V(S)$ for each coalition $S \subseteq N$. Because the process of deriving a coalition's joint payoff from all strategies is bypassed in this game form, it cannot be directly applied to economic problems that require player's strategies (like oligopolistic industries). However, TU games in normal form can be readily applied in these situations.

In order to define the core solution concept, we need the following notations. Let $\mathcal{N}$ denote the set of all coalitions (the set of all nonempty subsets of $N$). For each $S \in \mathcal{N}$, let $|S|$ denote the number of players in $S$, and $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space whose coordinates have as superscripts the members in $S$. For any $x = \{x^1, ..., x^n\} \in X$, where $x^i = \{x^i_1, x^i_2, ..., x^i_{m_i}\} \in X_i$, let $x_S = \{x^i \mid i \in S\} \in X_S = \prod_{i \in S} X_i$ be the strategies of a coalition $S$; and $x_{-S} = \{x^i \mid i \notin S\} \in X_{-S} = \prod_{i \notin S} X_i$ be the strategies of the players in the complementary coalition $N \setminus S$. Similar notation applies to any $u = \{u^1, ..., u^n\} \in \mathbb{R}^n$. We shall write $x_N = x$ and $u_N = u$ for simplicity. For any two vectors $u, v \in \mathbb{R}^n$, $u \geq v \iff u_i \geq v_i$, all $i$; $u > v \iff u \geq v$ and $u \neq v$; and $u >> v \iff u_i > v_i$, all $i$.

Now the guaranteed payoff function for a coalition $S$, $\bar{u}_S : X_S \rightarrow \mathbb{R}$, is defined as

\[ \bar{u}_S(x_S) = \min_{z \in X_{-S}} \sum_{i \in S} u_i(x_S, z_{-S}) = \sum_{i \in S} u_i(x_S, \bar{y}_{-i}(x_S)) \]

---

5 One example is the linear Cournot industry: $N = \{1, 2, ..., n\}$ is the set of firms, $X_i = [0, y^i]$ ($y^i > 0$ is firm $i$'s capacity constraint), and $u^i(x) = \pi^i(x) = P(x)x^i - C^i(x^i) = (a - b\sum x^j)x^i - (d^i + c^i x^i)$. Here $P(X) = a - b\Sigma x^j$ is the inverse demand function ($a, b > 0$), and $C^i(x^i) = d^i + c^i x^i$, $x^i \in X^i$, is $i$'s cost function, where $d^i > 0$ and $c^i > 0$ are its fixed and marginal cost respectively. Many other problems like cost allocation (see Moulin, 1988) and merger analysis (Salant, Switzer and Reynolds (1983), Perry and Porter (1985), Farrell and Shapiro (1990), Levin (1990)) can also be modeled as normal form TU games.

6 Here the superscript $\text{CF}$ refers to either characteristic form or coalition function form. Such games are function from the subsets of $N$ to $\mathbb{R}$.
for each $x_S \in X_S$, where $\bar{y}_S(x_S)$ is the solution to the above minimization problem\(^7\). In the above expression, $(x_S, z_S)$ denotes a vector $y \in X$ such that $y^i = x^i$ if $i \in S$ and $y^i = z^i$ if $i \notin S$. Thus for each $S \in \mathcal{N}$ and $x \in X$, we can write $x = (x_S, x_{-S})$ for convenience. The coalition's maximum guaranteed joint payoff is

$$v(S) = \max \{ u_S(x_S) \mid x_S \in X_S \} = \bar{u}_S(\bar{x}_S) = \sum_{i \in S} u^i(\bar{x}_S, \bar{y}_S(\bar{x}_S)),$$

where $\bar{x}_S$ is the maximal solution. This leads to the following $\alpha$-core concept:

**Definition 1** (Aumann, 1959): The $\alpha$-core of a normal form TU game $\Gamma = \{N, X^i, u^i\}$ is a pair of joint strategy and payoff vector $(\bar{x}, \sigma)$ such that (i) $\sum_{i \in N} u^i(\bar{x}) = \sum_{i \in N} \sigma^i = v(N)$, and (ii) for each coalition $S$, $\sum_{i \in S} \sigma^i \geq v(S)$.

That is, $(\bar{x}, \sigma)$ is in the $\alpha$-core if (i) the strategy $\bar{x}$ maximizes the grand coalition's joint payoff, and (ii) the payoff vector $\sigma$ splits the grand coalition's maximal payoff in such a way that no coalition $S$ is given less than its guaranteed payoff $v(S)$\(^8\).

Remark 1. Note that the core of characteristic form games is defined by a payoff vector, the $\alpha$-core of normal form NTU games is defined either by a joint strategy or by a feasible payoff vector, and the above $\alpha$-core of normal form TU games is defined by both a strategy and a payoff vector.

Recall that a coalition structure is a partition of $N$, $\Delta = \{S_1, S_2, \ldots, S_k\}$, satisfying $\bigcup S_i = N$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. Let $\Pi$ denote the set of all partitions of $N$, and for each $\Delta \in \Pi$,\(^7\) Implicitly, we assume that the minimization problem has an optimal solution. This is true under the usual conditions that all $u^i$ are continuous and all $X^i$ are compact. Otherwise, we should replace the "Min" by "Inf."

\(^8\) Or no coalition can guarantee a higher joint payoff than what it is currently receiving. Note also that the second requirement of the Definition 1, $\sum_{i \in S} \sigma^i \geq v(S)$ is equivalent to $\sum_{i \in T} \sigma^i \leq v(N) - v(N \setminus T)$ for any coalition $T$. This means that no coalition $T$ is paid more than its marginal contribution in the grand coalition ($v(N) - v(N \setminus T)$).
let $k = k(\Delta) = |\Delta|$ denote the number of coalitions in $\Delta$. Then a coalition structure $\Delta = \{S_1, S_2, ..., S_k\}$ induces $k$ parametric normal form TU games:

$$
\Gamma_S(x_{-S}) = \{S, X^i, u^i(, x_{-S})\}
$$

for $S \in \Delta$. For each $x_{-S}$, $\Gamma_S(x_{-S})$ has $|S|$ players and is a new game with fewer players.

A coalition structure implies naturally that players within a coalition will cooperate and different coalitions behave non-cooperatively. Thus for each $S \in \Delta$, players within $S$ shall take the complementary choice $x_{-S}$ as given and chose a cooperative solution $x_S$ as its response. This idea is formally presented in the next definition.

**Definition 2:** Given a coalition structure $\Delta = \{S_1, S_2, ..., S_k\}$ in the game (1). The **hybrid solution** associated with $\Delta$ is a pair of $(x_S, \sigma_S)$ such that for each $S \in \Delta$, $(x_S, \sigma_S)$ is in the $\alpha$-core of $\Gamma_S(x_{-S})$.

That is, a hybrid equilibrium is reached if each coalition maximizes its joint payoff in response to the outsiders' choice and it divides its joint payoff in the $\alpha$-core fashion among its members. Clearly, Nash equilibrium is the hybrid solution for the finest coalition structure, and the $\alpha$-core is the hybrid solution for the coarsest coalition structure.

This extends the individual best response of Nash equilibrium to the coalitional "best response" of hybrid solution. We adopt $\alpha$-core as the "best response" of a coalition because "the core concept is often regarded as basic to the theory, and other solutions gain some support if it can be shown that they are in some way related to it" (Maschler, Peleg and Shapley, 1979), and because other core-like solutions (the $\beta$-core and strong equilibrium, see Figure 1 below) generally do not exist in normal form games.

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9 Such behavioral implication of a coalition structure has been present in Thrall and Lucas (1963), Aumann and Dreze (1974), Myerson (1977), Hart and Kurz (1983), Zhao (1992), and Ray and Vohra (1993). However, a coalition structure has been interpreted to have different behavioral implications (see Block and Ghosal, 1994).

10 Note that best response is a well defined concept for individual players, it is only loosely defined here for a coalition, because there is no best way to divide a coalition's joint payoff. Here, each $S$ chooses a "best response" to the complementary choice $x_{-S}$ in the following sense: (i) $\tilde{x}_S$ maximizes its joint payoff given $\tilde{x}_{-S}$; and (ii) $\sigma_S$ is an $\alpha$-core payoff of $\Gamma_S(x_{-S})$. 
Obviously, a coalition might choose, in addition to the \( \alpha \)-core, other cooperative solutions as a "best response" to a fixed complementary choice. Let

\[
D = \{ \alpha \text{-core, } \beta \text{-core, Strong equilibrium, Pareto efficient outcome; Kernel, Bargaining set; Shapely value, Nucleolus, } \tau \text{-value, Equal share outcome} \}
\]

denote the set of possible distribution rules (for definitions, see the references given in the next remark). The relationship among these concepts are illustrated in Figure 1.

(Figure 1 about here)

Remark 2. Note that the \( \alpha \)-core, \( \beta \)-core and Strong equilibrium (Aumann, 1959), the Pareto efficient, and the Equal share outcomes are defined for the game (1), but other concepts like kernel (Davis & Maschler, 1965), bargaining set (Aumann & Maschler, 1964), the Shapely value (1953), the nucleolus (Schmeidler, 1969), and the \( \tau \)-value (Tijs, 1981) can only be defined for a characteristic form game. In this paper, we shall define these concepts from the following TU game in characteristic form

\[
\Gamma^\cf_\alpha = \{N, v^\alpha(\cdot)\},
\]

which is derived from the original game (1) in the \( \alpha \)-core fashion. A coalition's joint payoff \( v^\alpha(S) \) in (6) is the same as that in (3), which is rewritten as

\[
v^\alpha(S) = \max_{x_S \in X_S} \min_{z_S \in X_S} \sum_{i \in S} u^i(x_S, z_S).
\]

Thus \((\bar{x}, \sigma)\) is in the \( \alpha \)-core of (1) if and only if (i) \( \sum_{i \in N} \sigma^i = \sum_{i \in N} u^i(\bar{x}) = V(N) \), and (ii) \( \sigma \) is a core vector of \( \Gamma^\cf_\alpha = \{N, v^\alpha(\cdot)\} \). Now let

\[
v^\beta(S) = \min_{z_S \in X_S} \max_{x_S \in X_S} \sum_{i \in S} u^i(x_S, z_S),
\]

\[
\Gamma^\cf_\beta = \{N, v^\beta(\cdot)\}
\]
denote the characteristic form TU game derived from the original game \textit{in the }\beta\text{-core fashion}, then the \(\beta\)-core of \((1)\) is a pair \((\bar{x}, \sigma)\) such that (i) \(\sum_{i \in N} \sigma^i = \sum_{i \in N} u^i(\bar{x}) = V(N)\), and (ii) \(\sigma\) is a core vector of \(\Gamma^\text{CF}_\beta = \{N, v^\beta(\cdot)\}\).\footnote{In the \(\alpha\)-core, \(S\) can "guarantee" the joint payoff \(v^\alpha(S)\), i.e., they can find a choice \(x_S\) such that their joint payoff is at least \(v^\alpha(S)\) regardless of the choices of the outsiders. While in the \(\beta\)-core, \(S\) "can not be prevented from" receiving the joint payoff \(v^\beta(S)\), i.e., for every complementary choice \(z_{-S}\), they can counteract it (to choose some choice \(x_S = x_S(z_{-S})\)) so as to make their joint payoff no less than \(v^\beta(S)\).}

Given a coalition structure \(\Delta = \{S_1, S_2, ..., S_k\}\), a distribution rule (DR) for \(\Delta\) is a function from \(\Delta\) to \(D\) such that for each \(S \in \Delta\), \(DR(S) \in D\). That is, a division rule specifies that a coalition \(S\) distribute its payoffs according to \(DR(S)\). The concept of hybrid solution with a given distribution rule (or HSDR for short) can be given as:

**Definition 3**: Given a coalition structure \(\Delta = \{S_1, S_2, ..., S_k\}\) and a DR for \(\Delta\) in the game \((1)\). The \textit{hybrid solution with DR for }\(\Delta\) is a pair of \((\bar{x}, \sigma)\) such that for each \(S \in \Delta\), (i) \(\bar{x}_S\) is a solution of \(\max \{\sum_{i \in S} u^i(x_S, \bar{x}_S) \mid x_S \in X_S\}\); and (ii) \(\sigma_S\) splits the joint payoff \(\sum_{i \in S} u^i(\bar{x}) = \sum_{i \in S} \sigma^i\) according to \(DR(S)\).\footnote{As mentioned in Remark 2, if \(DR(S) \in \{\alpha\text{-core, }\beta\text{-core, Strong equilibrium, Pareto efficient outcome, Equal share outcome}\}\), \(\sigma_S\) is a payoff vector of \(\Gamma^S_{S(\bar{x}_{-S})}\) according to \(DR(S)\). If \(DR(S) \in \{\text{Kernel, Bargaining set, Shapely value, Nucleolus, }\tau\text{-value}\}\), \(\sigma_S\) is a payoff vector of \(\Gamma^S_{S(\bar{x}_{-S})}^\text{CF}\) according to \(DR(S)\), where \(\Gamma^S_{S(\bar{x}_{-S})}^\text{CF}\) is derived from \((4)\) in the \(\alpha\text{-core} \) fashion (similar to \((6)\)).}

That is, a HSDR is reached if each \(S\) maximizes its payoff in response to outside choice and divides its payoff according to \(DR(S)\). Note that Definition 3 is more general than Definition 2 (the special case of \(DR(S) = \alpha\)-core). Note also that a given coalition structure has multiple HSDR's, which are caused by at least three sources: (a) the hybrid equilibrium strategy might not be unique for \(\Delta\), (b) there are multiple cooperative solutions (except the values) for a given DR, and (c) different coalitions in \(\Delta\) might choose different DR's.

**Remark 3.** The economic applications and behavioral implications of HSDR are very broad. In fact, HSDR has been used in both applied and theoretical works without its formal definition. For example, when \(DR(S) = \text{efficient outcome}\), the HSDR is the equilibrium in
international monetary coordination (Kehoe, 1987), the equilibrium in cooperative labor supply (Chiappori, 1992), the binding equilibrium agreements in Ray and Vohra (1994), and the quasi-hybrid solution in Zhao (1991).

Remark 4. When the hybrid equilibrium strategy for a given $\Delta$ is unique, the two solutions in Definitions 2 and 3 shall have the same equilibrium strategy for the same $\Delta$ and a coalition $S (S \in \Delta)$ has the same joint payoff in both cases, though the payoff vectors might be different. This is so because a coalition always maximizes its joint payoff in response to the outsiders' choice, and a coalition in Definition 3 just splits the same joint payoff according different distribution rules. Precisely, let $\bar{x}(\Delta) = \{\bar{x}_{S_1}, ..., \bar{x}_{S_k}\}$ denote the unique hybrid strategy for a given $\Delta$ in Definitions 2 and 3, then the unique vector of coalition's payoff, $\phi: \Pi \rightarrow \bigcup_{\Delta \in \Pi} R^{\left|\Delta\right|}$, is given by

$$\phi(\Delta, S) = \sum_{i \in S} u^i(\bar{x}(\Delta))$$

for each $\Delta \in \Pi$ and each $S \in \Delta$.

We are now ready to define the concept of stable coalition structures. The main purpose in studying the stability of coalition structures is to identify those structures upon which the players can not improve, or they can not do better than the current outcome. The precise meaning of "do better" depends on how the members of the deviating coalition $S$ perceive the actions of the outsiders. This is the source of variations in different stability concepts. For TU games in normal form, the meaning of "do better" depends on what new HSDR be chosen, this in turn depends on two issues: (a) What is the new coalition structure

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13 This becomes clear if we obtain a new normal form game by replacing each coalition with a new player, whose payoff and choice are those of the coalition replaced. The Nash equilibrium strategy and payoff of the new game are the equilibrium strategy and a coalition's payoff in both Definitions 2 and 3.

14 This defines a game in partition function form (Thrall and Lucas, 1963), which is given as $\Gamma^{PF} = \{N, \phi\}$, where for each $\Delta \in \Pi$, $\phi(\Delta) \in R^{\left|\Delta\right|}$, specifying a joint value $\phi(\Delta, S)$ for each coalition $S \in \Delta$, and the superscript PF stands for partition function.

15 As discussed in Remark 4, when there is a unique hybrid equilibrium strategy, the joint payoff for each coalition structure can be uniquely computed. What distribution rule a coalition uses has no effect on its own joint payoff nor others', but it does affect the stability of a coalition structure.
(of which the deviating coalition is a member)? (b) What distribution rule will the deviating coalition choose?

For each \( B \in \Pi \) in the game (1), let

\[
(11) \quad HSD(B) = \{ (\bar{x}, \sigma) \mid (\bar{x}, \sigma) \text{ be a HSDR for } B \}
\]

denote the set of all HSDR's for \( B \). Let

\[
(12) \quad \Delta = \{ S_1, S_2, \ldots, S_k \}
\]

be the current coalition structure, \( S \) be the deviating coalition, and

\[
(13) \quad \Pi(S) = \{ \Delta \in \Pi \mid \Delta = \{ S, T_1, T_2, \ldots, T_m \} \}
\]

denote the set of partitions of which \( S \) is a member. The next definition defines five stable concepts for a given coalition structure (i.e., the \( \alpha, \gamma, \delta, \epsilon, \) and \( \zeta \)-stabilities, following Aumann's (1961) tradition (see also Hart & Kurz (1983)) of characterizing stable concepts by Greek letters), which are based on the following three coalition structures:

\[
(14) \quad B^\alpha(\Delta, S) = B^\alpha(S) = \{ S, T_1, T_2, \ldots, T_m' \}
\]

is the worst coalition structure in \( \Pi(S) \) for \( S \), representing the pessimistic belief that the worst outcome might happen to the deviating coalition. That is, it is a Min solution in

\[
(15) \quad \min_{B \in \Pi(S)} \left\{ \min_{(\bar{x}, \sigma) \in HSD(B)} \sum_{i \in S} u^i(\bar{x}(i)) \right\}.
\]

\[
(16) \quad B^\gamma(\Delta, S) = \{ S, T_1, T_2, \ldots, T_m'' \}
\]

is the new structure in \( \Pi(S) \) such that if an old coalition \( S_i \in \Delta \) has members to join \( S \), then the rest of \( S_i \) become singletons, otherwise, the coalition \( S_i \) remains unchanged.

\[
(17) \quad B^\delta(\Delta, S) = \{ S, T_1, T_2, \ldots, T_m^* \}
\]

is the new structure in \( \Pi(S) \) such that if an old coalition \( S_i \in \Delta \) has members to join \( S \), then the rest of \( S_i \) remain as a coalition, otherwise, the coalition \( S_i \) remains unchanged.\(^{17}\)

---

\(^{16}\) When there are multiple solutions, introduce some arbitrary rule and pick up one.

\(^{17}\) Precisely, \( B^\gamma(\Delta, S) = \{ S, T_1, \ldots, T_m'' \} \) satisfies (a) \( T_1 \subseteq S_j \) for some \( S_j \in \Delta \) for each \( i = 1, 2, \ldots, m'' \), (b) \( T_i = S_j \) if \( T_i \subseteq S_j \) and \( S_j \cap S = \emptyset \), and (c) \( T_1 \) is a singleton if \( T_1 \subseteq S_j \) and \( S_j \cap S \neq \emptyset \). \( B^\delta(\Delta, S) = \{ S, T_1, \ldots, T_m^* \} \)
For example, let $N = (1, 2, 3, 4, 5)$, $\Delta = \{1, (2,3,4,5)\}$, and $S = (1,2)$, then $\Pi(S) = \{ B_1, B_2, B_3, B_4, B_5 \}$, where $B_1 = \{(1,2), 3, 4, 5\}$, $B_2 = \{(1,2), (3, 4), 5\}$, $B_3 = \{(1,2), (3, 5), 4\}$, $B_4 = \{(1,2), 3, (4, 5)\}$, and $B_5 = \{(1,2), (3, 4, 5)\}$. So $B_1 = \{ (1,2), 3, 4, 5 \}$, $B_2 = \{ (1,2), (3, 4), 5 \}$, $B_3 = \{ (1,2), (3, 5), 4 \}$, $B_4 = \{ (1,2), 3, (4, 5) \}$, and $B_5 = \{ (1,2), (3, 4, 5) \}$.

If $\phi(B_1) = (0.5, 0.4, 0.4, 0.4)$, $\phi(B_2) = \phi(B_3) = \phi(B_4) = (0.6, 0.6, 0.6)$, and $\phi(B_5) = (0.7, 0.7)$, then $\phi(B_1, S) = 0.5$, $\phi(B_2, S) = \phi(B_3, S) = \phi(B_4, S) = 0.6$, and $\phi(B_5, S) = 0.7$. So $\min \{ \phi(B, S) \mid B \in \Pi(S) \} = 0.5$, and $B^\delta(S) = B_1 = \{(1,2), 3, 4, 5\}$.

Now the concepts of stable coalition structures can be introduced as follows.

**Definition 4:** Let $\Delta = \{ S_1, S_2, ..., S_k \}$ be a coalition structure in the game (1), and $(\bar{x}, \bar{\sigma})$ be a fixed HSDR for $\Delta$. The fixed coalition structure $\Delta$ with the payoff $\bar{\sigma}$ is

(a) $\alpha$-stable if there is no $S \in \mathcal{N}$ such that $\sigma_S >> \bar{\sigma}_S$ for all $(x, \sigma) \in \text{HSD}(\delta^\alpha(S))$;

(b) $\gamma$-stable if there is no $S \in \mathcal{N}$ such that $\sigma_S >> \bar{\sigma}_S$ for all $(x, \sigma) \in \text{HSD}(\delta^\gamma(\Delta, S))$;

(c) $\delta$-stable if there is no $S \in \mathcal{N}$ such that $\sigma_S >> \bar{\sigma}_S$ for all $(x, \sigma) \in \text{HSD}(\delta^\delta(\Delta, S))$;

(d) $\varepsilon$-stable if there is no $S \in \mathcal{N}$ such that $\sigma_S >> \bar{\sigma}_S$ for some $(x, \sigma) \in \text{HSD}(\delta^\varepsilon(S))$;

(e) $\zeta$-stable if there is no $S \in \mathcal{N}$ such that $\sum_{i \in S} \sigma^i > \sum_{i \in S} \bar{\sigma}^i$ for all $(x, \sigma) \in \text{HSD}(\delta^\zeta(S))$.

To put it differently, a coalition structure $\Delta$ with the equilibrium payoff $\bar{\sigma}$ is $\alpha$-stable ($\gamma$-, $\delta$-, and $\varepsilon$-stable, respectively) if no coalition $S$ can split its joint payoff at the new coalition satisfies (a) $T_i \subseteq S_j$ for some $S_j \in \Delta$ for each $i = 1, 2, ..., m^*$, (b) $T_1 = S_j$ if $T_1 \subseteq S_j$ and $S_j \cap S = \emptyset$, and (c) $T_1 = S_j \setminus S$ if $T_1 \subseteq S_j$ and $S_j \cap S \neq \emptyset$.

As discussed in section 3 of Hart and Kurz (1983), $\Delta^\gamma(S)$ represents the idea that coalition formation is a result of a unanimous agreement "among all its members," i.e., if some players leave, the agreement breaks down and the rest become singletons; and $\Delta^\delta(S)$ represents the idea that coalition formation is a result of "majority voting", i.e., if some players leave, the willingness to cooperate are still effective among loyal members and the rest remain as a coalition (a smaller one).

The concept of $\alpha$-stability is different from $\alpha$-core, though it is based on the similarly cautious belief underlying the $\alpha$-core. The concept of $\beta$-stability is omitted, because it is identical to the $\alpha$-stable concept in this situation.

There are three alternative ways in defining the stability. The first is the coalition formation games used in Hart and Kurz (1983). Each concept can be defined as a strong equilibrium (or the $\alpha$, $\beta$-core) of a normal form game similar to that discussed in Section 5. The second alternative is to transform the game (1) into a partition
structure $\mathcal{B}^\alpha(S)$, $\mathcal{B}^\gamma(\Delta, S)$, and some $\mathcal{B} \in \Pi(S)$) in such a way that all its members are better off, and it is $\zeta$-stable if no coalition $S$ can increase its joint payoff by moving to $\mathcal{B}^\alpha(S)$.

It is useful to consider what an unstable structure means. A coalition structure $\Delta$ with the payoff $\bar{\sigma}$ is not $\alpha$-stable (or equivalently, not $\beta$-stable, see Footnote 18) if there exists $S \in N$ such that at any new structure $\mathcal{B} \in \Pi(S)$, all its members are better off at all HSDR of $\mathcal{B}$ (i.e., at all $(x, \sigma) \in \text{HSD}(\mathcal{B})$). Similarly, it is not $\gamma$-stable (or $\delta$-stable) if there exists $S \in N$ such that at the new structure $\mathcal{B}^\gamma(\Delta, S)$ ($\mathcal{B}^\delta(\Delta, S)$), all its members are better off at all HSDR of the new structure, and it is not $\varepsilon$-stable if there exists $S \in N$ such that at some new structure $\mathcal{B} \in \Pi(S)$, all its members are better off at some HSDR of $\mathcal{B}$. Finally, it is not $\zeta$-stable if there exists $S \in N$ such that at any new structure $\mathcal{B} \in \Pi(S)$, its joint payoff is increased at all HSDR of the new structure.

Remark 5. The stability of a coalition structure $\Delta$ is essentially the stability of a HSDR for $\Delta$, since the distribution rule adopted by a coalition and the existence of HSDR can both affect the stability. For example, let $(\bar{x}(\Delta), \bar{\sigma}(\Delta))'$ and $(\bar{x}(\Delta), \bar{\sigma}(\Delta))''$ be two HSDR's for the same $\Delta$ according to two different DR's, then it is possible that $\Delta$ with the payoff $\bar{\sigma}(\Delta)'$ is stable but not the other. On the other hand, suppose (1) $(\bar{x}, \bar{\sigma})$ is a HSDR for a fixed $\Delta$, (2) the HSDR does not exists for all $\mathcal{B} \neq \Delta$, then $\Delta$ with the payoff $\bar{\sigma}$ will be stable. These provide some insights into the refinement of Nash equilibria (see Section 3).

Remark 6. The $\alpha$, $\gamma$ and $\delta$-stabilities are extensions of Hart and Kurz (1983), the $\varepsilon$-stability is a variation of Shenoy (1979), and the $\zeta$-stability is the same as that in Thrall and

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20 Though the coalition is better off at any new structure, it may not be able to split its joint payoff (according some rules in D) in all cases such that all its members are better off.

21 The idea of $\alpha$-stability can be found in Aumann (1961), Thrall and Lucas (1963). Though these were not referenced in Riezman (1985), his formation of customs unions adopts precisely the $\alpha$-stability.
Lucas (1963). Consequently, the above stability concept can also be defined alternatively by using these earlier results.

Remark 7. The $\delta$-stability has been used in the earlier literature, though the authors phrased their concepts differently. Three such examples are listed here. The first one is the coalition proof Nash equilibrium (see Section 3), which is equivalent to the $\delta$-stability of the finest coalition structure when $\text{DR}(S) \equiv \text{Undominated Self-enforcing equilibrium}$.

The second one is the stability of dominant cartel in d'Aspremont, Jacquemin, Gabszewicz and Weymark (1983). Let $\Delta = \{T, i_1, i_2, ..., i_m\} \in \Pi$, and $\text{DR}(S) \equiv \text{Equal share outcome}$, where $T$ is the dominant cartel, and $i_1, i_2, ..., i_m$ are singletons. The stability of the dominant cartel $T$ is the $\delta$-stability of $\Delta$ when the deviating coalition can only be $S = \{i\} (i \in T)$ or $S = T \cup i (i \notin T)$.  

The third one is the stable binding equilibrium agreement in Ray and Vohra (1993). Let $\Delta = \{S_1, S_2, ..., S_k\}$ and $\text{DR}(S) \equiv \text{Pareto efficient outcome}$. If the deviating coalitions are restricted to $T \subset S$ and $T \neq S$ for some $S \in \Delta$, the $\delta$-stability is exactly the stability of a binding equilibrium agreement.

3. THE REFINEMENT OF NASH EQUILIBRIA

The concepts of a HSDR and a stable coalition structure can both be used to refine Nash equilibria. We shall add Nash equilibrium to the set of distribution rules given by (5). Consequently, the definition of HSDR can be modified as follows.

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22 It is assumed that all firms are identical, and the singletons are called fringe firms. The cartel $T$ has internal stability if no member $i$ has incentive to leave (i.e., the new structure is restricted to the forms of $\Delta^I = \{T \setminus i, i, i_1, i_2, ..., i_m\}$), external stability if no outsider has incentive to join $T$ (i.e., the new structure is restricted to the forms of $\Delta^E = \{T \cup i_k, i_1, ..., i_{k-1}, i_{k+1}, ..., i_m\}$). It is stable if it has both the internal and external stabilities.

23 Here, deviation is only allowed to make the structure finer and no coalition cutting through two or more members of $\Delta$ is allowed to form. Thus our new concept extends Ray and Vohra's work in two aspects: a coalition's "best" response and possible deviations. We allow a variety of responses for a coalition in addition to Pareto efficient outcome, and we allow any new coalition structure in $\Pi(S)$. 

13
**Definition 5:** Given a coalition structure \( \Delta = \{ S_1, S_2, \ldots, S_k \} \) and a DR for \( \Delta \) in the game (1). The **hybrid solution with DR for** \( \Delta \) is a pair of \((\bar{x}, \sigma)\) such that (a) for each \( S \in \Delta \) with \( DR(S) = \) Nash equilibrium, \( \sigma_S = u(\bar{x})_S \) and \( \bar{x}_S \) is a Nash equilibrium of \( \Gamma_S(\bar{x}_S) \); (b) for each \( S \in \Delta \) with \( DR(S) = \) cooperative solution, \( \bar{x}_S \) is a solution of \( \max \{ \sum_{i \in S} u^i(x_{\bar{S}}, \bar{x}_S) \mid x_{\bar{S}} \in X_S \} \) and \( \sigma_S \) splits the joint payoff \( \sum_{i \in S} u^i(\bar{x}) = \sum_{i \in S} \sigma^i \) according to \( DR(S) \).

The new insights on Nash equilibrium are discussed in three propositions (for more discussions, see Zhao (1996)): Proposition 1 provides new definitions of Nash equilibrium, Proposition 2 shows the equivalence between coalition-proof Nash equilibrium (CPNE for short) and the \( \delta \)-stability of the finest coalition structure, and Proposition 3 provides a new refinement in the spirit of CPNE.

**Proposition 1.** Consider the game (1). (a) \( \bar{x} \) is a Nash equilibrium if and only if \((\bar{x}, u(\bar{x}))\) is a HSDR (with any distribution rule) for the finest coalition structure;

(b) \( \bar{x} \) is a Nash equilibrium if and only if \((\bar{x}, u(\bar{x}))\) is a **HSDR for any a coalition structure** \( \Delta \) and \( DR(S) \equiv \) Nash equilibrium;

(c) If \((\bar{x}, u(\bar{x}))\) is a HSDR for a given \( \Delta \) with \( DR(S) \equiv \) undominated Nash equilibrium for all \( S \in \Delta \), then \( \bar{x} \) is a refined Nash equilibrium.

For any \( S \in \mathcal{N} \), let \( \tilde{S} \) denotes the finest partition of \( S \). Then Part (a) says the following: \( \bar{x} \) is a Nash equilibrium \( \iff (\bar{x}, u(\bar{x})) \in \text{HSD}(\Delta^*), \) where \( \Delta^* = \tilde{N} \) is the finest partition of \( N \), and \( \text{HSD}(\Delta) \) is the set of all HSDR's for \( \Delta \). Let \( \text{HSD}(\Delta)^* \) be the set of undominated HSDR for \( \Delta \) (i.e., \((\bar{x}, \sigma) \in \text{HSD}(\Delta)^* \) if there is no \((\bar{x}', \sigma') \in \text{HSD}(\Delta) \) such that \( \sigma < \sigma' \)). Then \( \bar{x} \) is an undominated Nash equilibrium \( \iff (\bar{x}, u(\bar{x})) \in \text{HSD}(\Delta^*)^* \), and \( \bar{x} \) is a CPNE if for any \( S \in \mathcal{N}, (\bar{x}_S, u(\bar{x})_S) \in \text{HSD}(\tilde{S}) \) and \( \bar{x}_S \) is an undominated self-enforcing equilibrium for the game \( \Gamma_S(\bar{x}_S) \) (see (4)), where self-enforcing equilibrium is defined in Bernheim, Peleg and Whinston (1987).

**Proposition 2.** \( \bar{x} \) is a CPNE \( \iff \Delta^* \) with the payoff \( u(\bar{x}) \) is \( \delta \)-stable and \( DR(S) \equiv \) undominated self-enforcing equilibrium.
Note for the finest partition $\Delta^*$, $B^\hat{\Delta}(\Delta^*, S) = \{S, i_1, i_2, ..., i_m\}$ for any $S \in \mathcal{N}$. A modified CPNE can be defined as a Nash equilibrium $\bar{x}$ such that for any $S \in \mathcal{N}$, $(\bar{x}_S, u(\bar{x})_S) \in HSD(\tilde{S})^*$ for the game $\Gamma_{\bar{x}_S}(\bar{x})$. That is, a modified CPNE is reached if all coalitions choose an undominated Nash equilibrium as a "best response."

**Proposition 3.** Then $\bar{x}$ is a modified CPNE $\iff \Delta^*$ with the payoff $u(\bar{x})$ is $\delta$-stable and $DR(S)\equiv$undominated Nash equilibrium.

Let $HSD(n)^{MC}$ and $HSD(n)^C$ denote respectively the above modified and the original CPNE in the game (1), then $HSD(2)^{MC} = HSD(2)^C$. It can be checked that $HSD(n)^{MC} \subseteq HSD(n)^C$ for $n = 3, 4, ..., \,$ so the modified CPNE is stronger than the original CPNE.

**4. EXISTENCE RESULTS**

This section provides sufficient conditions for the existence of HSDR (Theorems 1, 2 and 3). The relationships among the stable concepts are provided in Propositions 4 and 5. All results are proved in Section 6. Our existence results require the weak separability assumption defined below. For each $T \in \mathcal{N}$ and $T \neq N$ in the game (1), its guaranteed joint payoff function is given by (2) and has the form

$$\overline{u}_T(x_T) = \sum_{i \in T} u^i(x_T, \overline{y}_{-T}(x_T)) = \min_{y_{-T} \in X_{-T}} \sum_{i \in T} u^i(x_T, y_{-T})$$

for each $x_T \in X_T$, and its maximum guaranteed joint payoff is (see (3) or (7))

$$v(T) = \max \{ \overline{u}_T(x_T) | x_T \in X_T \} = \overline{u}_T(\tilde{x}_T) = \sum_{i \in T} u^i(\tilde{x}_T, \overline{y}_{-T}(\tilde{x}_T)),$$

where $\tilde{x}_T$ is the maximum solution.

**Definition 6:** (i) The guaranteed payoff function $\overline{u}_T(x_T)$ for a coalition $T$ is **weakly separable** at a point $x_T$ if for all $i \in T$,

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24 We now use $T$ instead of $S$, in order to make the future proofs clear. In the proofs, a balanced collection is given by $B = \{T_1, T_2, ..., T_k\}$, and a coalition structure is given by $\Delta = \{S_1, S_2, ..., S_k\}$.  


15
\[ u^i(x_T, \bar{y}_{-T}(x_T)) = \min_{y_{-T} \in X_{-T}} u^i(x_T, y_{-T}); \]

(ii) The maximum joint payoff \( v(T) \) for a coalition \( T \) is weakly separable if its guaranteed payoff function \( \bar{u}_T \) is weakly separable at \( \bar{x}_T \).

Roughly speaking, the weak separability of \( \bar{u}_T \) at \( x_T \) requires that if \( \bar{y}_{-T}(x_T) \) minimizes the joint payoff \( \sum_{i \in T} u^i(x_T, y_{-T}) \), it also minimizes all of its individual payoff \( u^i(x_T, y_{-T}) \) for all \( i \in T \).

**Assumption I:** Assumption I holds in a normal form TU game if for each \( S \in N \) and \( S \)

**Theorem 1:** A normal form TU game \( \Gamma = \{N, X^i, u^i\} \) has at least one \( \alpha \)-core if for each player \( i \in N \), (a) \( X^i \) is compact and convex; (b) \( u^i(x) \) is continuous in \( x \); (c) \( u^i(x) \) is concave in \( x \); and (d) Assumption I is satisfied in the game.

Note that Scarf's \( \alpha \)-core existence theorem in an NTU normal form game (1971) only requires Conditions (a), (b) and the quasiconcavity. But in a TU normal form game, much stronger condition,\(^{25}\) the concavity rather than the quasi-concavity of the payoff functions and Assumption I are required for balancedness. Though Assumption I is strong, it is satisfied in oligopoly markets, which is shown in the next corollary.\(^{26}\)

**Corollary 1.** Consider an oligopoly game \( \Gamma = \{N, X^i, \pi^i\} \) defined by \( X^i = [0, \bar{y}_i^i] \) and \( \pi^i(x) = P(X) x^i - C_i(x^i) \) for each \( i \in N \), where \( P(X) = p(\Sigma x^i) \) is the decreasing inverse demand function. The game has at least one \( \alpha \)-core if for each \( i \in N \), its profit function \( \pi^i(x) \) is continuous and concave in \( x \).

Because a coalition with side-payments has a larger blocking power, one might, at the first glance, reason that the set of \( \alpha \)-core payoffs for a TU normal form game is smaller than

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\(^{25}\) The conditions are only sufficient conditions. As shown in Zhao (1995a), there exists a large class of linear oligopolistic industries in which the associated TU games (in the \( \alpha \)-core fashion) are convex and thus have non-empty \( \alpha \)-core, but the profit functions in these games are not even quasi-concave.

\(^{26}\) This is so because \( \bar{y}_{-S} \), the capacity constraints of outsiders (see footnote 5) minimizes both the joint profit \( \pi_T = \sum_{i \in T} [P(X) x^i - C_i(x^i)] \) and any member \( i \)'s profit \( \pi^i = P(X) x^i - C_i(x^i) \) for all \( x_T \).
that in an NTU game and thus stronger conditions are required for its existence. However, this is not a solid argument, because one could argue, quite oppositely, that the set of the grand coalition's possible splits also becomes larger. As a result, no stronger conditions should be required for existence. As shown in Example 1, there is no general relationship between TU $\alpha$-core and NTU $\alpha$-core. Thus, it remains to be seen if the conditions of Theorem 1 can be replaced by the weak conditions of Scarf's theorem for NTU $\alpha$-core.

Example 1. Consider a duopoly industry: $P = 6 - x^1 - x^2$, $\bar{y}^1 = 3$, $\bar{y}^2 = 4$, $c^1 = 1$, $c^2 = 2$, $d^1 = d^2 = 0$. Then the two guaranteed profit functions are:

$$\pi^1(x^1) = [P(x^1, \bar{y}^2) - c^1] x^1 = [1 - x^1] x^1, \text{ and}$$

$$\pi^2(x^2) = [P(\bar{y}^1, x^2) - c^2] x^2 = [1 - x^2] x^2,$$

and they both attain a maximum value of $1/4$ at $x^1 = x^2 = 1/2$. Let $E$ and $F = \{ (\pi^1, \pi^2) \in \mathbb{R}^2_+ \mid \pi^1 + \pi^2 = 6.25 \}$ denote respectively the NTU and TU efficient profit frontiers, then

$$\alpha\text{-core}_{\text{NTU}} = \{ (\pi^1, \pi^2) \in E \mid \pi^1 \geq \frac{1}{4}, \pi^2 \geq \frac{1}{4} \}, \text{ and}$$

$$\alpha\text{-core}_{\text{TU}} = \{ (\pi^1, \pi^2) \in F \mid \pi^1 \geq \frac{1}{4}, \pi^2 \geq \frac{1}{4} \}.$$

These are represented in Figure 2.

(Figure 2 about here )

Theorem 2: Given a coalition structure $\Delta= \{ S_1, S_2, ..., S_k \}$ in the game (1). There is at least one hybrid solution associated with $\Delta$ if (a) for each $i \in N$, $X^i$ is compact and convex; (b) for each $i \in N$, $u^i(x)$ is continuous in $x$; (c) for each $S \subseteq \Delta$, all $u^i(x_S, x_{-S})$, $i \in S$, are concave in $x_S$; and (d) for each $S \subseteq \Delta$, Assumption I is satisfied in the parametric game $\Gamma_S(x_{-S})$ for all $x_{-S}$.

That is, if all strategy sets are closed, bounded and convex, the payoffs of each coalition are concave in the coalition's own strategies and are continuous in all strategies, and each coalition's game satisfies Assumption I, then there exists at least one pair of joint strategy
and payoff vector \((\bar{x}, \sigma)\) such that for each \(S \in \Delta\), \(\bar{x}_S\) maximizes its joint payoff and \(\sigma_S\) is an \(\alpha\)-core payoff given \(\bar{x}_S\).

**Corollary 2.** Given a market structure \(\Delta = \{S_1, S_2, \ldots, S_k\}\) in the oligopoly game \(\Gamma = \{N, X^i, \pi^i\}\). There is at least one hybrid solution associated with \(\Delta\) if (a) for all \(i \in N\), \(\pi^i(x)\) is continuous, and (b) for each \(S \in \Delta\), all \(\pi^i(x_S, x_{-S})\), \(i \in S\) are concave in \(x_S\).

Remark 8. Theorem 2 is the TU counter part of the hybrid solution existence in normal form NTU game by Zhao(1992), while Theorem 1 is the TU counter part of the \(\alpha\)-core existence in normal form NTU game by Scarf (1971). Similarly as in Theorem 1, It remains to be proved or disproved if the conditions in Theorem 2 can be replaced by the weak conditions for NTU normal form games in Zhao (1992). 27

Remark 9. Theorem 2 becomes Theorem 1 when the coalition structure is the coarsest. However, it does not become the Nash equilibrium existence theorem when the coalition structure is the finest, because Nash equilibrium only requires the quasi-concavity of a player’s payoff function on his own strategies.

In the next theorem, we shall limit the set of possible distributions to be \(DR(S) \in \{\alpha-\)core, efficient outcome, Shapely value, Nucleolus, \(\tau\)-value, and Equal share outcome\}. Other cooperative concepts (\(\beta\)-core, Strong equilibrium, Kernel, and Bargaining set) are excluded because their existence are not known and are reserved for future studies.

**Theorem 3:** Given \(\Delta = \{S_1, S_2, \ldots, S_k\}\) and a DR for \(\Delta\) in game (1). There is at least one HSDR for \(\Delta\) if (a) for each \(i \in N\), \(X^i\) is compact and convex; (b) for each \(i \in N\), \(u^i(x)\) is continuous; (c) for each \(S \in \Delta\) with \(DR(S) = \alpha\)-core, its game \(\Gamma_S(x_S)\) satisfies Assumption I for all \(x_S\) and all \(u^i(x_S, x_{-S})\), \(i \in S\), are concave in \(x_S\); and (d) for each \(S\) with \(DR(S) \neq \alpha\)-core, its joint payoff \(\sum_{i \in S} u^i(x_S, x_{-S})\) is quasi-concave in \(x_S\).

**Corollary 3.** Given a market structure \(\Delta = \{S_1, S_2, \ldots, S_k\}\) and a distribution rule \(DR\) for \(\Delta\) in an oligopoly game \(\Gamma = \{N, X^i, \pi^i\}\). There is at least one HSDR for \(\Delta\) if (a) for all

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27 In fact, the author has carelessly claimed (p. 158, Zhao(1992)) that this is true, which should remain as a conjecture, though it is unlikely to be true.
i ∈ N, \( \pi^i(x) \) is continuous, (b) for each \( S \in \Delta \) with \( DR(S) = \alpha \)-core, all \( \pi^i(x_S, x_{-S}) \), \( i \in S \) are concave in \( x_S \) and (c) for each \( S \) with \( DR(S) \neq \alpha \)-core, its joint payoff \( \sum_{i \in S} u_i^i(x_S, x_{-S}) \) is quasi-concave in \( x_S \).

Remark 10. For each coalition \( S \) with \( DR(S) = \alpha \)-core, the conditions imposed on \( S \) in Theorem 3 is the same as that in Theorem 2, and for each \( S \) with \( DR(S) \neq \alpha \)-core (i.e., \( DR(S) \in \{ \text{efficient outcome, Shapely value, Nucleolus, } \tau \text{-value, and Equal share outcome} \} \)), the conditions are weakened to the quasi-concavity of \( \sum_{i \in S} u_i^i(x_S, x_{-S}) \) in \( x_S \). Thus Theorem 3 is stronger than Theorem 2. It becomes Theorem 2 when \( DR(S) \equiv \alpha \)-core for all \( S \in \Delta \), it becomes Theorem 1 when the coalition structure is the coarsest and \( DR(N) = \alpha \)-core, and it becomes the Nash equilibrium existence theorem when the coalition structure is the finest and \( DR(i) \neq \alpha \)-core for all \( i \). These subtle differences are useful in applications. For example, a coalition's joint profits in a linear industry is quasi-concave in \( x_S \) but individual profit functions are not concave in \( x_S \).

In the rest of this section we shall present two propositions about the stable coalition structures. The first proposition establishes the relationship between the stability concepts. The second one provides an assessment on the difficulty of finding stable structures.

Let the sets of stable coalition structures for the original game (1) be denoted as:

\[
\Pi_\alpha = \{ \Delta \in \Pi \mid \Delta \text{ is } \alpha \text{-stable} \},
\]

\[
\Pi_\gamma = \{ \Delta \in \Pi \mid \Delta \text{ is } \gamma \text{-stable} \},
\]

\[
\Pi_\delta = \{ \Delta \in \Pi \mid \Delta \text{ is } \delta \text{-stable} \},
\]

\[
\Pi_\epsilon = \{ \Delta \in \Pi \mid \Delta \text{ is } \epsilon \text{-stable} \},
\]

\[
\Pi_\zeta = \{ \Delta \in \Pi \mid \Delta \text{ is } \zeta \text{-stable} \}.
\]

By "\( \Delta \) is \( \alpha \)-stable" we mean that \( \Delta \) with some hybrid equilibrium payoff \( \bar{\sigma} \) is \( \alpha \)-stable as defined in Definition 4. The relationships among these sets are:

**Proposition 4.**

(a) \( \Pi_\epsilon \subseteq \{ \Pi_\gamma \cup \Pi_\delta \} \subseteq \Pi_\alpha = \Pi_\beta \)

(b) \( \Pi_\zeta \subseteq \Pi_\alpha = \Pi_\beta \)
In other words, the $\varepsilon$ and $\zeta$-stabilities are stronger than other concepts, the $\alpha$-stability is the weakest, and the $\gamma$ and $\delta$-stabilities are between the $\varepsilon$ and $\alpha$-stabilities. As discussed in Footnote 18, the $\alpha$ and $\beta$ stability concepts are identical. We added $\Pi_{\beta}$ in the propositions so they can be compared with earlier results shown in the next remark.

Remark 11. Hart and Kurz (1983) have observed "{\Pi_\gamma \cup \Pi_\delta} \subset \Pi_\beta \subset \Pi_\alpha". As can be seen in the following Example 2, there is no general inclusion relationship between $\Pi_\varepsilon$ and $\Pi_\zeta$ (nor between $\Pi_\gamma$ and $\Pi_\zeta$, nor between $\Pi_\delta$ and $\Pi_\zeta$).

Example 2. Let $N=\{1, 2, 3, 4\}$, $\Delta=\{(1,2,3,4)\}$, and $S=\{1,2\}$. Then $B_\gamma = B_\gamma^{\delta}(\Delta, S) = \{(1,2), 3, 4\}$, $B_\delta = B_\delta^{\delta}(\Delta, S) = \{(1,2), (3, 4)\}$, and $\Pi(S) = \{B_\gamma, B_\delta\}$. For simplicity, assume that the payoffs of $S = \{1,2\}$ associated with each partition are uniquely given by $\sigma(B_\gamma, S) = \{3, 5\}$ and $\sigma(B_\delta, S) = \{4, 2\}$. By using the alternative definition of stable structures in the proof of Proposition 4, we have $V_\gamma(S) = \{v \in \mathbb{R}^4 \mid v_1 \leq 3, v_2 \leq 5\}$, $V_\delta(S) = \{v \in \mathbb{R}^4 \mid v_1 \leq 4, v_2 \leq 2\}$, $V_\alpha(S) = V_\beta(S) = V_\gamma(S) \cap V_\delta(S) = \{v \in \mathbb{R}^4 \mid v_1 \leq 3, v_2 \leq 2\}$, $V_\varepsilon(S) = V_\gamma(S) \cup V_\delta(S)$, $V_\zeta(S) = \{v \in \mathbb{R}^4 \mid v_1 + v_2 \leq 6\}$. These sets are represented in Figure 3. It can be seen that there exists no inclusion relation between $V_\varepsilon(S)$ and $V_\zeta(S)$ (nor between $V_\gamma(S)$ and $V_\zeta(S)$). In turn, there exists no general relationship between the $\varepsilon$-stability and the $\zeta$-stability (see the precise proof of Proposition 4 in Section 6).

(Figure 3 about here )

There is no general result for the existence of stable coalition structures in the previous literature on partition function form games. Consequently, the general existence for stable structure in normal form games remains a difficult open problem. The next proposition provides an assessment on how difficult it is to find sufficient conditions for a stable structure---The conditions for a stable structure are generally stronger than that for $\beta$-core but weaker than that for strong equilibrium.
Proposition 5. (a) Assume the hybrid equilibrium strategy for any partition is unique and \( \text{DR}(S) \equiv \text{efficient outcome for all } S \). Let \( \Delta^0 = \{N\} \) denote the coarsest partition of \( N \) and \((\bar{x}, \bar{\sigma})\in \text{HSD}(\Delta^0)\) be a fixed HSDR for \( \Delta^0 \). If the coarsest coalition structure \( \Delta^0 \) with the payoff \( \bar{\sigma} \) is \( \alpha \)-stable, then the payoff \( \bar{\sigma} \) is a \( \beta \)-core vector for the original game (1).

(b) If the original game (1) has a non-empty strong equilibrium \( \bar{x} \), then the pair \((\bar{x}, \sigma) = (\bar{x}, u(\bar{x}))\) is the hybrid solution for all coalition structures, and any coalition structure will be stable by every criterion listed in Definition 4.

Note that under the assumptions of Part (a), all coalitions including the grand coalition only maximize their joint payoff and do not care how to split this joint payoff among its members. As a result, the \( \alpha \) and \( \zeta \)-stabilities for the grand coalition are identical.

Remark 12. The proposition implies that a sufficient condition for the grand coalition to be \( \alpha \)-stable shall be weaker than that for strong equilibrium existence and stronger than that for \( \beta \)-core existence. By Proposition 4, the \( \gamma \), \( \delta \), and \( \varepsilon \)-stabilities shall require stronger conditions than that of \( \alpha \)-stability.

5. THE FORMATION OF COALITION STRUCTURES

In this section we briefly discuss the formation of coalition structures. We shall assume that the hybrid equilibrium strategy \( \bar{x}(\Delta) \) is unique for each coalition structure \( \Delta \). More discussions or extensions to general case are beyond the purpose of this paper and is reserved for future studies. We present three models: the first two use the coalition structure formation game, and the third uses the cooperation structure formation game.

Suppose a coalition structure \( \Delta \in \Pi(S) \) is formed. Then, under the above assumption, the joint payoff of the coalition \( S \) is uniquely given by (See (10) or Remark 4)

\[
\phi(\Delta, S) = \sum_{i \in S} u_i(\bar{x}(\Delta))
\]

For individual payoffs, we need to consider, however, the following three cases:

(a) Suppose \( \Delta \) is the finest. Then each player receives the Nash equilibrium payoff.
(b) Suppose $S$ is not a singleton and a $\text{DR}(S)$ is chosen. If $\text{DR}(S)$ splits $\phi(\Delta, S)$ uniquely, then each player $i$ in $S$ gets his share given by $\text{DR}(S)$.

(c) Suppose $S$ is not a singleton, a $\text{DR}(S)$ is chosen, and the splits of $\phi(\Delta, S)$ by $\text{DR}(S)$ are not unique. Then players in $S$ choose a split function (like the bargaining solutions) defined below.

For each $S \in \mathcal{N}$ and a $\text{DR}=d$ for $S$, a split function for $S$ is a function,\footnote{Precisely, it satisfies the following: (i) $f(\Delta; S, d) = \{ f(j, \Delta; S, d) \mid j \in S \} \in \mathbf{R}^S$ for each $\Delta \in \Pi(S)$; (ii) $\sum_{j \in S} f(j, \Delta; S, d) = \phi(\Delta, S)$; and (iii) $f(\Delta; S, d)$ is a payoff vector according to the given $\text{DR} = d$.}

\begin{equation}
(19) \quad f(S, d): \Pi(S) \rightarrow \mathbf{R}^S,
\end{equation}

that splits $\phi(\Delta, S)$ according to $\text{DR} = d$ for each $\Delta \in \Pi(S)$. Let

\begin{equation}
(20) \quad F(S, d) = \{ f(S, d) \mid f(S, d) \text{ is a split function for } S \text{ with } \text{DR} = d \}
\end{equation}

denote the set of all split functions with $\text{DR} = d$ for $S$, and

\begin{equation}
(21) \quad [\mathbf{N} \times \Delta] \otimes F = \{ (S, d, f) \mid S \in \mathcal{N}, d \in D, f \in F(S, d) \}
\end{equation}

denote the set of all $(S, d, f)$ such that $f \in F(S, d)$. For each $i$, let $\mathcal{N}^i = \{ S \in \mathcal{N} \mid i \in S \}$ denote the set of coalitions of which $i$ is a member. Then the $\gamma$-stability is implemented in the next model, where a player $i$ proposes a coalition $S^i (\in \mathcal{N}^i)$, a distribution rule $d^i (\in D)$ and a split function $f^i (\in F(S^i, d^i))$.

**Model 1.** The normal form game

\begin{equation}
(22) \quad \Gamma_1 = \{ \mathbf{N}, \Sigma^i, v^i \}
\end{equation}
is defined as follows:

(a) The set of players is $\mathbf{N}$, for each $i \in \mathbf{N}$, its strategy set is $\Sigma^i = \{ \mathcal{N}^i \times D \} \otimes F$.

For each play $\lambda = (\lambda^1, ..., \lambda^n) \in \prod_{i \in \mathbf{N}} \Sigma^i (\lambda^i = (T^i, d^i, f^i)$, all $i$), a coalition structure $\mathcal{B}(\lambda) = \{ S_1, S_2, ..., S_k \}$, a distribution rule $\text{DR}(\lambda) = \text{DR}(\mathcal{B}(\lambda))$, and a payoff vector $v(\lambda) = \{ v(\lambda)^1, ..., v(\lambda)^n \} = \{ v(\lambda)_S \mid S \in \mathcal{B}(\lambda) \}$ are determined below. For each $i \in \mathbf{N}$, let $S \in \mathcal{B}(\lambda)$ be the coalition formed, of which $i$ is a member, then

(b) $S = T^i$ if $(T^i, d^i) = (T^i, d^i)$ and $f^i(\mathcal{B}(\lambda); S, d^i) = f^i(\mathcal{B}(\lambda); S, d^i)$ for all $j \in T^i$,

= $\{ i \}$ otherwise;
(c) \( \text{DR}(S) = d^i; \)

(d) \( v(\lambda)_S = f^i(\mathcal{B}(\lambda); S, d^i). \)

**Proposition 6.** (a) Let \( \Delta = \{S_1, S_2, ..., S_k\} \) be a fixed coalition structure, \( \text{DR}(\Delta) \) be a fixed distribution rule for \( \Delta \), and let \( (\bar{x}, \overline{\sigma}) \in \text{HSD}(\Delta) \). Let \( \lambda = \{\lambda^1, ..., \lambda^n\} \) in (22) be defined as follows. For each \( S \in \Delta \) and all \( i \in S \), \( \lambda^i = (T^i, d^i, f^i) = (S, \text{DR}(S), f^i_{\sigma}(S, d^i)) \). Then \( \Delta \) with \( \overline{\sigma} \) is \( \gamma \)-stable if and only if \( \lambda \) is a strong equilibrium of \( \Gamma_1 \).

(b) If a choice \( \lambda = \{\lambda^1, ..., \lambda^n\} \) is a strong equilibrium of \( \Gamma_1 \), the coalition structure \( \beta(\lambda) \) with the payoff \( v(\lambda) \) is \( \gamma \)-stable.

Note that after a player \( i \) submits \( \lambda^i = (T^i, d^i, f^i) \), he knows that either \( T^i \) or \( \{i\} \) will be formed. This is why Model 1 only implements the \( \gamma \)-stability----the model does not allow a proper subset (other than \( \{i\} \)) of \( T^i \) be formed. In the following Model 2, the \( \delta \)-stability is implemented by allowing any subset of \( T^i \) be formed. A player needs to propose, in addition to \( (d^i, T^i, f^i) \), a split function for every subset of \( T^i \).

For each \( T \in \mathcal{N} \) and a \( \text{DR} = d \) for \( T \), let \( \Psi(T, d) = \prod_{S \subseteq T} F(S, d) \) and \( \{\mathcal{N} \times D\} \otimes \Psi = \{ (T, d, \psi) \mid T \in \mathcal{N}, d \in D, \psi \in \Psi(T, d) \} \) be set of all \( (T, d, \psi) \) such that \( \psi \in \Psi(T, d) \).

**Model 2.** The normal form game (23) \( \Gamma_2 = \{N, \Sigma^i, \nu^i\} \)

is defined as follows:

(a) The set of players is \( N \), for each \( i \in N \), its strategy set is \( \Sigma^i = \{\mathcal{N}^i \times D\} \otimes \Psi \).

For each play \( \lambda = \{\lambda^1, ..., \lambda^n\} \in \prod_{i \in N} \Sigma^i \) (\( \lambda^i = (T^i, d^i, \psi^i) \), all \( i \)), a coalition structure \( \mathcal{B}(\lambda) = \{S_1, S_2, ..., S_k\} \), a distribution rule \( \text{DR}(\lambda) = \text{DR}(\mathcal{B}(\lambda)) \), and a payoff vector \( v(\lambda) = \{v(\lambda)^1, ..., v(\lambda)^n\} = \{v(\lambda)_S \mid S \in \mathcal{B}(\lambda)\} \) are determined in the following way. For each player \( i \in N \), let \( S \in \mathcal{B}(\lambda) \) be the coalition formed, of which \( i \) is a member. Then

(b) \( S \) is formed if for all \( j \in S \), \( (T^j, d^j) = (T^i, d^i) \) and

29 \( f_{\mathcal{B}}^i(S, d^i) \) is defined as \( f^i(\mathcal{B}; S, d^i) = \overline{\sigma}_S \) if \( \mathcal{B} = \Delta \), = any split of \( \phi(\mathcal{B}, S) \) according to \( d^i \) if \( \mathcal{B} \neq \Delta \).

30 Note that \( F(S, d) \) is the set of all split functions with \( \text{DR} = d \) for \( S \) ( \( f(S, d): \Pi(S) \to \mathcal{R}^S \)). So \( \psi \in \Psi(T, d) \) \( \iff \psi(T, d) = \{v(S, d; T) = f(S, d) \mid S \subseteq T\} \), which is a list of split functions with \( \text{DR} = d \) (\( f(S,d): \Pi(S) \to \mathcal{R}^S \)) for every coalition \( S \subseteq T \).
\[ \psi^i(B(\lambda); S, d^i; T^i) = \psi^i(B(\lambda); S, d^i; T^i). \]

(c) \( \text{DR}(S) = d^i. \)

(d) \( v(\lambda)_{S^i} = \psi^i(B(\lambda); S, d^i; T^i). \)

**Proposition 7.** (a) Let \( \Delta = \{S_1, S_2, \ldots, S_k\} \) be fixed, \( \text{DR}(\Delta) \) be a fixed distribution rule for \( \Delta \), and let \( (\bar{x}, \bar{\sigma}) \in \text{HSD}(\Delta) \). Let \( \lambda = \{\lambda^1, \ldots, \lambda^n\} \) in (23) be defined as follows. For each \( S \in \Delta \) and all \( i \in S \), \( \lambda^i = (T^i, d^i, \psi^i) = (S, \text{DR}(S), \psi^i_{\bar{\sigma}}(S, d^i)). \)

Then \( \Delta \) with \( \bar{\sigma} \) is \( \delta \)-stable if and only if \( \lambda \) is a strong equilibrium of \( \Gamma_2 \).

(b) If a choice \( \lambda = \{\lambda^1, \ldots, \lambda^n\} \) is a strong equilibrium in \( \Gamma_2 \), the coalition structure \( B(\lambda) \) with the payoff \( v(\lambda) \) is \( \delta \)-stable.

**Remark 13.** The above results are similar to that in Hart and Kurz (1983): a stable coalition structure is equivalent to the strong equilibrium in the coalition structure formation game. Players in the Hart and Kurz model propose only a coalition \( T^i \), because a player's payoff is uniquely given once a partition is formed. However, players here need to propose a triplet of \( (T^i, d^i, \psi^i) \) or \( (T^i, d^i, \psi^i) \), because they need to decide what coalition structure to form, what strategies to choose and how to split a coalition's joint payoff.

The next model allows more coalitions to be formed by using Myerson's cooperation structure (1977), which is a set of unconnected bilateral links between players. It is useful here to define it as a collection \( g = \{(i_1, j_1), \ldots, (i_k, j_k)\} \) of two-member coalitions \( (1 \leq k \leq n(n-1)/2) \). Two players \( i \) and \( j \) are connected by \( g \) in a coalition \( S \) if \( i = j \) or if there is some \( m \geq 1 \) and a coalition \( (i_0, i_1, \ldots, i_m) \subseteq S \) such that \( i_0 = i, i_m = j \) and \( (i_{t-1}, i_t) \in g \) for \( t = 1, \ldots, m \). A cooperation structure \( g \) leads to unique partition of \( N \)

\[ N/g = \{ (i \mid i \text{ and } j \text{ are connected by } g \text{ in } N ) \mid j \in N \}. \]

---

\( \psi^i_{\bar{\sigma}}(T^i, d^i) \) is defined as follows. For each \( T \subseteq T^i \), \( \psi^i_{\bar{\sigma}}(T, d^i; T^i) = f_{\bar{\sigma}}(T, d^i), \) that is, \( \psi^i_{\bar{\sigma}}(B, T, d^i; T^i) = \bar{\sigma}_T \) if \( B = \Delta \).
For each $i \in N$ and $DR = d$, let $\Phi(N^i, d) = \prod_{S \in N^i} F(S, d)$, and $D \otimes \Phi^i = \{(d, \phi) \mid d \in D, \phi \in \Phi(N^i, d)\}$ be set of all $(d, \phi)$ such that $\phi \in \Phi(N^i, d)$.\(^{32}\)

**Model 3.** The normal form game

(25) \quad \Gamma_3 = \{N, \Sigma^i, v^i\}

is defined as follows:

(a) The set of players is $N$, for each $i \in N$, its strategy set is $\Sigma^i = N^i \times \{D \otimes \Phi^i\}$.

For each $\lambda = \{\lambda^1, \ldots, \lambda^n\} \in \prod_{i \in N} \Sigma^i$ ($\lambda^i = (T^i, d^i, \phi^i)$, all $i$), a cooperation structure $g(\lambda)$, a coalition structure $B(\lambda) = B(g(\lambda)) = \{S_1, S_2, \ldots, S_k\}$, a distribution rule $DR(\lambda) = DR(B(\lambda))$, and a payoff vector $v(\lambda) = \{v(\lambda)^1, \ldots, v(\lambda)^n\} = \{v(\lambda)_S \mid S \in B(\lambda)\}$ are determined as follows.

(b) \quad g(\lambda) = \{(i, j) \mid (i, j) \in T^i \cap T^j, i \neq j\}.

For each $i \in N$, let $S \in B(\lambda)$ be the coalition formed, of which $i$ is a member. Then

(c) \quad S is formed if for all $j \in S$, $j$ and $i$ are connected by $g(\lambda)$ in $S$, $d^i = d^j$, and $\phi^i(B(\lambda); S, d^i) = \phi^i(B(\lambda); S, d^j)$.

(d) \quad DR(S) = d^i.

(e) \quad v(\lambda)_S = \phi^i(B(\lambda); S, DR(S)).

In order to implement the partition $N/g(\lambda)$ (see (24)) in $\Gamma_3$, the members in each coalition should not only be connected but also propose the same distribution rule and the same split at $B(\lambda)$. The difference between the above three models can be seen in the following example, which removes the DR and Split functions parts. Let $N = \{1, 2, 3\}$, $T^1 = T^2 = (1, 2, 3)$, $T^3 = (2, 3)$. Then the coalition structures formed in $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are respectively $B^1 = \{1, 2, 3\}$, $B^2 = \{(1, 2), 3\}$, and $B^3 = \{(1, 2, 3)\}$.

Remark 14. It remains an open problem to study what stability is associated with Model 3, which uses the cooperation structure.

\(^{32}\) $\phi \in \Phi(N^i, d) \Leftrightarrow \phi = \{(\phi(S, d) \mid S \in N^i\}$, which is a list split functions ($\phi(S, d) : \Pi(S) \rightarrow \mathbb{R}^S$) for every sub-coalition $S$ in $N^i$.  

25
6. PROOFS

**Proof of Propositions 1, 2 and 3.** These proofs are straightforward as they follow directly from the definitions.

A necessary and sufficient condition for the existence of a core in the game (5) is that the game be balanced, this is defined as follows. Let \( B = \{ T_1, T_2, \ldots, T_k \} \) be a collection of coalitions. For each firm \( i \in N \), let \( B(i) = \{ T \in B \mid i \in T \} \) denote the set of those coalitions of which \( i \) is a member. Then \( B \) is a balanced collection if there are non-negative numbers \( w_T \) for each \( T \in B \) such that for each player \( i \), \( \sum_{T \in B(i)} w_T = 1 \). A TU game \( \Gamma = \{ N, v(S) \} \) is balanced if

\[ \sum_{S \in B} w_S v(S) \leq v(N) \]

for any balanced collection of coalitions \( B \), with weight \( w_S \) for each \( S \in B \).

**Proof of Theorem 1:** It follows from the assumptions that the product of all players’ strategy sets is convex and compact, and that the grand coalition’s joint payoff is continuous. Thus there exists an \( \bar{x} \) that maximizes the grand coalition’s joint payoff. The existence of an \( \alpha \)-core payoff is equivalent to the non-emptiness of the core in the TU game \( \Gamma^{CF}_\alpha = \{ N, v_\alpha(S) \} \) defined in (5). Since this game has non-empty core if and only if it is balanced, we only need to show

\[ \sum_{S \in B} w_S v(S) \leq v(N) \]

for any balanced collection of coalitions \( B = \{ T_1, T_2, \ldots, T_k \} \) with weight \( w_T \) for each \( T \in B \).

It follows from (2) and (3) that (note that \( v(S) = v_\alpha(S) \))

\[ v(N) = \max_{x \in X} \sum_{i \in N} u^i(x) = \sum_{i \in N} u^i(\bar{x}) \],

\[ v(S) = \max_{x_S \in X_S} \min_{z_S \in X_S} \sum_{i \in S} u^i(x_S, z_S) = \sum_{i \in S} u^i(x_S^*, y_S(x_S^*)) \]

where \( x_S^* \) is the solution to the above maximization problem and \( y_S(x_S^*) \) is the solution to the minimization problem. Thus

\[ (P1) \quad \sum_{S \in B} w_S v(S) = \sum_{S \in B} w_S \sum_{i \in S} u^i(x_S^*, y_S(x_S^*)) \]
\[
= \sum_{i=1}^{n} \sum_{S \in \mathcal{B}(i)} w_S u^i(x_S^*, \bar{y}_S(x_S^*))
\]
where \( \mathcal{B}(i) = \{ T \in \mathcal{B} \mid i \in T \} \) is the set of those coalitions of which \( i \) is a member. For each player \( i \in N \), let

\[(P2) \quad x^1 = \sum_{S \in \mathcal{B}(i)} w_S x_{S(i)}^* \in X^1, \text{ and} \]

\[x = \{x^1, \ldots, x^n\} \in X = \prod_{i \in N} X^i \]

where \( x_{S(i)}^* \) is the \( i \)-th component of \( x_S^* \). We shall show that

\[(P3) \quad \sum_{S \in \mathcal{B}(i)} w_S u^i(x_S^*, \bar{y}_S(x_S^*)) \leq u^i(x) \]

for each player \( i \). Without loss of generality, we only need to show the above inequality for player 1, as the arguments for other players are essentially the same. In order to show

\[\sum_{S \in \mathcal{B}(1)} w_S u^i(x_S^*, \bar{y}_S(x_S^*)) \leq u^1(x),\]

we shall define, for each \( S \in \mathcal{B}(1) = \{ T \in \mathcal{B} \mid 1 \in T \} \), \( y(S) = (x_S^*, z_S) \in X = \prod_{i \in N} X^i \) as follows.

For the fixed \( S \in \mathcal{B}(1) \) and each \( i \in N \backslash S \), let

\[z^i = \sum_{T \in \mathcal{B}(i) \backslash \mathcal{B}(1)} \hat{w}_T x_T^i \in X^i,\]

and for each \( T \) in the above summation,

\[\hat{w}_T = \frac{w_T}{\sum_{E \in \mathcal{B}(i) \backslash \mathcal{B}(1)} w_E}.
\]

It follows from Hypothesis I, the concavity of \( u^1 \) and the following equality

\[x = \sum_{S \in \mathcal{B}(1)} w_S y(S) \quad (\text{see Scarf (1971)}) \]

that

\[\sum_{S \in \mathcal{B}(1)} w_S u^1(x_S^*, \bar{y}_S(x_S^*)) \leq \sum_{S \in \mathcal{B}(1)} w_S u^1(y(S)) \quad u^1(x).\]

By (P1) and (P3),

\[\sum_{S \in \mathcal{B}(1)} w_S v(S) = \sum_{i=1}^{n} \sum_{S \in \mathcal{B}(i)} w_S u_i(x_S^*, \bar{y}_S(x_S^*)) \leq \sum_{i=1}^{n} u_i(x) \leq \sum_{i=1}^{n} u_i(\bar{x}) = v(N).\]

Thus the game is balanced. 

\[Q.E.D.\]
**Proof of Theorem 2:** For each coalition $S \in \Delta$, let $\delta_S(x_S)$ denote its set of optimal responses given the complementary choice $x_{-S}$, that is, the set of solutions to the optimization problem

$$\max \sum_{i \in S} u_i(x_S, x_{-S})$$

for a fixed $x_{-S}$. By Conditions a, b and c, the correspondence $\delta: X \rightarrow 2^X$, defined by

$$\delta(x) = \prod_{S \in \Delta} \delta_S(x_S) = \{ \delta_S(x_{-S}) \}$$

for each $x = \{ x_1, ..., x_n \} = \{ x_{-S}, ..., x_S \} \in X = \prod_{i \in N} X_i$, satisfies the conditions of Kakutani's fixed point theorem, thus it has a fixed point $\bar{x}$ such that for each coalition $S \in \Delta$, $\bar{x}_S$ maximizes its joint payoff given $\bar{x}_{-S}$. Now each parametric game $\Gamma_S(\bar{x}_{-S})$ (defined in (4)) satisfies the conditions of Theorem 1, thus it has an $\alpha$-core payoff vector $\sigma_S$. Let

$$\sigma = \{ \sigma_1, ..., \sigma_n \} = \{ \sigma_S | S \in \Delta \} \in \mathbb{R}^n,$$

then $(\bar{x}, \sigma)$ is a hybrid solution associated with the coalition structure $\Delta$. \textbf{Q.E.D.}

**Proof of Theorem 3:** Similarly as in the proof of Theorem 2, there exists a joint strategy $\bar{x}$ such that for each $S \in \Delta$, $\bar{x}_S$ maximizes the coalition's joint payoff given $\bar{x}_{-S}$. Now consider each $S \in \Delta$. If $\text{DR}(S) = \alpha$-core, by Theorem 1, there is an $\alpha$-core payoff vector $\sigma_S$ for the parametric normal form game $\Gamma_S(\bar{x}_S)$. Otherwise, for any $\text{DR}(S) \in \{ \text{efficient outcome, the Shapely value, Nucleolus, } \tau \text{-value, and the Equal share outcome} \}$, there is always a payoff vector $\sigma_S$ in the TU game $\Gamma_{\alpha}^{CF} = \{ N, v_{\alpha}(S) \}$ according to $\text{DR}(S)$. Let

$$\sigma = \{ \sigma_1, ..., \sigma_n \} = \{ \sigma_S | S \in \Delta \} \in \mathbb{R}^n,$$

then $(\bar{x}, \sigma)$ is a HSDR associated with the coalition structure $\Delta$. \textbf{Q.E.D.}

**Proof of Proposition 4:** As discussed in the second paragraph following Definition 4, a coalition structure $\Delta$ with the payoff $\bar{\sigma}$ is not $\alpha$-stable (or equivalently, not $\beta$-stable) if there exists $S \in \mathcal{N}$ such that at any new structure $\mathcal{B} \in \Pi(S)$, all its members are better off at all HSDR of $\mathcal{B}$, it is not $\gamma$-stable ($\delta$-stable) if there exists $S \in \mathcal{N}$ such that at the new structure $\tilde{\mathcal{B}}(\Delta, S)$ ($\tilde{\mathcal{B}}(\Delta, S)$), all its members are better off at all HSDR of $\tilde{\mathcal{B}}(\Delta, S)$ ($\tilde{\mathcal{B}}(\Delta, S)$), it is not $\varepsilon$-stable if there exists $S \in \mathcal{N}$ such that at some new structure $\mathcal{B} \in \Pi(S)$, all its members are
better off at some HSDR of $\mathcal{B}$. Thus $\alpha$-instability is included in both the $\gamma$ and $\delta$-instability, which are both included in the $\varepsilon$-instability. Consequently, the $\varepsilon$-stability is included in both the $\gamma$ and $\delta$-stability, which are both included in the $\alpha$-stability. This proves Part (a). Part (b) can be similarly proved.

The above intuitive arguments can be precisely verified in the following alternative definitions for the stability, which are also used in the proof of Proposition 2. Let $\Delta = \{S_1, S_2, \ldots, S_k\} \in \Pi$, $DR(\Delta)$, and $(\vec{x}, \vec{\sigma}) \in HSD(\Delta)$ be fixed in the original game (1), and let the NTU games in characteristic form

$$(P4) \quad \Gamma_{\alpha} = \{N, V_{\alpha}\}, \quad \Gamma_{\gamma} = \{N, V_{\gamma}\}, \quad \Gamma_{\delta} = \{N, V_{\delta}\}, \quad \Gamma_{\varepsilon} = \{N, V_{\varepsilon}\}, \quad \Gamma_{\zeta} = \{N, V_{\zeta}\},$$ respectively be defined as follows. For each $S \neq N$,

$$(P5) \quad V_{\alpha}(S) = \bigcap_{\theta \in \Pi(S)} \left\{ \cap_{(x, \sigma) \in HSD(\theta)} \{ v \in \mathbb{R}^n \mid \sigma_S \geq v_S \} \right\};$$

$$(P6) \quad V_{\gamma}(S) = \bigcap_{(x, \sigma) \in HSD(\mathcal{B}(\Delta, S))} \{ v \in \mathbb{R}^n \mid \sigma_S \geq v_S \};$$

$$(P7) \quad V_{\delta}(S) = \bigcap_{(x, \sigma) \in HSD(\mathcal{B}(\Delta, S))} \{ v \in \mathbb{R}^n \mid \sigma_S \geq v_S \};$$

$$(P8) \quad V_{\varepsilon}(S) = \bigcup_{\theta \in \Pi(S)} \bigcup_{(x, \sigma) \in HSD(\theta)} \{ v \in \mathbb{R}^n \mid \sigma_S \geq v_S \};$$

$$(P9) \quad V_{\zeta}(S) = \bigcap_{\theta \in \Pi(S)} \left\{ \bigcap_{(x, \sigma) \in HSD(\theta)} \{ v \in \mathbb{R}^n \mid \sum_{i \in S} \sigma_i^i \geq \sum_{i \in S} v_i \} \right\};$$

and for $S = N$,

$$(P10) \quad V_{\alpha}(N) = V_{\gamma}(N) = V_{\delta}(N) = V_{\varepsilon}(N) = V_{\zeta}(N) = \{ v \in \mathbb{R}^n \mid v \leq \vec{\sigma} \}.$$
Core of $\Gamma_\zeta \subseteq \text{Core of } \Gamma_\alpha$ which lead to Proposition 1.

**Proof of Proposition 5.** Under the assumptions of Part (a), the coarsest structure $\Delta^0$ with a payoff $\bar{\sigma}$ is $\alpha$-stable ($\gamma$-stable, $\delta$-stable, $\epsilon$-stable, and $\zeta$-stable) if and only if $\bar{\sigma}$ belongs to the core of the following TU games in characteristic form

\[
\Gamma^\alpha = \{N, v^\alpha\}, \Gamma^\gamma = \{N, v^\gamma\}, \Gamma^\delta = \{N, v^\delta\}, \Gamma^\epsilon = \{N, v^\epsilon\}, \Gamma^\zeta = \{N, v^\zeta\},
\]

which are defined as follows. For each $S \neq N$,

\[
\begin{align*}
V^\alpha(S) &= \sum_{i \in S} \sigma^i \left[= \sum_{i \in S} u^i(\bar{x}(B^\alpha(S)))\right], \quad (\bar{x}, \sigma) \in \text{HSD}(B^\alpha(S)); \\
V^\gamma(S) &= \sum_{i \in S} \sigma^i \left[= \sum_{i \in S} u^i(\bar{x}(B^\gamma(\Delta^0,S)))\right], \quad (\bar{x}, \sigma) \in \text{HSD}(B^\gamma(\Delta^0,S)); \\
V^\delta(S) &= \sum_{i \in S} \sigma^i \left[= \sum_{i \in S} u^i(\bar{x}(B^\delta(\Delta^0,S)))\right], \quad (\bar{x}, \sigma) \in \text{HSD}(B^\delta(\Delta^0,S)); \\
V^\epsilon(S) &= \max_{\beta \in \Pi(S)} \sum_{i \in S} \sigma^i(\beta) = \sum_{i \in S} u^i(\bar{x}(\beta)), \quad (\bar{x}(\beta) \sigma(\beta)) \in \text{HSD}(\beta) \quad \text{for each } \beta \in \Pi(S); \quad \text{and}
\end{align*}
\]

\[
V^\alpha(N) = v^\zeta(N) = v^\gamma(N) = v^\delta(N) = v^\epsilon(N) = v(N),\]

$v(N)$ is given by (3). It follows from (7) and the above definitions that

\[
V^\beta(S) \leq V^\alpha(S) = v^\zeta(S) \leq \min \{v^\gamma(S), v^\delta(S)\} \leq \max \{v^\gamma(S), v^\delta(S)\} \leq v^\epsilon(S)
\]

for all $S \neq N$. Thus

\[
\text{Core of } \Gamma^\epsilon \subseteq \{\text{Core of } \Gamma^\gamma \cup \text{Core of } \Gamma^\delta\} \subseteq \text{Core of } \Gamma^\alpha = \text{Core of } \Gamma^\zeta \subseteq \beta\text{-Core}.
\]

This leads to Part (a) of Proposition 1. The proof of Part (b) is obvious.

**Proofs of Propositions 6 and 7.** These two propositions can be proved by simply checking the definitions. The proofs are straightforward and are omitted.

### 7. CONCLUDING REMARKS
The concept of HSDR and the concept of stable coalition structures for normal form TU games have been used in the literature before they are formally defined. Our formal definitions make these tools available to scholars who are interested in situations in which cooperative behavior within a coalition coexists with strategic behavior across the coalitions. Since we studied normal form TU games, the results can be readily applied to the oligopoly market (Corollaries 1, 2, and 3), another benefit is the new way to refine the multiple Nash equilibria (Section 3).

We have provided sufficient conditions for the existence of HSDR (Theorems 1, 2, and 3). The conditions become very mild when coalitions do not choose core as their "best" response. The sufficient conditions for stable coalition structures remain as open problems, they shall be somewhat weaker than that for strong equilibrium and stronger than that for β-core. We also constructed three models of endogenizing stable coalition structures. A stable structure is equivalent to a strong equilibrium in the coalition structure formation game, and players in normal form TU games need to decide not only what coalition structure to form and what strategies to choose but also how to split a coalition's joint payoff.

Finally, this paper opens up a list of future topics in the contexts of normal form TU games and oligopoly markets. Some of these are listed below: (a) The existence of stable coalition structure; (b) The existence of kernel and Bargaining set; (c) The bargaining solutions; (d) What stability are there in Myerson's cooperation structure formation (Model 3)? (e) How to implement a coalition structure when there are multiple hybrid equilibrium strategies? (f) The refinement of hybrid solutions (like Selten, 1975; Myerson, 1978; Kalai & Samet, 1984; Kohlberg & Mertens, 1986; Van Damme, 1989).

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33 One direction is that of Ray and Vohra (1993) and d'Aspremont et al. (1983). By limiting the set of possible deviating coalitions, we might get some general existence results.


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Figure 1. The solutions in a normal form TU game, where the bargaining solutions are not known. $A \rightarrow B$ means $A$ leads to $B$ or $B$ is defined from $A$. $A \rightarrow \leftarrow B$ means $B$ is included in $A$. 
Figure 2. The $\alpha$-core payoffs in 2-person TU and NTU games.

Figure 3. Different payoffs of the deviating coalition $S=\{1, 2\}$, as defined in Definitions 4 and 5.