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ON A PROBLEM IN ALGEBRAIC MODEL THEORY

In Andréka-Németi [1] the class $STr(C)$ of all small trees over C is defined for an arbitrary category C. Throughout the present paper C denotes an arbitrary category. In Def. 4 of [1] on p. 367 the injectivity relation $\models \subseteq (Ob\ C) \times (STr(C))$ is defined. Intuitively the members of $STr(C)$ represent the formulas and $\models$ represents the validity relation between objects of C considered as models and small trees of C considered as formulas. If $\varphi \in STr(C)$ and $a \in Ob\ C$ then the statement $a \models \varphi$ is associated to the model theoretic statement "the formula φ is valid in the model a". It is proved there that the Łoś lemma is true in every category C if we use the above quoted concepts. To this the notion of an ultraproduct $P_{i \in I} a_i/U$ of objects $< a_i : i \in I > \in {}^I Ob\ C$ of C if we use the above quoted concepts. To this the notion of an ultraproduct $P_{i \in I} a_i/U$ of objects $< a_i : i \in I > \in {}^I Ob\ C$ of C is defined in [1], in [2] and in [7] Def. 12. Then the problem was asked there ("Open problem 1" on p. 375) "for which categories is the characterization theorem of axiomatizable hulls of classes of models $Mod\ Th\ K = Uf\ Up\ K$ true?", where the operators Uf and Up on classes of models is defined on p. 319 of the book [3]. Of course, here in the definition of Uf and Up on classes $K \subseteq Ob\ C$ of objects of C we have to replace the standard notion of ultraproducts of models by the above quoted category theoretic ultraproduct $P_{i \in I} a_i/U$ of objects of C. That is for any C and $K \subseteq Ob\ C$ we define $Up\ K$ exactly as in [2] p. 136, but a more detailed definition is found in [7] Def. 17.

Definition 1. Let C be an arbitrary category and let $K \subseteq Ob\ C$ and $T \subseteq STr(C)$ be arbitrary classes. Let $a \in Ob\ C$ and $\varphi \in Str(C)$. Then we define

(i) $K \models T$ iff $(\forall b \in K)(\forall \psi \in T)\ b \models \psi$.

(ii) $K \models \varphi$ iff $K \models \{\varphi\}$, and $a \models T$ iff $\{a\} \models T$.
(iii) $Mod\ T =^d \{b \in Ob\ C\ :\ b \models T\}$.
(iv) $Th\ K =^d \{\psi \in STr(C)\ :\ K \models \psi\}$.
(v) $a\ \equiv_{ee} b$ iff $Th\{a\} = Th\{b\}$.
(vi) $Ee\ K =^d \{b \in Ob\ C\ :\ (\exists a \in K) b \equiv_{ee} a\}$.

□

In the present paper we characterize those categories in which $Mod\,Th\ K = Ee\ Up\ K$ holds for all $K \subseteq Ob\ C$.

Note that the above introduced notations $Mod\ T$ and $Th\ K$ are sloppy since the precise notation would be $Mod_C T$ and $Th_C K$ since e.g. $Mod_C T$ is a function of both C *and* T. We hope that context will help.

Strongly small objects of C were defined in [1], [7] Def. 13, and [2]. We shall use this notion without recalling it, but we note, that in the textbook [4] in item 22E there on p. 155 strongly small objects were defined under the name *strongly finitary objects*.

Definition 2. Let C be an arbitrary category. We say that C has only *set many nonisomorphic strongly small objects* iff there is a *set* $B \subseteq Ob\ C$ such that every strongly small object of C is isomorphic to some element of B.

□

Definition 3. We say that *ultraproducts exist* in C iff for every *set* I and for all $a \in {}^I(Ob\ C)$ and for every ultrafilter U on I the ultraproduct $P_{i \in I} a_i / U$ exists in C.

Theorem 1. *Let C be an arbitrary category. Assume that conditions (i)-(iii) below hold. Let $K \subseteq Ob\ C$ be an arbitrary class. Then*
$Mod\ Th\ K = Ee\ Up\ K$. (That is "$Mod\ Th$" = "$Ee\ Up$" on C.)

(i) C has only set many nonisomorphic strongly small objects.
(ii) Ultraproducts exist in C (the small ones only, see Def. 3).
(iii) C has an initial object.

The proof is that of Thm. 6.13 in Hien [5], p. 145.

□

Theorems 2-3 below state that both conditions (i) and (ii) are needed in Thm. 1 above. The proof of Thm. 2 below is that of Statement 6.12.1

on p. 142 of Hien [5].

THEOREM 2. (Necessity of (i) in Thm. 1) *There exists a category C in which all ultraproducts exist and C has an initial object but*

$Mod\ Th\ K \neq Ee\ Up\ K$ *for some* $K \subseteq ObC$. *That is C satisfies (ii) and (iii) of Thm. 1 but not its conclusion.*

□

THEOREM 3. (Necessity of (ii) in Thm. 1) *There exists a category C satisfying (i) and (iii) of Thm. 1 in which the conclusion of Thm. 1 fails.*

PROOF: Let $Injs$ be the category of all sets with injective mappings between them. That is $Ob(Injs) =$ "all sets" and for two sets X, Y the function $f : X \longrightarrow Y$ is in $Mor(Injs)$ iff $(\forall x, y \in X)[f(x) = f(y) \Rightarrow x = y]$. Let $K =^d \{X \in Ob(Injs) : |X| < \omega\}$. We claim that $Mod\ Th\ K \neq K$ but $Ee\ Up\ K = K$. Directed colimits in $Injs$ coincide with those in $Sets$. In particular directed unions are directed colimits in $Injs$. Therefore strongly small objects are finite sets in $Injs$. Using this fact, it is not hard to prove that there is an infinite $X \in Mod\ Th\ K$. But $Up\ K = K$ since $P_{i\in I}A_i$ does not exist in $Injs$ if $|I| > 1$ and $(\exists i \in I)|A_i| > 1$ and $0 \notin \{A_i : i \in I\}$, because of the following. There are two cones k and h from 1 into $< A_i : i \in I >$ and $i, j \in I$ such that $k_i(0) \neq h_i(0)$ but $k_j(0) = h_j(0)$. Let k^+ and h^+ be the maps induced into the product. Then $k^+(0) \neq h^+(0)$ hence the j-th projection is not one-one since $pj_j(k^+(0)) = k_j(0) = h_j(0) = pj_j(h^+(0))$. Hence ultraproducts cannot exist either. Further $Ee\ K = K$ is obvious since if $|X| \geq \omega > |Y|$ then $X \not\equiv_{ee} |Y|$ because the tree $< Y \cup \{Y\}, 0 >\in STr(Injs)$ is injective in Y but not in X (actually it is not injective in any set Z if $|Z| > |Y|$). We have proved $Ee\ Up\ K = K \neq Mod\ Th\ K$.

□

FACT: Let C be an arbitrary category and let $K \subseteq Ob\,C$. Then $Mod\,Th\,K \supseteq Ee\ Up\ K \supseteq Uf\ Up\ K$.

PROOF: Immediate by [1].

PROPOSITION: *The conditions of Thm. 1 are not the best possible. Namely: There exists a category C such that all three conditions ((i), (ii), (iii)) of Thm. 1 fail for C but the conclusion of Thm. 1 is true for C.*

Proof: Let C be a large discrete category. That is $Ob\ C$ is a proper class (i.e. $Ob\ C$ is not a set) and $(\forall a, b \in Ob\ C0\ [a \neq b \Rightarrow Hom(a, b) = 0]$ and $(\forall a \in Ob\ C)\ |Hom(a, a)| = 1$. Then every object of C is strongly small. Thus there is a proper class of nonisomorphic strongly small objects in C. Further ultraproducts do not exist in C since there are no morphisms there. Let $K \subseteq Ob\ C$. We claim that $Mod\ Th\ K = K$. Let $a \in Ob\ C$. Assume $a \notin K$. Then $< a, 0 > \in STr(C)$ namely $< a, 0 >$ is the one element tree with root a and no branches. Clearly $a \not\models < a, 0 >$ and $(\forall b \in Ob\ C)[b \neq a \Rightarrow b \models < a, 0 >]$. Thus $K \models < a, 0 >$ proving that $a \notin Mod\ Th\ K$.

□

Problems:

(i) Improve Thm. 1. Find a sharper characterization of those categories in which $Mod\ Th = Ee\ Up$.

(ii) Under what conditions is $Mod\ Th = Uf\ Up$ true?

(iii) Is there a category C satisfying (i) and (ii) of Thm. 1 in which $Mod\ Th\ K \neq Ee\ Up\ K$, for some $K \subseteq Ob\ C$? That is, can condition (iii) be dropped from Thm. 1?

□

For the category Lf_α of locally finite cylindric algebras see [6] or in the textbook on representable cylindric algebras [3], p. 321. The following is a corollary of results in [6] and Thm. 1 above. For a motivation we note that Lf_α is exactly the class of algebras obtainable from classical first order theories as it was proved in Pro. 1 of [6].

Corollary: *Let α be any ordinal. Let $K \subseteq Lf_\alpha$ be arbitrary. Then in the category Lf_α we have*

$$Mod\ ThK = Ee\ Up\ K.$$

□

Problem: Is $Mod\ Th\ K = Uf\ Up\ K$ true in Lf_α?

We note that in the above problem and corollary ultraproducts (Up and Uf) are understood in the category theoretic sense (as they are throughout the present paper).

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