

## HOMOGENEOUS COUNTABLE CONNECTED HAUSDORFF SPACES

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In 1925, P. Urysohn gave an example of a countable connected Hausdorff space [4]. Other examples have been contributed by R. Bing [1], M. Brown [2], and E. Hewitt [3]. Relatively few of the properties of such spaces have been examined. In this paper the question of homogeneity is studied. Theorem I shows that there exists a bihomogeneous countable connected Hausdorff space. Theorems II and III deal with other questions related to homogeneity.

A space  $Z$  is *homogeneous* if and only if for every pair of elements  $x$  and  $y$  of  $Z$  there exists a homeomorphism  $f$  of  $Z$  onto itself such that  $f(x) = y$ . A space  $Z$  is *bihomogeneous* if and only if for every pair of elements  $x$  and  $y$  of  $Z$  there exists a homeomorphism  $f$  of  $Z$  onto itself such that  $f(x) = y$  and  $f(y) = x$ .

**THEOREM I.** *There exists a bihomogeneous countable connected Hausdorff space.*

**PROOF.** Let  $X_0$  be a countable connected Hausdorff space  $\{x_1^0, x_2^0, \dots\}$  and let  $\{X_1, X_2, \dots\}$  be a countably infinite family of mutually disjoint homeomorphic images of  $X_0$ . For each  $i$ , let  $h_i$  be a homeomorphism of  $X_0$  onto  $X_i$ , and for each  $i$  and  $j$ , let  $x_j^i$  be  $h_i(x_j^0)$ .

Let  $p_1^1$  be  $(x_1^2, x_2^1)$ , and  $p_2^1$  be  $(x_1^1, x_2^2)$ . Let  $p_1^2$  be  $(x_1^2, x_2^1, x_3^3)$ ,  $p_2^2$  be  $(x_1^1, x_2^2, x_3^4)$ ,  $p_3^2$  be  $(x_1^4, x_2^3, x_3^1)$ , and  $p_4^2$  be  $(x_1^3, x_2^4, x_3^2)$ . Suppose that  $n$  is a positive integer such that if  $i \leq 2^n$ , (1)  $p_i^n$  exists and (2) for some positive integers  $k_1, k_2, \dots$ , and  $k_{n+1}$ ,  $p_i^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}})$ . Then let  $p_i^{n+1}$  be  $(x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}}, x_{n+2}^{i+2^n})$  and  $p_{i+2^n}^{n+1}$  be

$$(x_1^{k_1+2^n}, x_2^{k_2+2^n}, \dots, x_{n+1}^{2^n+k_{n+1}}, x_{n+2}^i).$$

For each  $n$  let  $Z_n$  be  $\bigcup_{i=1}^{2^n} ([X_i - \bigcup_{j=1}^{n+1} \{x_j^i\}] \cup \{p_i^n\})$ . A neighborhood system for  $Z_n$  is defined as follows: If  $z$  is a point of some  $X_i$ , then  $\mathfrak{N}(z)$  denotes the neighborhood system of  $z$  in  $X_i$ . If  $z \in \bigcup_{i=1}^{2^n} [X_i - \bigcup_{j=1}^{n+1} \{x_j^i\}]$ , then for some  $i$  and  $j$ ,  $i \leq 2^n$  and  $j > n+1$ ,  $z = x_j^i$ , and in this case the neighborhood system  $\mathfrak{N}^n(z)$  of  $z$  in  $Z_n$  is  $\{U: U \in \mathfrak{N}(z) \text{ and } [\bigcup_{j=1}^{n+1} \{x_j^i\}] \cap U = \emptyset\}$ . Suppose that  $k \leq 2^n$  and  $k_1, k_2, \dots$ , and  $k_{n+1}$  are positive integers such that

Presented to the Society, January 29, 1960; received by the editors April 26, 1960.

$$p_k^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}}).$$

If  $i \leq n+1$ , let  $G_{k_i}$  be  $\{U : U \in \mathcal{H}(x_i^{k_i}) \text{ and if } r \neq i \text{ and } r \leq n+1, \text{ then } x_r^{k_r} \notin U\}$ . Then the neighborhood system  $N^n(p_k^n)$  of  $p_k^n$  in  $Z_n$  is  $\{\bigcup_{i=1}^{n+1} (U_i - \{x_i^{k_i}\})\} \cup \{p_k^n : U_i \in G_{k_i}\}$ .

If  $n$  is a positive integer,  $k \leq 2^n$  and  $p_k^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}})$ , then the points  $x_1^{k_1}, x_2^{k_2}, \dots$ , and  $x_{n+1}^{k_{n+1}}$  are the *coordinates* of  $p_k^n$ .

LEMMA 1. For each  $n$ ,  $Z_n$  is a countable connected Hausdorff space.

PROOF. Let  $n$  be a positive integer. It is clear that  $Z_n$  is a countable Hausdorff space. Suppose that  $Z_n$  is not connected. Then there exist disjoint nonempty open sets  $A$  and  $B$  such that  $A \cup B = Z_n$ . For each  $i, i \leq 2^n$ , let  $K_i$  be  $\{x_j^i : x_j^i \in Z_n\} \cup \{p_k^n : p_k^n \text{ has a coordinate which belongs to } X_i\}$ . For each  $i, K_i$  is connected since it is homeomorphic with  $X_i$ . Assuming that the notation is chosen so that  $K_1 \subset A$ , let  $j$  be  $\min\{i : K_i \subset B\}$ . Now there exist integers  $k$  and  $r, r < j$ , such that  $p_k^n$  has a coordinate belonging to  $X_r$  and a coordinate belonging to  $X_j$ . Then  $p_k^n \in K_r \cap K_j$  and hence  $p_k^n \in A \cap B$ . This contradicts the fact that  $A \cap B = \emptyset$ . This establishes the lemma.

Suppose that  $n \leq r, U_n$  is an open set in  $Z_n$ , and  $U_r$  is an open set in  $Z_r$ . Then the notation

$$U_n \tilde{C} U_r$$

means that (1) if  $p_j^n \in U_n$ , then  $p_j^r \in U_r$  and (2) if  $x_i^r \in U_n$  then either (i)  $x_i^r \in U_r$  or (ii) for some  $k, p_k^n \in U_r$  and  $x_i^r$  is a coordinate of  $p_k^n$ .

Certain permutations are to be defined now. Let  $q_1^1$  be the identity, and  $q_2^1$  be (1, 2) each on  $\{1, 2\}$ . Let  $q_1^2$  be the identity,  $q_2^2$  be (1, 2)(3, 4),  $q_3^2$  be (1, 3)(2, 4), and  $q_4^2$  be (1, 4)(2, 3), each on  $\{1, 2, 3, 4\}$ . Suppose that  $n$  is a positive integer and if  $i \leq 2^n$ , there is a permutation  $q_i^n$  on  $\{1, 2, \dots, 2^n\}$  such that for distinct positive integers  $k_1, k_2, \dots$ , and  $k_2^n, q_i^n = (k_1, k_2)(k_3, k_4) \dots (k_{2^n-1}, k_{2^n})$ . Then if  $i \leq 2^n, q_i^{n+1} = (k_1, k_2)(k_3, k_4) \dots (k_{2^n-1}, k_{2^n})(k_1+2^n, k_2+2^n)(k_3+2^n, k_4+2^n) \dots (k_{2^n-1}+2^n, k_{2^n}+2^n)$  and  $q_{i+2^n}^{n+1} = (k_1, k_2+2^n)(k_3, k_4+2^n) \dots (k_{2^n-1}, k_{2^n}+2^n)(k_1+2^n, k_2)(k_3+2^n, k_4) \dots (k_{2^n-1}+2^n, k_{2^n})$ . Notice that each permutation is the product of disjoint 2-cycles and if  $r \leq 2^n$  and  $s \leq 2^n$  then there is one and only one positive integer  $i$  such that  $q_i^n(r) = s$  and  $q_i^n(s) = r$ .

LEMMA 2. If  $i \leq 2^n, k \leq 2^n$ , and  $p_k^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}})$ , then  $p_{q_i^n(k)}^n = (x_1^{q_i^n(k_1)}, x_2^{q_i^n(k_2)}, \dots, x_{n+1}^{q_i^n(k_{n+1})})$ .

PROOF. The result is clear if  $n = 1$ . Assume the result holds for  $n$  and suppose that  $i \leq 2^n, k \leq 2^n$ , and  $p_k^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}})$ . If

$j \leq n+1$ , then  $q_i^{n+1}(k_j) = q_i^n(k_j)$ ,  $q_i^{n+1}(k_j + 2^n) = q_i^{n+1}(k_j) + 2^n$ ,  $q_{i+2^n}^{n+1}(k_j) = q_i^{n+1}(k_j) + 2^n$ , and  $q_{i+2^n}^{n+1}(k_j + 2^n) = q_i^{n+1}(k_j)$ . Therefore, if  $m = i + 2^n$ , then

- (1) 
$$p_{q_i^{n+1}(k)}^{n+1} = (x_1^{q_i^{n+1}(k_1)}, \dots, x_{n+1}^{q_i^{n+1}(k_{n+1})}, x_{n+2}^{q_i^{n+1}(k+2^n)}),$$
- (2) 
$$p_{q_i^{n+1}(k+2^n)}^{n+1} = (x_1^{q_i^{n+1}(k_1+2^n)}, \dots, x_{n+1}^{q_i^{n+1}(2^n+k_{n+1})}, x_{n+2}^{q_i^{n+1}(k)}),$$
- (3) 
$$p_{q_m^{n+1}(k)}^{n+1} = (x_1^{q_m^{n+1}(k_1)}, \dots, x_{n+1}^{q_m^{n+1}(k_{n+1})}, x_{n+2}^{q_m^{n+1}(k+2^n)}),$$
- (4) 
$$p_{q_m^{n+1}(k+2^n)}^{n+1} = (x_1^{q_m^{n+1}(k_1+2^n)}, \dots, x_{n+1}^{q_m^{n+1}(2^n+k_{n+1})}, x_{n+2}^{q_m^{n+1}(k)}).$$

This establishes Lemma 2.

For each  $n$  and each  $i$ ,  $i \leq 2^n$ , let  $g_i^n$  be the function from  $Z_n$  onto  $Z_n$  defined as follows: If  $k \leq 2^n$  and  $j > n+1$ , then  $g_i^n(x_j^k) = x_j^{q_i^n(k)}$  and  $g_i^n(p_k^n) = p_{q_i^n(k)}^n$ .

LEMMA 3. For each  $n$  and each  $i$ ,  $i \leq 2^n$ ,  $g_i^n$  is a homeomorphism of  $Z_n$  onto  $Z_n$ .

PROOF. Suppose that  $i \leq 2^n$ . It is clear that  $g_i^n$  is one-to-one and onto. Suppose that  $k \leq 2^n$ ,  $j > n+1$  and  $U \in N^n(x_j^k)$ . It is clear that  $g_i^n(U) \in N^n(g_i^n(x_j^k))$ . Suppose that  $k_1, k_2, \dots$ , and  $k_{n+1}$  are positive integers such that  $p_k^n = (x_1^{k_1}, x_2^{k_2}, \dots, x_{n+1}^{k_{n+1}})$ . Let  $V$  be an element of  $N^n(p_k^n)$ . Then for each  $r$ ,  $r \leq n+1$ , there exists an element  $V_r$  of  $G_k$  such that  $V = \bigcup_{r=1}^{n+1} (V_r - \{x_r^{k_r}\}) \cup \{p_k^n\}$ . Now for each  $r$ ,  $r \leq n+1$ ,  $g_i^n(V_r) \in G_{q_i^n(k_r)}$ . By Lemma 2,

$$g_i^n(p_k^n) = (x_1^{q_i^n(k_1)}, \dots, x_{n+1}^{q_i^n(k_{n+1})}),$$

and hence

$$g_i^n(V) = \bigcup_{r=1}^{n+1} (g_i^n(V_r) - \{x_r^{q_i^n(k_r)}\}) \cup \{p_{q_i^n(k)}^n\}$$

which is an element of  $N^n(g_i^n(p_k^n))$ . This shows that  $(g_i^n)^{-1}$  is continuous. Since  $g_i^n = (g_i^n)^{-1}$  it follows that  $g_i^n$  is a homeomorphism.

LEMMA 4. For each  $n$ ,  $r$ , and  $s$  such that  $r \leq 2^n$  and  $s \leq 2^n$ , there exists one and only one positive integer  $i$ ,  $i \leq 2^n$ , such that  $g_i^n(p_s^n) = p_r^n$  and  $g_i^n(p_r^n) = p_s^n$ .

PROOF. Suppose that  $r \leq 2^n$  and  $s \leq 2^n$ . Now there exists one and only one positive integer  $i$ ,  $i \leq 2^n$ , such that  $q_i^n(r) = s$  and  $q_i^n(s) = r$ . Then  $g_i^n(p_r^n) = p_{q_i^n(r)}^n = p_s^n$  and  $g_i^n(p_s^n) = p_{q_i^n(s)}^n = p_r^n$ .

For each  $k$ , let  $p_k$  be the sequence whose  $i$ th term is, for sufficiently large  $n$ , the  $i$ th coordinate of  $p_k^n$ . Let  $Z$  be  $\{p_k: k \text{ is a positive integer}\}$ . A neighborhood system for  $Z$  is defined as follows: If for some  $i$ ,  $U_i$  is an open set in  $Z_i$  let  $U_i^*$  denote the set of all points  $p_j$  of  $Z$  such that either (1)  $p_j^i \in U_i$  or (2)  $p_j$  has a coordinate in  $U_i$ . Now for each  $k$ ,  $N(p_k)$ , the neighborhood system of  $p_k$  in  $Z$  is  $\{(U_{i=n}^\infty U_i^*): k \leq 2^n, U_i \text{ is an open set in } Z_i, p_k^i \in U_i, \text{ and } U_i \tilde{C} U_{i+1}\}$ .

For each  $r$ , let  $g_r$  be the function from  $Z$  onto  $Z$  defined as follows: Suppose that  $k \leq 2^n$ . For some  $s, s \leq 2^n, g_r^n(p_k^n) = p_s^n$ . Set  $g_r(p_k) = p_s$ . Then  $g_r$  is well defined since if  $t \geq n, g_r^t(p_k^t) = p_s^t$ .

It follows from Lemma 4 and the definition of  $g_i$  that for each  $r$  and each  $s$  there exists one and only one positive integer  $i$  such that  $g_i(p_r) = p_s$  and  $g_i(p_s) = p_r$ .

LEMMA 5.  $Z$  is a countable connected Hausdorff space.

PROOF.  $Z$  is clearly a countable Hausdorff space. Suppose that  $Z$  is not connected. Then there exist disjoint nonempty open sets  $A$  and  $B$  such that  $A \cup B = Z$ . There exist positive integers  $k$  and  $j$  such that  $p_k \in A$  and  $p_j \in B$ . Let  $n$  be a positive integer such that  $k \leq 2^n$  and  $j \leq 2^n$ . Let  $A_n$  be  $\{p_i^n: p_i \in A\} \cup \{x_i^n: s \leq 2^n \text{ and there exists an element } p_i \text{ of } A \text{ such that } x_i^n \text{ is a coordinate of } p_i\}$ . Let  $B_n$  be  $\{p_i^n: p_i \in B\} \cup \{x_i^n: s \leq 2^n \text{ and there exists an element } p_i \text{ of } B \text{ such that } x_i^n \text{ is a coordinate of } p_i\}$ . Now  $A_n$  and  $B_n$  are nonempty disjoint open sets in  $Z_n$ , and  $A_n \cup B_n = Z_n$ . This contradicts the fact that  $Z_n$  is connected, and hence  $Z$  is connected.

LEMMA 6. For each  $r, g_r$  is a homeomorphism of  $Z$  onto  $Z$ .

PROOF. Let  $r$  be a positive integer. It is clear that  $g_r$  is one-to-one and onto. Let  $k$  be a positive integer. There exists a  $j$  such that  $g_r(p_k) = p_j$ . Suppose that  $U \in N(p_k)$ . Now there exists an  $n$  and sets  $U_n, U_{n+1}, \dots$  such that if  $s \geq n, U_s$  is open in  $Z_s, p_k^s \in U_s, U_s \tilde{C} U_{s+1}$ , and such that  $U = \bigcup_{s=n}^\infty U_s^*$ . Let  $m$  be a positive integer such that  $j \leq 2m$ . Now since  $U_s \tilde{C} U_{s+1}$ , then  $\bigcup_{s=n}^\infty U_s^* = \bigcup_{s=m}^\infty U_s^*$ , and hence  $U = \bigcup_{s=m}^\infty U_s^*$ . Now for each  $s, s \geq m, g_r^s$  is a homeomorphism and hence  $g_r^s(U_s) \in N^s(p_j^s)$ . Also  $g_r^s(U_s) \tilde{C} g_r^{s+1}(U_{s+1})$ . Now since  $g_r(U_s^*) = [g_r^s(U_s)]^*$  and  $g_r(U) = \bigcup_{s=m}^\infty g_r(U_s^*)$ , then  $g_r(U) \in N(p_j)$ . This shows that  $(g_r)^{-1}$  is continuous. Since  $g_r = (g_r)^{-1}, g_r$  is continuous and it follows that  $g_r$  is a homeomorphism.

The above lemmas establish the theorem.

THEOREM II. There exists a countable connected Hausdorff space  $Z$  such that each point of  $Z$  is a cut point of  $Z$ .

PROOF. Let  $Y$  be a countable connected Hausdorff space and let  $\{X_{ij}: \text{each of } i \text{ and } j \text{ is a positive integer}\}$  be a countably infinite family of mutually disjoint countable connected Hausdorff spaces, each disjoint from  $Y$ . Let  $y^1$  be a one-to-one sequence onto  $Y$ , i.e., let a fixed denumeration  $y_1^1, y_2^1, y_3^1, \dots$  be placed on the elements of  $Y$ , and for each  $(i, j)$  let  $x^{ij}$  be a one-to-one sequence onto  $X_{ij}$ .

If  $p \in Y$ , then  $\mathfrak{N}(p)$  denotes the neighborhood system of  $p$  in  $Y$ , and if  $p$  is a point of some  $X_{ij}$ , then  $\mathfrak{N}(p)$  denotes the neighborhood system of  $p$  in  $X_{ij}$ .

Let  $Z_1$  be  $\bigcup_{i=1}^\infty (X_{1i} - \{x_1^{1i}\}) \cup \{(y_i^1, x_1^{1i}): i \text{ is a positive integer}\}$ . A neighborhood system for  $Z_1$  is defined as follows: If  $z \in \bigcup_{i=1}^\infty (X_{1i} - \{x_1^{1i}\})$ , then for some  $i$  and some  $k, k \neq 1, z = x_k^{1i}$ , and in this case  $N^1(x_k^{1i})$ , the neighborhood system of  $x_k^{1i}$  in  $Z_1$ , is  $\{U: U \in \mathfrak{N}(x_k^{1i}) \text{ and } x_1^{1i} \notin U\}$ . In order to define neighborhoods in  $Z_1$  of the remaining points of  $Z_1$ , certain functions will be introduced. Suppose that  $i$  is a positive integer and that  $V \in \mathfrak{N}(y_i^1)$ . Let  $V^*$  be  $\{(y_i^1, x_1^{1i}): y_i^1 \in V\}$ .  $V^*$  is a one-to-one function. Let  $W(V)$  be the set of all functions  $w$  on  $(\text{range } V^*)$  such that if  $x_1^{1i} \in (\text{range } V^*)$ , then  $w(x_1^{1i}) \in \mathfrak{N}(x_1^{1i})$ . If  $w \in W(V)$  let  $U(V, w)$  be

$$V^* \cup [U \{ [w(x_1^{1i}) - \{x_1^{1i}\}]: x_1^{1i} \in \text{dom } w \}].$$

Now if for some  $i, z = (y_i^1, x_1^{1i})$ , then  $N^1(z)$ , the neighborhood system in  $Z_1$  of  $z$ , is  $\{U(V, w): V \in \mathfrak{N}(y_i^1) \text{ and } w \in W(V)\}$ .  $Z_1$  with the resulting topology is a countable connected Hausdorff space.

LEMMA 1. For each  $i, (y_i^1, x_1^{1i})$  is a cut point of  $Z_1$ .

PROOF. Let  $i$  be a positive integer. Let  $A$  be  $\{x_j^{1i}: j > 1\}$ . Then  $A$  and its complement relative to  $Z_1 - \{(y_i^1, x_1^{1i})\}$  are disjoint nonempty open sets whose union is  $Z_1 - \{(y_i^1, x_1^{1i})\}$ . This establishes Lemma 1.

Now for each  $n, n \neq 1$ , let  $y^n$  be a one-to-one sequence onto  $\bigcup_{i=1}^\infty (X_{(n-1)i} - \{x_1^{(n-1)i}\})$ , and let  $Z_n$  be  $\bigcup_{i=1}^\infty (X_{ni} - \{x_1^{ni}\}) \cup \{(y_j^n, x_1^{nj}): \text{each of } j \text{ and } k \text{ is a positive integer and } j \leq n\}$ . A neighborhood system for  $Z_n$  is defined as follows: If  $z \in \bigcup_{i=1}^\infty (X_{ni} - \{x_1^{ni}\})$ , then for some  $i$  and some  $k, k \neq 1, z = x_k^{ni}$ , and in this case  $N^n(x_k^{ni})$ , the neighborhood system in  $Z_n$  for  $x_k^{ni}$ , is  $\{U: U \in \mathfrak{N}(x_k^{ni}) \text{ and } x_1^{ni} \notin U\}$ . In order to define neighborhoods in  $Z_n$  of the remaining points of  $Z_n$ , certain functions will be introduced. Suppose that  $i$  is a positive integer and that  $V \in N^{n-1}(y_i^n)$ . Let  $V^*$  be  $\{(y_j^n, x_1^{nj}): y_j^n \in V\}$ .  $V^*$  is a one-to-one function. Let  $W(V)$  be the set of all functions  $w$  on  $(\text{range } V^*)$  such that if  $x_1^{ni} \in (\text{range } V^*)$ , then  $w(x_1^{ni}) \in \mathfrak{N}(x_1^{ni})$ . If  $w \in W(V)$ , let  $U(V, w)$  be  $V^* \cup [U \{ [w(x_1^{ni}) - \{x_1^{ni}\}]: x_1^{ni} \in \text{dom } w \}].$  Now if for some  $i, z$

$= (y_i^n, x_1^{n1})$ , then  $N^n(z)$ , the neighborhood system in  $Z_n$  of  $z$  is  $\{U(V, w) : V \in N^{n-1}(y_i^n)$  and  $w \in W(V)\}$ . Suppose that  $i$  and  $j$  are positive integers,  $j \leq n-1$ , and that  $V \in N^{n-1}(y_i^j, x_1^{j1})$ . Let  $V^*$  be  $\{(y_s^j, x_1^{j1}) : (y_s^j, x_1^{j1}) \in V$  or  $y_s^j \in V\}$ . Let  $W(V)$  be the set of all functions  $w$  on  $\{x_1^{ns} : (y_s^j, x_1^{j1}) \in V^*\}$  such that if  $x_1^{ns} \in \{x_1^{ns} : (y_s^j, x_1^{j1}) \in V^*\}$ , then  $w(x_1^{ns}) \in \mathfrak{N}(x_1^{j1})$ . If  $w \in W(V)$ , let  $U(V, w)$  be

$$V^* \cup [ \cup \{ [w(x_1^{ns}) - \{x_1^{ns}\}] : x_1^{ns} \in \text{dom } w \} ] .$$

Now if for some  $i$  and some  $j, j \leq n-1, z = (y_i^j, x_1^{j1})$ , then  $N^n(z)$ , the neighborhood system in  $Z_n$  of  $z$ , is  $\{U(V, w) : V \in N^{n-1}(y_i^j, x_1^{j1})$  and  $w \in W(V)\}$ . Now for each  $n, Z_n$ , with the resulting topology, is a countable connected Hausdorff space.

LEMMA 2. For each  $i, j$ , and  $n, i \leq n, (y_i^j, x_1^{j1})$  is a cut point of  $Z_n$ .

PROOF. Lemma 1 shows that the conclusion is true if  $n = 1$ . Let  $n$  be a positive integer greater than 1, and suppose that if  $k$  is a positive integer less than  $n$  and  $i$  and  $j$  are integers,  $i \leq k$ , then  $(y_i^j, x_1^{j1})$  is a cut point of  $Z_{n-1}$ . Let  $i$  and  $j$  be positive integers,  $i \leq n$ .

Now if  $i < n$ , there exist disjoint nonempty open subsets  $A$  and  $B$  of  $Z_{n-1} - \{(y_i^j, x_1^{j1})\}$  such that  $A \cup B = Z_{n-1} - \{(y_i^j, x_1^{j1})\}$ . Let  $J_0$  be a subset of the positive integers such that  $k \in J_0$  if and only if  $y_i^k \in A$ . Let  $C$  be  $\{(y_s^j, x_1^{j1}) : (y_s^j, x_1^{j1}) \in A$  or  $y_s^j \in A\} \cup [ \cup \{ (X_{nk} - \{x_1^{nk}\}) : k \in J_0 \} ]$ , and  $D$  be  $\{(y_s^j, x_1^{j1}) : (y_s^j, x_1^{j1}) \in B$  or  $y_s^j \in B\} \cup [ \cup \{ (X_{nk} - \{x_1^{nk}\}) : k \notin J_0 \} ]$ . Now  $C$  and  $D$  are disjoint nonempty open subsets of  $Z_n - \{(y_i^j, x_1^{j1})\}$  such that  $C \cup D = Z_n - \{(y_i^j, x_1^{j1})\}$ .

If  $i = n$ , the proof that  $(y_i^j, x_1^{j1})$  is a cut point of  $Z_n$  is similar to the proof that  $(y_i^j, x_1^{j1})$  is a cut point of  $Z_1$ , and hence is omitted.

Let  $Z$  be  $\{(y_i^j, x_1^{j1}) : \text{each of } i \text{ and } j \text{ is a positive integer}\}$ . A neighborhood system for  $Z$  is defined as follows: Suppose that  $z \in Z$ . Then for some  $i$  and some  $j, z = (y_i^j, x_1^{j1})$ . Let  $S(z)$  be the set of all functions  $V$  on  $\{i, i+1, i+2, \dots\}$  such that (1)  $V_i \in N^i(z)$  and (2) if  $k > i$ , then  $V_k \in N^k(z)$  and for some element  $w$  of  $W(V_{k-1}), V_k = U(V_{k-1}, w)$ . Now if for some  $i$  and some  $j, z = (y_i^j, x_1^{j1})$ , then  $N(z)$ , the neighborhood system in  $Z$  of  $z$  is  $\{ [ \cup_{k=i}^\infty (V_k \cap Z) ] : V \in S(z) \}$ .  $Z$  with the resulting topology is a countable connected Hausdorff space.

LEMMA 3. Each point of  $Z$  is a cut point of  $Z$ .

PROOF. Suppose that  $z \in Z$ . Then there exist integers  $i$  and  $j$  such that  $z = (y_i^j, x_1^{j1})$ . Now there exist sequences  $A_i, A_{i+1}, A_{i+2}, \dots$  and  $B_i, B_{i+1}, B_{i+2}, \dots$  such that for every positive integer  $k$  greater than  $i-1, (1) A_k$  and  $B_k$  are nonempty disjoint open subsets of  $Z_k$

$-\{(y_j^i, x_1^i)\}$  such that  $A_k \cup B_k = Z_k - \{(y_j^i, x_1^i)\}$ , (2) if  $v \in A_k$  and  $V$  is an element of  $N^k(v)$  such that  $V \subset A_k$ , then for every element  $w$  of  $W(V)$ ,  $U(V, w) \subset A_{k+1}$ , and (3) if  $v \in B_k$  and  $V$  is an element of  $N^k(v)$  such that  $V \subset B_k$ , then for every element  $w$  of  $W(V)$ ,  $U(V, w) \subset B_{k+1}$ . Let  $A$  be  $\bigcup_{k=t}^{\infty} (A_k \cap Z)$  and  $B$  be  $\bigcup_{k=t}^{\infty} (B_k \cap Z)$ . Then  $A$  and  $B$  are nonempty disjoint open subsets of  $Z - \{(y_j^i, x_1^i)\}$  such that  $A \cup B = Z - \{(y_j^i, x_1^i)\}$ .

The above lemmas establish the theorem.

**THEOREM III.** *There exists a countable connected Hausdorff space  $Z$  such that the only homeomorphism of  $Z$  onto itself is the identity.*

**OUTLINE OF PROOF.** If  $Z$  is a connected space and  $z \in Z$ , then the statement that  $z$  has component number  $n$  means that  $n$  is a positive integer and that  $Z - \{z\}$  has exactly  $n$  components. Using a construction similar to that of Theorem II, a countable connected Hausdorff space  $Z$  can be constructed such that (1) every point of  $Z$  is a cut point of  $Z$  and (2) distinct points of  $Z$  have distinct component numbers. It follows that the only homeomorphism of  $Z$  onto itself is the identity.

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