ON HIGHER LEVEL GREEN CORRESPONDENCE

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Abstract. This paper generalizes some aspects of J. A. Green's work on the cuspidal representations of general linear groups over finite fields to general linear groups over finite quotients of discrete valuation rings.

1. Introduction

The irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ were computed by J. A. Green in 1955 [Gre55]. In Green's work, parabolic induction was used to construct many irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ from irreducible characters of smaller general linear groups over $\mathbf{F}_q$. The representations which could not be obtained in this way, known as *cuspidal representations*, were shown to be in canonical bijective correspondence with Galois orbits of norm-primitive characters of $\mathbf{F}_{q^n}^\times$ (these are characters which do not factor through the norm map $\mathbf{F}_{q^n}^\times \to \mathbf{F}_{q^d}^\times$ for any $d|n$).

Let F be a non-Archimedean local field with ring of integers $\mathfrak{o}$. Let $\mathfrak{p}$ be the maximal ideal in $\mathfrak{o}$, and $\mathfrak{o}_k = \mathfrak{o}/\mathfrak{p}^k$ for $k \geq 1$. Thus $\mathfrak{o}_1$ is a finite field, the residue field of F, which we take to be $\mathbf{F}_q$. In contrast with $\mathrm{GL}_n(\mathfrak{o}_1)$, not much is known in general about the representation theory of $\mathrm{GL}_n(\mathfrak{o}_k)$. Unlike general linear groups over fields, for which conjugacy classes are parameterized by Jordan canonical forms, the classification of conjugacy classes in $\mathrm{GL}_n(\mathfrak{o}_k)$ for all n and k is known to be a *wild* problem in the sense that it contains the matrix pair problem.

The representations of $\mathrm{GL}_n(\mathfrak{o})$ received considerable attention after supercuspidal representations of $\mathrm{GL}_n(F)$ were constructed by induction from a compact subgroup modulo its center [Shi68, How77, Kut78]. A class of representations (*représentations très cuspidales*) of the maximal compact subgroups modulo center which give rise to irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ were identified by Carayol [Car84]. When the maximal compact subgroup modulo center in question is $F^\times \mathrm{GL}_n(\mathfrak{o})$, the restrictions of these representations to $\mathrm{GL}_n(\mathfrak{o})$ correspond to what we call *strongly cuspidal representations* of $\mathrm{GL}_n(\mathfrak{o}_k)$ for some k (Definition 4.1). Carayol used these representations to construct all the supercuspidal representations of $\mathrm{GL}_n(F)$ when n is prime. A remarkable body of work towards the classification of supercuspidal representations of $\mathrm{GL}_n(F)$ continues up to the present time, to which we have not done justice in this introduction. With respect to $\mathrm{GL}_n(\mathfrak{o})$, this body of work considers only those very special representations that are needed to understand the representations of the p-adic group itself, since the general representation theory is unmanageably complicated.

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In this article, we take the point of view that the representation theory of $\mathrm{GL}_n(\mathfrak{o})$ is interesting in its own right, and while extremely complicated, does display a certain structure. To this end, a new definition of cuspidality is introduced for representations of $\mathrm{GL}_n(\mathfrak{o}_k)$. This definition is closer in spirit to the characterization in [Gre55] of cuspidal representations as those which do not occur in representations induced by parabolic induction. More specifically, for any $\mathfrak{o}$-module $\mathfrak{o}_\lambda = \oplus_{i=1}^m \mathfrak{o}_{\lambda_i}$ of type $\lambda = (\lambda_1, \dots, \lambda_m)$ let $G_\lambda = G_{\lambda,F}$ stand for its automorphism group. Thus, for example, $G_{k^n} = \mathrm{GL}_n(\mathfrak{o}_k)$. Say that $\lambda \leq \mu$ if $\mathfrak{o}_\lambda$ can be embedded in $\mathfrak{o}_\mu$. We call an irreducible representation of G_{k^n} *cuspidal* (see Definition 4.2) if it cannot be constructed from lower building blocks. By lower building blocks we mean the representations of G_λ, where $\lambda \leq k^n$. These automorphism groups play the role of Levi components of proper parabolic subgroups of $\mathrm{GL}_n(\mathfrak{o}_1)$. Representations of G_{k^n} are constructed from those of G_λ using *infinitesimal* and *geometric* induction (Section 3). Our first result, which is proved in Section 4.2, compares cuspidality with strong cuspidality.

Theorem A. *Every strongly cuspidal representation is cuspidal. When n is prime every cuspidal representation is strongly cuspidal.*

The construction of strongly cuspidal representations is well-known [Shi68, Gér75, How77]. When n is prime, then by Theorem A, all cuspidal representations are obtained in this manner. Moreover, for all n, these representations have properties analogous to cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_1)$. Firstly, they can be parameterized in an analogous fashion. Suppose that E is an unramified extension of F of degree n, and $\mathfrak{O}$ is the integral closure of $\mathfrak{o}$ in E. Let $\mathfrak{P}$ denote the maximal ideal in $\mathfrak{O}$ and $\mathfrak{O}_k$ denote the finite quotient ring $\mathfrak{O}/\mathfrak{P}^k$. For $k > 1$, a character $\mathfrak{O}_k^\times \to \mathbf{C}^\times$ is said to be *strongly primitive* if its restriction to $\ker \mathfrak{O}_k^\times \to \mathfrak{O}_1 \cong \mathfrak{O}_1$ does not factor through any proper subfield via the trace map. Say that a character of $\mathfrak{O}_1^\times$ is strongly primitive if it is norm-primitive. In Section 5.3 we prove

Theorem B. *There is a canonical bijective correspondence between strongly cuspidal representations of G_{k^n} and* $\mathrm{Gal}(E/F)$*-orbits of strongly primitive characters of $\mathfrak{O}_k^\times$.*

The units group $\mathfrak{O}_k^\times$ is embedded in G_{k^n} since $\mathfrak{O}_k \simeq \mathfrak{o}_k^n$ as $\mathfrak{o}_k$-modules. An element of G_{k^n} is said to be cuspidal if it is conjugate to an element of $\mathfrak{O}_k^\times$ whose image in $\mathfrak{O}_1^\times$ lies in no proper subfield. In section 5.2, we establish another property that strongly cuspidal representations share with cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_1)$, which is that the correspondence of Theorem B is well-behaved with respect to character values on strongly cuspidal elements.

Theorem C. *Let ω be a strongly primitive character of $\mathfrak{O}_k^\times$ and let Θ_ω be the corresponding strongly cuspidal character of G_{k^n}. Then for all cuspidal elements $u \in \mathfrak{O}_k^\times \subset G_{k^n}$*

$$\Theta_\omega(u) = (-1)^{(n-1)k} \sum_{\gamma \in \mathrm{Gal}(E/F)} \omega({}^\gamma u),$$

Moreover, Θ_ω vanishes on conjugacy classes which do not intersect $\mathfrak{O}_k^\times \cdot \mathrm{Ker}\{G_{k^n} \to G_{\lceil k/2 \rceil^n}\}$.

Remark. Theorems B and C are due to Green when $k = 1$. For $k > 1$, although the main ideas of the proofs can be found in the existing literautre on supercuspidal representations, the detailed account in Section 5 gives the complete picture. In particular, Theorem C is deduced from [Gér75, Theorem 1]. It is closely related to the result obtained by Henniart in [Hen93, Section 3.7].

There already is evidence that the representation theory of a group such as G_λ can be studied by breaking up the problem into two parts. The first is to correctly define and understand the cuspidal representations. The second is to construct the remaining representations from cuspidal representations of G_μ with $\mu < \lambda$. This approach has been implemented successfully in [Onn07] for automorphism groups of modules of rank two. Theorems A, B and C provide further evidence of the validity of this approach when $\lambda = k^n$ and n is a prime.

The inevitability of the family of groups G_λ in the representation theory of G_{k^n} or even G_{2^n} can be seen from another perspective. In Section 6, we prove

Theorem D. *Let F be a local function field. Computing the irreducible characters of the family of groups $\{G_{2^n,F} = \mathrm{GL}_n(\mathfrak{o}_2) \mid n \in \mathbf{N}\}$ is equivalent to computing the irreducible characters of the family $\{G_{\lambda,E} \mid \lambda \in \Lambda,\ E/F \text{ unramified extension}\}$.*

Finally, we point out a suggestive connection to the Macdonald correspondence which might admit a higher level incarnation as well. Macdonald has established a correspondence between irreducible representations of G_{1^n} and equivalence classes of n-dimensional tamely ramified representations of the Weil-Deligne group W_F' [Mac80]. One ingredient in this correspondence is a functional equation for the zeta function associated to G_{1^n}. It admits a straightforward generalization to G_{k^n} for $k > 1$. Let $\hat{f}$ denote a properly normalized additive Fourier transform of $f \in \mathbf{C}\left(\mathrm{M}_n(\mathfrak{o}_k)\right)$ with respect to $\psi\left(\mathrm{tr}(\cdot)\right)$, where $\psi : \mathfrak{o}_k \to \mathbf{C}$ is an additive character which does not factor through $\mathfrak{o}_{k-1}$. Let $\mathcal{Z}(f,\rho) = \sum_{g \in G_{k^n}} f(g)\rho(g) \in \mathrm{End}_{\mathbf{C}}(V)$ where $f \in \mathbf{C}\left(\mathrm{M}_n(\mathfrak{o}_k)\right)$ and (ρ, V) is an irreducible representation of G_{k^n}. Denote by $\check{\rho}$ the contragredient representation of ρ. In Section 7, we prove

Theorem E. *If ρ is not infinitesimally induced (in particular if ρ is cuspidal) there exist $\varepsilon(\rho,\psi)$ such that*

$$^{t}\mathcal{Z}(\hat{f},\check{\rho}) = \varepsilon(\rho,\psi)\mathcal{Z}(f,\rho).$$

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2. Notations and preliminaries

2.1. **Automorphism groups.** Let Λ denote the set of partitions. Any $\lambda \in \Lambda$ can be written in the form $(\lambda_1^{r_1}, \dots, \lambda_l^{r_l})$, where $\lambda_1 > \cdots > \lambda_l$ and $r_1, \dots, r_l$ are positive integers. The sum $r_1 + \cdots + r_l$ will be called the *length* of the partition, and λ_1 will be called the *height* of the partition.

Every finitely generated torsion $\mathfrak{o}$-module is of the form $\mathfrak{o}_\lambda = \mathfrak{o}_{\lambda_1}^{r_1} \oplus \cdots \oplus \mathfrak{o}_{\lambda_l}^{r_l}$ for some $\lambda \in \Lambda$. Consider the group $G_\lambda = \mathrm{Aut}_{\mathfrak{o}}(\mathfrak{o}_\lambda)$. In particular, taking $\lambda = (k^n)$, we have $G_{k^n} = \mathrm{GL}_n(\mathfrak{o}_k)$. When it is necessary to specify the underlying non-Archimedean local field F, the notation $G_{\lambda,F}$ will be used for G_λ.

When F is a function field, the groups $G_{\lambda,F}$ are the groups of units of finite dimension algebras over $\mathfrak{o}_1$ which seem to have a rich and interesting module theory, largely not understood at the present time (see [DPS06]).

Let N_r denote the kernel of the natural map $G_{k^n} \to G_{r^n}$. Then, if $r \geq k/2$, the map $\mathrm{M}_n(\mathfrak{o}_{k-r}) \to N_r$ defined by $A \mapsto \mathrm{I} + \varpi^r A$, is an isomorphism of groups (it is a bijection of sets for all $r < k$). The result is a short exact sequence

$$0 \to \mathrm{M}_n(\mathfrak{o}_{k-r}) \to G_{k^n} \to G_{r^n} \to 1, \tag{2.1}$$

for every $r \geq k/2$.

Consider for a moment the special case where $r = 1$. According to [Ser68, II.4, Proposition 8], the map $\mathfrak{o}_k \to \mathfrak{o}_1$ has a unique multiplicative section (which is also additive when F is a local function field) $s\colon \mathfrak{o}_1 \to \mathfrak{o}_k$. This can be used to define a section $s\colon G_{1^n} \to G_{k^n}$ by applying it to each entry of a matrix in G_{1^n}. The section s is a homomorphism when F is a function field.

2.2. **Similarity classes associated to representations.** Assume that $r \geq k/2$. The action of G_{k^n} on its normal subgroup $\mathrm{M}_n(\mathfrak{o}_{k-r})$ factors through $G_{(k-r)^n}$. In fact, this is just the usual action by similarity transformations

$$g \cdot A = gAg^{-1}, \quad g \in G_{(k-r)^n},\ A \in \mathrm{M}_n(\mathfrak{o}_{k-r}).$$

It results in an action of $G_{(k-r)^n}$ on the set of all characters of $\mathrm{M}_n(\mathfrak{o}_{k-r})$.

Now suppose that ρ is an irreducible representation of G_{k^n} on a vector space V. The restriction of ρ to $\mathrm{M}_n(\mathfrak{o}_{k-r})$ gives rise to a decomposition $V = \oplus V_\chi$, where χ ranges over the set of characters of $\mathrm{M}_n(\mathfrak{o}_{k-r})$. Clifford theory then tells us that the set of characters χ for which V_χ is non-trivial consists of a single orbit for the action of $G_{(k-r)^n}$ on the characters of $\mathrm{M}_n(\mathfrak{o}_{k-r})$.

The group $\mathrm{M}_n(\mathfrak{o}_{k-r})$ can be identified with its Pontryagin dual (as a $G_{(k-r)^n}$-space). For this, pick an additive character ψ of $F \to \mathbf{C}^\times$ whose restriction to $\mathfrak{o}$ is trivial, but whose restriction to $\mathfrak{p}^{-1}$ is non-trivial. For each $A \in \mathrm{M}_n(\mathfrak{o}_{k-r})$, define a character $\psi_A\colon \mathrm{M}_n(\mathfrak{o}_{k-r}) \to \mathbf{C}^\times$ by $\psi_A(B) = \psi(\varpi^{r-k}\mathrm{tr}(AB))$. The map $A \mapsto \psi_A$ identifies $\mathrm{M}_n(\mathfrak{o}_{k-r})$ with its Pontryagin dual, and preserves the action of $G_{(k-r)^n}$.

Thus we associate, for each $r \geq k/2$, to each irreducible representation ρ of G_{k^n}, a similarity class $\Omega_{k-r}(\rho) \subset \mathrm{M}_n(\mathfrak{o}_{k-r})$.

3. Induction and restriction functors

This section introduces the functors that will play the role of parabolic induction and restriction in the context of $\mathrm{GL}_n(\mathfrak{o}_k)$. They were introduced in [Onn07, Section 2]. Geometric induction is an obvious analogue of parabolic induction in the case of a field. Infinitesimal induction has no analogue in that setting.

3.1. **Geometric induction and restriction functors.** Given a direct sum decomposition $\mathfrak{o}_k^n = \mathfrak{o}_k^{n_1} \oplus \mathfrak{o}_k^{n_2}$, define P_{n_1,n_2} to be the subgroup of G_{k^n} which preserves $\mathfrak{o}_k^{n_1}$. There is a natural surjection $\varphi\colon P_{n_1,n_2} \to G_{k^{n_1}} \times G_{k^{n_2}}$. Denote the kernel by U_{n_1,n_2}. Define the functor $i_{n_1,n_2}\colon \mathrm{Rep}(G_{k^{n_1}}) \times \mathrm{Rep}(G_{k^{n_2}}) \to \mathrm{Rep}(G_{k^n})$ taking representations σ_1 and σ_2 of $G_{k^{n_1}}$ and $G_{k^{n_2}}$ respectively to the induction to G_{k^n} of the pullback under φ of $\sigma_1 \boxtimes \sigma_2$. The functor $r_{n_1,n_2}\colon \mathrm{Rep}(G_{k^n}) \to \mathrm{Rep}(G_{k^{n_1}}) \times \mathrm{Rep}(G_{k^{n_2}})$ is defined by restricting a representation ρ of G_{k^n} to P_{n_1,n_2} and then taking the invariants under U_{n_1,n_2}. By Frobenius reciprocity, these functors form an adjoint pair:

$$\mathrm{Hom}_{G_{k^n}}(\rho, i_{n_1,n_2}(\sigma_1,\sigma_2)) = \mathrm{Hom}_{G_{k^{n_1}} \times G_{k^{n_2}}}(r_{n_1,n_2}(\rho), \sigma_1 \boxtimes \sigma_2).$$

Following [Onn07], the functors i_{n_1,n_2} and r_{n_1,n_2} are called *geometric induction* and *geometric restriction* functors, respectively.

3.2. **Infinitesimal induction and restriction functors.** For two partitions λ and μ, say that $\lambda \leq \mu$ if there exists an embedding of $\mathfrak{o}_\lambda$ in $\mathfrak{o}_\mu$ as an $\mathfrak{o}$-module. This is equivalent to the existence of a surjective $\mathfrak{o}$-module morphism $\mathfrak{o}_\mu \to \mathfrak{o}_\lambda$. If $\lambda \leq k^n$, then the pair (λ, k^n) has the *unique embedding* and *unique quotient* properties, i.e., all embeddings of $\mathfrak{o}_\lambda$ in $\mathfrak{o}_{k^n}$ and all surjections of $\mathfrak{o}_{k^n}$ onto $\mathfrak{o}_\lambda$ lie in the same G_{k^n}-orbit. As a consequence the functors that are defined below will, up to isomorphism, not depend on the choices of embeddings and surjections involved (in terms of [BO07, Section 2], k^n is a *symmetric type*).

Given $\lambda \leq k^n$, take the obvious embedding of $\mathfrak{o}_\lambda$ in $\mathfrak{o}_k^n$ given on standard basis vectors by $\mathbf{f}_i \mapsto \pi^{k-\lambda_{h(i)}}\mathbf{e}_i$, where $h(i)$ is such that $r_1 + \cdots + r_{h(i)-1} < i \leq r_1 + \cdots + r_{h(i)}$. Define

$$P_{\lambda \hookrightarrow k^n} = \{g \in G_{k^n} \mid g \cdot \mathfrak{o}_\lambda = \mathfrak{o}_\lambda\},$$

Restriction to $\mathfrak{o}_\lambda$ gives rise to a homomorphism $P_{\lambda \hookrightarrow k^n} \to G_\lambda$ which, due to the unique embedding property, is surjective. Let $U_{\lambda \hookrightarrow k^n}$ be the kernel. One may now define an induction functor $i_{\lambda \hookrightarrow k^n} \colon \mathrm{Rep}(G_\lambda) \to \mathrm{Rep}(G_{k^n})$ as follows: given a representation of G_λ, pull it back to a representation of P_λ via the homomorphism $P_{\lambda \hookrightarrow k^n} \to G_\lambda$, and then induce to G_{k^n}. Its adjoint functor $r_{\lambda \hookrightarrow k^n} \colon \mathrm{Rep}(G_{k^n}) \to \mathrm{Rep}(G_\lambda)$ is obtained by taking a representation of G_{k^n}, restricting to $P_{\lambda \hookrightarrow k^n}$, and taking the vectors invariant under $U_{\lambda \hookrightarrow k^n}$. The adjointness is a version of Frobenius reciprocity: there is a natural isomorphism

$$\mathrm{Hom}_{G_{k^n}}\left(\rho, i_{\lambda \hookrightarrow k^n}(\sigma)\right) = \mathrm{Hom}_{G_\lambda}\left(r_{\lambda \hookrightarrow k^n}(\rho), \sigma\right)$$

for representations ρ and σ of G_{k^n} and G_λ respectively. In terms of matrices, the groups $P_{\lambda \hookrightarrow k^n}$ and $U_{\lambda \hookrightarrow k^n}$ are

$$P_{\lambda \hookrightarrow k^n} = \{(a_{ij}) \in G_{k^n} \mid a_{ij} \in \pi^{\min\{0, h(j)-h(i)\}}\mathfrak{o}_k\},$$
$$U_{\lambda \hookrightarrow k^n} = \{(a_{ij}) \in P_{\lambda \hookrightarrow k^n} \mid a_{ij} \in \delta_{ij} + \pi^{h(j)}\mathfrak{o}_k\}.$$

Dually, fix the surjection of $\mathfrak{o}_k^n$ onto $\mathfrak{o}_\lambda$ given by $\mathbf{e}_i \mapsto \mathbf{f}_i$ and define

$$P_{k^n \twoheadrightarrow \lambda} = \{g \in G_{k^n} \mid g \cdot \ker(\mathfrak{o}_k^n \to \mathfrak{o}_\lambda) = \ker(\mathfrak{o}_k^n \to \mathfrak{o}_\lambda)\}.$$

Taking the induced map on the quotient gives rise to a homomorphism $P_{k^n \twoheadrightarrow \lambda} \to G_\lambda$ which, by the unique quotient property, is surjective. Let $U_{k^n \twoheadrightarrow \lambda}$ denote the kernel. An adjoint pair of functors $i_{k^n \twoheadrightarrow \lambda} \colon \mathrm{Rep}(G_\lambda) \to \mathrm{Rep}(G_{k^n})$ and $r_{k^n \twoheadrightarrow \lambda} \colon \mathrm{Rep}(G_{k^n}) \to \mathrm{Rep}(G_\lambda)$ are defined exactly as before. $P_{k^n \twoheadrightarrow \lambda}$ is conjugate to $P_{\lambda' \hookrightarrow k^n}$ and $U_{k^n \twoheadrightarrow \lambda}$ is conjugate to $U_{\lambda' \hookrightarrow k^n}$, where λ' is the partition that is complementary to λ in k^n, i.e., the partition for which $\ker(\mathfrak{o}_k^n \to \mathfrak{o}_\lambda) \cong \mathfrak{o}_{\lambda'}$. Therefore, the collection of irreducible representations obtained as summands after applying either of the functors $i_{\lambda \hookrightarrow k^n}$ or $i_{k^n \twoheadrightarrow \lambda}$ is the same. Following [Onn07], the functors $i_{\lambda \hookrightarrow k^n}$ and $i_{k^n \twoheadrightarrow \lambda}$ are called *infinitesimal induction functors*. The functors $r_{\lambda \hookrightarrow k^n}$ and $r_{k^n \twoheadrightarrow \lambda}$ are called *infinitesimal restriction functors*.

4. Cuspidality and strong cuspidality

4.1. **The definitions of cuspidality.** Recall from Section 2.2 that to every irreducible representation ρ of G_{k^n} is associated a similarity class $\Omega_1(\rho) \subset \mathrm{M}_n(\mathfrak{o}_1)$. The following definition was introduced in [Kut80] for $n = 2$ and in [Car84] for general n.

Definition 4.1 (Strong cuspidality)**.** An irreducible representation ρ of G_{k^n} is said to be *strongly cuspidal* if either $k = 1$ and ρ is cuspidal, or $k > 1$ and $\Omega_1(\rho)$ is an irreducible orbit in $\mathrm{M}_n(\mathfrak{o}_1)$.

In the above definition, one says that an orbit is irreducible if the matrices in it are irreducible, i.e., they do not leave any non-trivial proper subspaces of $\mathfrak{o}_1^n$ invariant. This is equivalent to saying that the characteristic polynomial of any matrix in the orbit is irreducible.

Another notion of cuspidality (which applies for any G_λ, however, we shall focus on $\lambda = k^n$) picks out those irreducible representations which can not be constructed from the representations of G_λ, $\lambda \leq k^n$ by using the functors defined in Section 3.

Definition 4.2 (Cuspidality)**.** An irreducible representation ρ of G_{k^n} is said to be *cuspidal* if it is neither a subrepresentation of a twist by a linear character of a geometrically induced representation nor of an infinitesimally induced representation.

4.2. **Comparison between the definitions.**

Theorem 4.3. *Every strongly cuspidal representation is cuspidal. When n is a prime, every cuspidal representation is strongly cuspidal.*

Proof. Let ρ be an irreducible non-cuspidal representation of G_{k^n}. The linear characters of G_{k^n} are of the form $\det\circ\chi$ for some character $\chi\colon \mathfrak{o}_k^\times \to \mathbf{C}^\times$. Using the identification of $N_{k-1} \simeq \mathrm{M}_n(\mathfrak{o}_1)$ with its dual from Section 2.2, the restriction of $\det\circ\chi$ to N_{k-1} is easily seen to be a scalar matrix. Thus ρ is strongly cuspidal if and only if $\rho(\chi) = \rho \otimes \det\circ\chi$ is, since adding a scalar matrix does not effect the irreducibility of the orbit $\Omega_1(\rho)$. Since ρ is non-cuspidal, there exist a character χ such that $\rho(\chi)^U$ is nonzero for some $U = U_{n_1,n_2}$ or $U = U_{\lambda\hookrightarrow k^n}$. In either case this implies that the orbit $\Omega_1(\rho(\chi))$ is reducible which in turn implies that $\rho(\chi)$ and hence ρ are not strongly cuspidal.

For the converse the following interesting result plays an important role. Call a similarity class in $\mathrm{M}_n(\mathfrak{o}_1)$ *primary* if its characteristic polynomial has a unique irreducible factor.

Proposition 4.4. *Let ρ be an irreducible representation of G_{k^n}. If $\Omega_1(\rho)$ is not primary then ρ is a subrepresentation of a geometrically induced representation.*

Proof. If $\Omega_1(\rho)$ is not primary then it contains an element $\varphi = \left(\begin{smallmatrix} \hat{w}_1 & 0 \\ 0 & \hat{w}_2 \end{smallmatrix}\right)$ with $\hat{w}_i \in \mathrm{M}_{n_i}(\mathfrak{o}_1)$ and $n = n_1 + n_2$, such that the characteristic polynomials of $\hat{w}_1$ and $\hat{w}_2$ have no common factor. It will be shown that $r_{n_1,n_2}(\rho) \neq 0$.

In what follows, matrices will be partitioned into blocks according to $n = n_1 + n_2$. Let $P_i = P_{(k^{n_1},(k-i)^{n_2})\hookrightarrow k^n}$ for $i = 0, \ldots, k$. Then P_i consists of matrices in G_{k^n} with blocks of the form $\left(\begin{smallmatrix} a & b \\ \varpi^i c & d \end{smallmatrix}\right)$. Let U_i be the normal subgroup of P_i consisting of block matrices of the form $\left(\begin{smallmatrix} \mathrm{I} & \varpi^{k-i} u \\ 0 & \mathrm{I} \end{smallmatrix}\right)$. The P_i's form a decreasing sequence of subgroups, while the U_i's form increasing sequences. Given a representation ρ_i of P_i/U_i define $r_i(\rho_i)$ to be the representation of P_{i+1}/U_{i+1} obtained by taking the vectors in the restriction of ρ_i to P_{i+1} that are invariant under U_{i+1}. That is,

$$r_i\colon \mathrm{Rep}(P_i/U_i) \to \mathrm{Rep}(P_{i+1}/U_{i+1}), \qquad r_i(\rho_i) = \mathrm{Inv}_{U_{i+1}/U_i} \circ \mathrm{Res}^{P_i/U_i}_{P_{i+1}/U_i}(\rho_i).$$

In particular, $P_k = P_{n_1,n_2}$ and $U_k = U_{n_1,n_2}$. Therefore, (see [Onn07, Lemma 7.1]) we have that $r_{n_1,n_2} = r_{k-1} \circ \cdots \circ r_0$. We argue by induction that $r_i \circ \cdots \circ r_0(\rho) \neq 0$ for all $i = 0, \ldots, k$.

If $i = 0$, then since $\varphi \in \Omega_1(\rho)$, we get that $\rho_{|U_1}$ contains the trivial character of U_1, hence, $r_0(\rho) \neq 0$. Denote $\rho_i = r_{i-1} \circ \cdots \circ r_0(\rho)$ and assume that $\rho_i \neq 0$. In order to show that $r_i(\rho_i) \neq 0$, consider the normal subgroup L_i of P_i which consists of block matrices of the form $\mathrm{I} + \left(\begin{smallmatrix} \varpi^{k-1}w_1 & \varpi^{k-i-1}u \\ \varpi^{k-1}v & \varpi^{k-1}w_2 \end{smallmatrix}\right)$. It is easily verified that $L_i/U_i \simeq \mathrm{M}_n(\mathfrak{o}_1)$, the isomorphism given by

$$\eta\colon \mathrm{I} + \begin{pmatrix} \varpi^{k-1}w_1 & \varpi^{k-i-1}u \\ \varpi^{k-1}v & \varpi^{k-1}w_2 \end{pmatrix} \mod U_i \ \mapsto \ \begin{pmatrix} w_1 & u \\ v & w_2 \end{pmatrix},$$

where w_1, w_2, u and v are appropriate block matrices over $\mathfrak{o}_1$. It follows that we can identify the dual of L_i/U_i with $\mathrm{M}_n(\mathfrak{o}_1)$: $\hat{x} \mapsto \psi_{\hat{x}} \circ \eta$, for $\hat{x} \in \mathrm{M}_n(\mathfrak{o}_1)$.

The action of P_i on the dual of L_i/U_i is given by $\hat{x} \mapsto g\hat{x}$ where $\psi_{g\hat{x}}(\eta(l)) = \psi_{\hat{x}}(\eta(g^{-1}lg))$. We shall not need the general action of elements of P_i, but rather of a small subgroup which is much easier to handle. If

$$g_c = \begin{pmatrix} \mathrm{I} & \\ \varpi^i c & \mathrm{I} \end{pmatrix}, \qquad \eta(l) = \begin{pmatrix} w_1 & u \\ v & w_2 \end{pmatrix}, \qquad \hat{x} = \begin{pmatrix} \hat{w}_1 & \hat{v} \\ \hat{u} & \hat{w}_2 \end{pmatrix},$$

then unraveling definitions gives

$$\hat{x} \mapsto g_c\hat{x} = \begin{pmatrix} \hat{w}_1 & \hat{v} \\ \hat{u} + c\hat{w}_1 - \hat{w}_2 c & \hat{w}_2 \end{pmatrix}. \tag{4.5}$$

As we have identifications $L_0/U_1 = \cdots = L_i/U_{i+1}$ we infer that the restriction of ρ_i to L_i/U_{i+1} contains a character

$$\psi_{\hat{x}} = (\varphi_{|L_0/U_1}, \hat{u})\colon L_i/U_{i+1} \times U_{i+1}/U_i = L_i/U_i \to \mathbf{C}^\times,$$

that is, $\psi_{\hat{x}}$ corresponds to $\hat{x} = \left(\begin{smallmatrix} \hat{w}_1 & 0 \\ \hat{u} & \hat{w}_2 \end{smallmatrix}\right)$. We claim that there exist g_c such that

$$g_c\hat{x} = \begin{pmatrix} \hat{w}_1 & 0 \\ 0 & \hat{w}_2 \end{pmatrix},$$

therefore $\rho_{i|U_{i+1}/U_i}$ contains the trivial character of U_{i+1}/U_i and hence $r_i(\rho_i) \neq 0$.

Indeed, using (4.5) it is enough to show that the map $c \mapsto c\hat{w}_1 - \hat{w}_2 c$ is surjective, hence $\hat{u}$ can be eliminated and the entry $(1,2)$ contains the trivial character. This map is surjective if and only if it is injective. So we show that its kernel is null. A matrix c is in the kernel if and only if

$$c\hat{w}_1 = \hat{w}_2 c. \tag{4.6}$$

Let p_i $(i = 1, 2)$ be the characteristic polynomials of $\hat{w}_i$. Our assumption on the orbits is that p_1 and p_2 have disjoint set of roots. Using (4.6) we deduce that

$$cp_1(\hat{w}_1) = p_1(\hat{w}_2)c.$$

By the Cayley-Hamilton theorem the left hand side of the above equation vanishes. Over an algebraic closure of $\mathfrak{o}_1$, $p_1(t) = \prod(t - \alpha_j)$, where the α_j are the roots of p_1. The hypothesis on $\hat{w}_1$ and $\hat{w}_2$ implies that none of these is an eigenvalue of $\hat{w}_2$. Therefore, $\hat{w}_2 - \alpha_j$ is invertible for each j. It follows that $p_1(\hat{w}_2) = \prod(\hat{w}_2 - \alpha_j)$ is also invertible, hence $c = 0$. This completes the proof of the proposition.

□

Returning now to the proof of Theorem 4.3, assume that ρ is not strongly cuspidal. There are two possibilities:

(a) Any element $\hat{\omega} \in \Omega_1(\rho)$ has eigenvalue in $\mathfrak{o}_1$. In such case, by twisting with a one-dimensional character χ, we get a row of zeros in the Jordan canonical form of $\hat{\omega}$. Therefore, $\rho(\chi)$ is infinitesimally induced from $G_{(k^{n-1},k-1)}$.
(b) Elements in $\Omega_1(\rho)$ have no eigenvalue in $\mathrm{M}_n(\mathfrak{o}_1)$. Since n is prime and since $\Omega_1(\rho)$ is reducible, the latter cannot be primary, and Proposition 4.4 implies that ρ is geometrically induced.

Thus, ρ is non-cuspidal.

□

5. Parameterization of strongly cuspidal representations

We now sketch the construction of strongly cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_k)$ when $k > 1$. Essentially the same construction can be found in [Shi68, How77, Car84, BK93, Hil95]. More generally, in [Gér75, Chapter IV], such representations are constructed for certain maximal compact subgroups of split reductive groups over non-Archimedean local fields whose derived groups are simply connected. We are careful to make sure that the construction here is canonical as we aim to prove Theorems B and C.

5.1. **Primitive characters.** Let E denote an unramified extension of F of degree n. Let $\mathfrak{O}$ be the integral closure of $\mathfrak{o}$ in E. The maximal ideal of $\mathfrak{O}$ is $\mathfrak{P} = \varpi\mathfrak{O}$. Let $\mathfrak{O}_k = \mathfrak{O}/\mathfrak{P}^k$. As an $\mathfrak{o}_k$-module, $\mathfrak{O}_k$ is free of rank n. Therefore, G_{k^n} can be identified with $\mathrm{Aut}_{\mathfrak{o}_k}(\mathfrak{O}_k)$. Left multiplication by elements of $\mathfrak{O}_k$ gives rise to $\mathfrak{o}_k$-module endomorphisms of $\mathfrak{O}_k$. Therefore, $\mathfrak{O}_k^\times$ can be thought of as a subgroup of G_{k^n}. Similarly, for each $r \geq k/2$, $\mathfrak{O}_{k-r}$ will be thought of as a subring of $\mathrm{M}_n(\mathfrak{o}_{k-r})$.

Strongly cuspidal representations of G_{k^n} will be associated to certain characters of $\mathfrak{O}_k^\times$ which we will call *strongly primitive*. In order to define a strongly primitive character of $\mathfrak{O}_k^\times$ it is first necessary to define a primitive character of $\mathfrak{O}_1$.

Definition 5.1 (Primitive character of $\mathfrak{O}_1$)**.** A *primitive character of* $\mathfrak{O}_1$ is a homomorphism $\phi\colon \mathfrak{O}_1 \to \mathbf{C}^\times$ which does not factor through any proper subfield via the trace map.

The map $\mathfrak{O}_k \to \mathfrak{O}_k^\times$ given by $a \mapsto 1+\varpi^r a$ induces an isomorphism $\mathfrak{O}_{k-r} \tilde{\to} \ker(\mathfrak{O}_k^\times \to \mathfrak{O}_r^\times)$, for each $1 \leq r < k$.

Definition 5.2 (Strongly primitive character of $\mathfrak{O}_k^\times$)**.** When $k > 1$, a *strongly primitive character of* $\mathfrak{O}_k^\times$ is a homomorphism $\omega\colon \mathfrak{O}_k^\times \to \mathbf{C}^\times$ whose restriction to $\ker(\mathfrak{O}_k^\times \to \mathfrak{O}_{k-1}^\times)$ is a primitive character when thought of as a character of $\mathfrak{O}_1$ under the above identification.

The above definition does not depend on the choice of uniformizing element $\varpi \in \mathfrak{p}$.
Suppose that $r \geq k/2$. An identification $A \mapsto \psi_A$ of $\mathrm{M}_n(\mathfrak{o}_{k-r})$ with its Pontryagin dual was constructed in Section 2.2. Given $a \in \mathfrak{O}_{k-r}$, view it as an element of $\mathrm{M}_n(\mathfrak{o}_{k-r})$. Let ϕ_a denote the restriction of ψ_a to $\mathfrak{O}_{k-r}$. Then $a \mapsto \phi_a$ is an isomorphism of $\mathfrak{O}_{k-r}$ with its Pontryagin dual.

5.2. **Construction of strongly cuspidal representations from strongly primitive characters.** The construction involves several stages. The reader may find it helpful to refer to (5.8) while navigating the construction. Let $l = \lceil k/2 \rceil$ be the smallest integer not less than $k/2$ and $l' = \lfloor k/2 \rfloor$ be the largest integer not greater than $k/2$. Let ω be a strongly primitive character of $\mathfrak{O}_k^\times$. Let $a \in \mathfrak{O}_{k-l}$ be such that the restriction of ω to $N_l \cap \mathfrak{O}_k^\times$ (when identified with $\mathfrak{O}_{k-l}$) is of the form ϕ_a. The strong primitivity of ω implies that the image of a in $\mathfrak{O}_1$ does not lie in any proper subfield. The formula

$$\tau_\omega(xu) = \psi_a(x)\omega(u) \text{ for all } x \in N_l \text{ and } u \in \mathfrak{O}_k^\times, \tag{5.3}$$

defines a homomorphism $\tau_\omega \colon N_l\mathfrak{O}_k^\times \to \mathbf{C}^\times$. Let L denote the kernel of the natural map $\mathfrak{O}_k^\times \to \mathfrak{O}_1^\times$ and let σ_ω denote the restriction of τ_ω to N_lL. Let q denote the order and p denote the characteristic of $\mathfrak{o}_1$.

Lemma 5.4. *There exists a unique irreducible representation σ'_ω of $N_{l'}L$ whose restriction to N_lL is σ_ω isotypic. This representation has dimension $q^{(l-l')(n^2-n)/2}$. Its character is given by*

$$\operatorname{tr}(\sigma'_\omega(x)) = \begin{cases} q^{(l-l')(n^2-n)/2}\sigma_\omega(x) & \text{if } x \in N_lL, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If k is even, then $l = l'$ and therefore, $\sigma'_\omega = \sigma_\omega$. Now, assume that k is odd. In this case the existence and uniqueness of σ'_ω, which is a version of the Stone-von Neumann theorem on representations of the Heisenberg group, is proved as follows:
N_lL is a normal subgroup of $N_{l'}\mathfrak{O}_k^\times$ (note that $N_l\mathfrak{O}_k^\times$ is not normal in $N_{l'}\mathfrak{O}_k^\times$). Moreover,

$$\sigma_\omega(yxy^{-1}) = \sigma_\omega(x) \text{ for all } y \in N_{l'}\mathfrak{O}_k^\times \text{ and } x \in N_lL. \tag{5.5}$$

The quotient $V = N_{l'}L/N_lL$ is naturally isomorphic to $\mathrm{M}_n(\mathfrak{o}_1)/\mathfrak{O}_1$ which[1], being an abelian group where every non-trivial element has order p, can be viewed as a vector space over $\mathbf{F}_p$ of dimension $(n^2-n)\log_p q$. Let μ_p denote the complex p^{th} roots of unity. Then $\beta \colon V \times V \to \mu_p$ defined by

$$\beta(xN_lL, yN_lL) = \sigma_\omega([x,y]) \text{ for all } x, y \in N_{l'}L,$$

is a well-defined alternating bilinear form. Since $N_{l'}N_lL = N_{l'}L$, assume without loss of generality that $x = 1 + \varpi^{l'}X$ and $y = 1 + \varpi^{l'}Y$, with $X, Y \in \mathrm{M}_n(\mathfrak{o}_k)$. Then,

$$[x,y] = 1 + \varpi^{k-1}(XY - YX).$$

By unwinding the definitions, we see that

$$\sigma_\omega([x,y]) = \psi(\varpi^{-l'}\operatorname{tr}(a(XY - YX))).$$

If, for some $x \in N_{l'}L$, $\sigma_\omega([x,y]) = 1$ for all $y \in N_{l'}L$, then the above equation implies that

$$\operatorname{tr}(aXY) \equiv \operatorname{tr}(aYX) \mod \mathfrak{p}^{l'} \text{ for all } Y \in \mathrm{M}_n(\mathfrak{o}_k).$$

Since $\operatorname{tr}(aYX) = \operatorname{tr}(XaY)$, the above equation means that X commutes with a modulo $\mathfrak{p}^{l'}$. Since the image of a in $\mathfrak{O}_1$ does not lie in any proper subfield, as an element of $\mathrm{M}_n(\mathfrak{o}_1)$ this image is a regular semisimple element, and its centralizer is $\mathfrak{O}_1$. It follows that $X \equiv 0 \mod \mathfrak{p}$, i.e., $x \in N_lL$. In other words, the form β is non-degenerate on V.

Let V_1 be a maximal totally isotropic subspace of (V, β). The pre-image G_1 of V_1 in $N_{l'}L$ is a maximal abelian subgroup of $N_{l'}L$ modulo the kernel of σ_ω. Pick any extension $\tilde\sigma_\omega$ of σ_ω to G_1 and set $\sigma'_\omega = \operatorname{Ind}_{G_1}^{N_{l'}L}\tilde\sigma_\omega$. The representation σ'_ω is irreducible and different choices of V_1 and $\tilde\sigma_\omega$ result in isomorphic representations.

[1]Here $\mathrm{M}_n(\mathfrak{o}_1)$ is identified with $\mathrm{End}_{\mathfrak{o}_1}(\mathfrak{O}_1)$.

Note that for any $x \in N_{l'}L$,

$$\operatorname{tr}(\sigma'_\omega(x)) = \frac{1}{|G_1|} \sum_{\substack{y \in N_{l'}L, \\ yxy^{-1} \in G_1}} \tilde{\sigma}_\omega(yxy^{-1}).$$

If $x \in N_l L$, then $yxy^{-1} \in N_l L$ as well. Hence, $\operatorname{tr}(\sigma'_\omega(x)) = [N_{l'}L : G_1]\sigma_\omega(x) = q^{(n^2-n)/2}\sigma_\omega(x)$ as claimed. If $x \notin N_l L$, one may always choose V_1 in such a way that $x \in G_1$. In that case

$$\tilde{\sigma}_\omega(yxy^{-1}) = \tilde{\sigma}_\omega(x)\sigma_\omega([x^{-1}, y]),$$

and $yxy^{-1} \in G_1$ if and only if $[x^{-1}, y] \in G_1$. But $[x^{-1}, y] \in N_l L \subset G_1$ for all $y \in N_{l'}L$. Therefore,

$$\operatorname{tr}(\sigma'_\omega(x)) = \frac{1}{|G_1|}\Big(\sum_{y \in N_{l'}L} \sigma_\omega([x^{-1}, y])\Big)\tilde{\sigma}(x) = 0,$$

for the sum in parentheses is a sum over a group of a non-trivial character. □

Recall (Section 2.1) that $s\colon \mathfrak{O}_1^\times \to \mathfrak{O}_k^\times$ denotes the unique multiplicative section, and that $x \in \mathfrak{O}_k^\times$ is called *cuspidal* if its image in $\mathfrak{O}_1^\times$ is not contained in any proper subfield.

Lemma 5.6. *When k is odd, there exists an irreducible representation τ'_ω of $N_{l'}\mathfrak{O}_k^\times$, which is unique up to isomorphism, whose restriction to $N_l L$ is σ_ω-isotypic, and such that for any $x \in N_{l'}\mathfrak{O}_k^\times$,*

$$\operatorname{tr}(\tau'_\omega(x)) = \begin{cases} 0 \text{ when } x \text{ is not conjugate to an element of } N_l\mathfrak{O}_k^\times \\ (-1)^{n-1}\omega(x) \text{ when } x \in \mathfrak{O}_k^\times \text{ is cuspidal.} \end{cases}$$

Proof. The lemma is a special case of [Gér75, Theorem 1].

The algebraic torus T defined over $\mathbf{F}_q$ such that $T(\mathbf{F}_q) = \mathfrak{O}_1^\times$ splits over the extension $\mathbf{F}_{q^n}$ of $\mathbf{F}_q$. The Galois group of this extension acts on the weights of $T(\mathbf{F}_{q^n})$ on $V \otimes \mathbf{F}_{q^n}$, which simply correspond to roots of GL_n. The Frobenius automorphism which generates this group acts as a Coxeter element on this root system. One may see that, in the language of [Gér75, 1.4.9(b)], this action has a unique symmetric orbit and $(n-2)/2$ non-symmetric orbits if n is even, and no symmetric orbits and $(n-1)/2$ non-symmetric orbits if n is odd. The symmetric orbits contribute a factor of (-1) to the character values. The hypothesis that u is not an element of any proper subfield of $\mathfrak{O}_1$ implies that u is regular semisimple, and no weight vanishes on it. □

When k is even, define the representation τ'_ω of $N_{l'}\mathfrak{O}_k^\times$ to be just τ_ω (see (5.3)). Then, for any $k > 1$, if $u \in \mathfrak{O}_k^\times$ is an element whose image in $\mathfrak{O}_1^\times$ is a generates $\mathfrak{O}_1^\times$, we have

$$\operatorname{tr}(\tau'_\omega(u)) = (-1)^{k(n-1)}\omega(u). \tag{5.7}$$

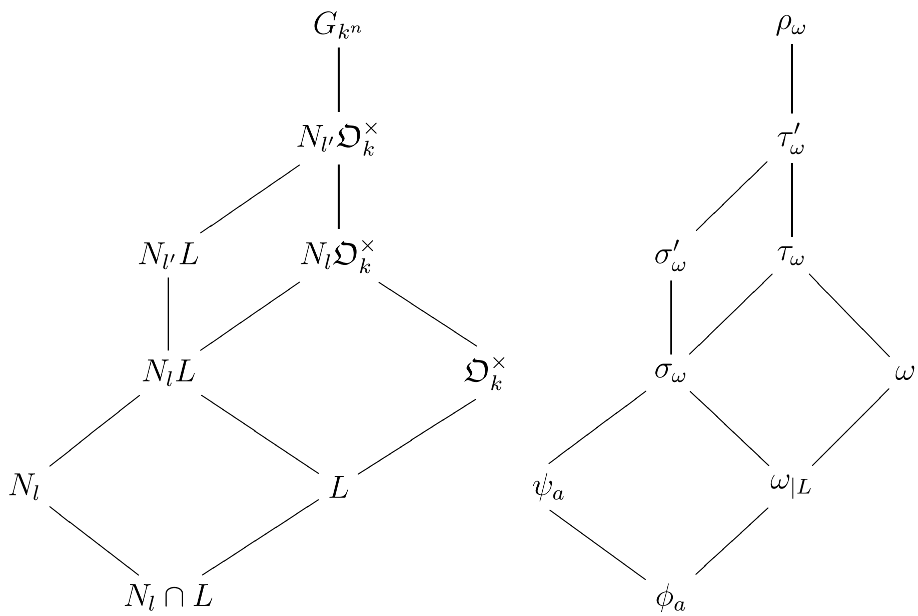
Finally, define

$$\rho_\omega = \operatorname{Ind}_{N_{l'}\mathfrak{O}_k^\times}^{G_{k^n}} \tau'_\omega.$$

This will be the strongly cuspidal representation associated to the strongly primitive character ω of $\mathfrak{O}_k^\times$. The representation ρ_ω is irreducible because $N_{l'}\mathfrak{O}_k^\times$ is the centralizer of σ_ω in G_{k^n}.

The steps in the construction of ρ_ω are described schematically below for the convenience of the reader. The diagram on the left describes the relation between the various groups involved. The position occupied by a group in the diagram on the left is occupied by the corresponding representation that appears in the construction in the diagram on the right.

(5.8)



Theorem 5.9. *For each strongly primitive character ω of $\mathfrak{O}_k^\times$, ρ_ω is an irreducible representation such that*

(1) $\mathrm{tr}(\rho_\omega(g)) = 0$ *if g is not conjugate to an element of $N_l\mathfrak{O}_k^\times$.*
(2) *if $u \in \mathfrak{O}_k^\times$ is such that its image in $\mathfrak{O}_1^\times$ is not contained in any proper subfield, then*

$$\mathrm{tr}(\rho_\omega(u)) = (-1)^{k(n-1)} \sum_{\gamma \in \mathrm{Gal}(E/F)} \omega({}^\gamma u)).$$

for every $u \in \mathfrak{O}_k^\times$, whose image in $\mathfrak{O}_1^\times$ lies in no proper subfield.

Proof. The first assertion follows from Lemma 5.4. The second follows from the fact that the intersection of the conjugacy class of u in G_{k^n} with $\mathfrak{O}_k^\times$ consists only of the elements ${}^\gamma u$, for $\gamma \in \mathrm{Gal}(E/F)$. $\square$

5.3. **The Green correspondence for strongly cuspidal representations of G_{k^n}.** The following is a detailed version of Theorem B.

Theorem 5.10.

(1) *For each strongly primitive character ω of $\mathfrak{O}_k^\times$, the representation ρ_ω of G_{k^n} is irreducible and strongly cuspidal.*
(2) *Every strongly cuspidal representation of G_{k^n} is isomorphic to ρ_ω for some strongly primitive character ω of $\mathfrak{O}_k^\times$.*
(3) *If ω' is another strongly primitive character of $\mathfrak{O}_k^\times$, then ρ_ω is isomorphic to $\rho_{\omega'}$ if and only if $\omega' = \omega \circ \gamma$ for some $\gamma \in \mathrm{Gal}(E/F)$.*

Proof of (1). It is clear from the construction that the restriction of ρ_ω to N_l contains ψ_a. This means that its restriction to N_{k-1} contains $\psi_{\overline{a}}$, where $\overline{a}$ is the image of a in $\mathfrak{O}_1$. Since this image does not lie in any proper subfield, its minimal polynomial is irreducible of degree n. Therefore, as an element of $\mathrm{M}_n(\mathfrak{o}_1)$, its characteristic polynomial must be irreducible. □

Proof of (2). Suppose that ρ is an irreducible strongly cuspidal representation of G_{k^n}. Unwinding the definitions, one see that $\Omega_1(\rho)$ is just the image of $\Omega_{k-l}(\rho)$ under the natural map $\mathrm{M}_n(\mathfrak{o}_{k-l}) \to \mathrm{M}_n(\mathfrak{o}_1)$. Let $p(t) \in \mathfrak{o}_{k-l}[t]$ be the characteristic polynomial of the matrices in $\Omega_{k-l}(\rho)$. Denote its image in $\mathfrak{o}_1[t]$ by $\overline{p}(t)$. The hypothesis on ρ implies that $\overline{p}(t)$ is irreducible. Let $\tilde{p}(t)$ be any polynomial in $\mathfrak{o}[t]$ whose image in $\mathfrak{o}_{k-l}[t]$ is $p(t)$. By Hensel's lemma, there is a bijection between the roots of $\tilde{p}(t)$ in E and the roots of $\overline{p}(t)$ in $\mathfrak{O}_1$. Consequently,

$$\mathrm{Hom}_F(F[t]/\tilde{p}(t), E) \cong \mathrm{Hom}_{\mathfrak{o}_1}(\mathfrak{o}_1[t]/\overline{p}(t), \mathfrak{O}_1).$$

But we know that $\mathfrak{O}_1$ is isomorphic to $\mathfrak{o}_1[t]/\overline{p}(t)$. In fact there are exactly n such isomorphisms. Each one of these gives an embedding of $F[t]/\tilde{p}(t)$ in E. Since both $F[t]/\tilde{p}(t)$ and E have degree n, these embeddings must be isomorphisms. Any root $\tilde{a}$ of $\tilde{p}(t)$ in E also lies in $\mathfrak{O}$. It is conjugate to the companion matrix of $\tilde{p}(t)$ in $\mathrm{GL}_n(\mathfrak{o})$. Therefore, its image $a \in \mathfrak{O}_{k-l}$ lies in $\Omega_{k-l}(\rho)$. It follows that $\rho_{|N_l}$ contains a ψ_a isotypic vector.

By applying the little groups method of Wigner and Mackey to the normal subgroup N_l of G_{k^n}, we see that every representation of ρ_k whose restriction to N_l has a ψ_a isotypic vector is induced from an irreducible representation of $N_{l'}\mathfrak{O}_k^\times$ whose restriction to N_l is ψ_a isotypic. It is not difficult then to verify (by counting extensions at each stage) that the construction of τ'_ω in Section 5.2 gives all such representations. □

Proof of (3). It follows from the proof of (2) that τ'_{ω_1} and τ'_{ω_2} are isomorphic if and only if $\omega_1 = \omega_2$. The Galois group $\mathrm{Gal}(E/F)$ acts by inner automorphisms of G_{k^n} (since we have identified it with $\mathrm{Aut}_{\mathfrak{o}_k}(\mathfrak{O}_k)$) preserving $N_{l'}\mathfrak{O}_k^\times$. Therefore, the restriction of ρ_{ω_1} to $N_{l'}\mathfrak{O}_k^\times$ also contains τ_{ω_2} whenever ω_2 is in the $\mathrm{Gal}(E/F)$-orbit of ω_1, hence ρ_{ω_1} is isomorphic to ρ_{ω_2}. If ω_1 and ω_2 do not lie in the same $\mathrm{Gal}(E/F)$-orbit then Theorem 5.9 implies that that ρ_{ω_1} can not be isomorphic to ρ_{ω_2}. □

5.4. **Connection with supercuspidal representations of $\mathrm{GL}_n(F)$.** In [BK93, Theorem 8.4.1], Bushnell and Kutzko proved that all the irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ can be obtained by compact induction from a compact subgroup modulo the center. One such subgroup is $F^\times\mathrm{GL}_n(\mathfrak{o})$. This group is a product of $\mathrm{GL}_n(\mathfrak{o})$ with the infinite cyclic group Z_1 generated by $\varpi\mathrm{I}$. Thus every irreducible representation of this group is a product of a character of Z_1 with an irreducible representation of $\mathrm{GL}_n(\mathfrak{o})$. An irreducible representation of $\mathrm{GL}_n(\mathfrak{o})$ is said to be of level $k-1$ if it factors through $\mathrm{GL}_n(\mathfrak{o}_k)$, but not through $\mathrm{GL}_n(\mathfrak{o}_{k-1})$. When n is prime, the representations of $\mathrm{GL}_n(\mathfrak{o})$ which give rise to supercuspidal representations are precisely those which are of level $k-1$, for some for $k > 1$, and, when viewed as representations of $\mathrm{GL}_n(\mathfrak{o}_k)$, are strongly cuspidal. For $k = 1$, they are just the cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_1)$. The corresponding representations of $Z\mathrm{GL}_n(\mathfrak{o})$ are called *très cuspidale de type* k by Carayol in [Car84, Section 4.1]. The construction that Carayol gives for these representations is the same as the one given here, except that the construction here is made canonical by using Gérardin's results.

Let χ be any character of Z_1. Set

$$\pi_{\omega,\chi} := \text{c-Ind}_{\mathrm{GL}_n(\mathfrak{o})F^\times}^{\mathrm{GL}_n(F)}(\rho_\omega \otimes \chi).$$

These are the supercuspidal representations of $\mathrm{GL}_n(F)$ associated to ρ_ω.

Let $r\colon \mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathfrak{o}_k)$ denote the homomorphism obtained by reduction modulo $\mathfrak{p}^k$. In the notation of [BK93], we have $r^{-1}(N_l L) = H^1(\beta, \mathfrak{A})$, $r^{-1}(N_{l'} L) = J^1(\beta, \mathfrak{A})$ and $r^{-1}(N_{l'} \mathfrak{O}_k^\times) = J(\beta, \mathfrak{A})$, where $\mathfrak{A} = \mathrm{M}_n(\mathfrak{o})$ and $\beta \in \mathrm{M}_n(F)$ is minimal (see [BK93, (1.4.14)]). These groups are very special cases of the groups defined in [BK93, (3.1.14)]. The inflation η of $\sigma_{\omega'}$ to $J^1(\beta, \mathfrak{A})$ is a special case of the Heisenberg representation defined in [BK93, Prop. 5.1.1].

We will say that a supercuspidal representation π of $\mathrm{GL}_n(F)$ belongs to the *unramified series* if the field extension $F[\beta]$ of F is unramified (by [BK93, (1.2.4), (6.2.3) (i)], this is equivalent to say that the $\mathfrak{o}$-order $\mathfrak{A}$ occurring in the construction of π is maximal). When n is a prime number, Carayol has proved (see [Car84, Theorem 8.1 (i)]) that the representations $\pi_{\omega,\chi}$ give all the supercuspidal representations of $\mathrm{GL}_n(F)$ which belong to the unramified series. However, when n is composite, the strongly cuspidal representations are not sufficient in order to build all the supercuspidal representations in the unramified series of $\mathrm{GL}_n(F)$ (see for instance Howe's construction in [How77]). It is natural to expect that, if one could achieve to construct representations of $\mathrm{GL}_n(\mathfrak{o})$ by using cuspidal representations in the sense of Definition 4.2 instead only the strongly cuspidal ones, one would be able to produce other (and perhaps all the) supercuspidal representations of $\mathrm{GL}_n(F)$ which belong to the unramified series.

6. Complexity of the classification problem

In this section it will be shown that the representation theory of the family of groups G_{k^n} actually involves the much larger family, $G_{\lambda,E}$ ($\lambda \in \Lambda$, E/F unramified), which was defined in Section 2.1, even when $k = 2$.

Theorem 6.1. *Let $F = \mathbf{F}_q((\varpi))$ be a local function field. Then the problems of finding all the irreducible characters of the following groups are equivalent:*

(1) *$G_{2^n,F}$ for all $n \in \mathbf{N}$.*
(2) *$G_{k^n,F}$ for all $k, n \in \mathbf{N}$.*
(3) *$G_{\lambda,E}$ for all partitions λ and all unramified extensions E of F.*

Proof. Obviously (3) implies (2), which implies (1). That (1) implies (3) follows from the somewhat more precise formulation in Theorem 6.2. □

Theorem 6.2. *Let F be a local function field. Then the problem of calculating all the irreducible representations of $G_{2^n,F}$ is equivalent to the problem of calculating all the irreducible representations of all the groups $G_{\lambda,E}$, where E ranges over all unramified extensions of F of degree d and λ ranges over all partitions such that $d(\lambda_1 r_1 + \cdots + \lambda_l r_l) \leq n$.*

Proof. The theorem is proved by applying the *little groups method* of Wigner and Mackey to the split short exact sequence (2.1) with $r = 1$ and $k = 2$. Pick any χ for which the χ-isotypic spaceV_χ is non-zero. Let G_χ be the stabilizer in G_{1^n} of χ. Then, $\sigma_\chi\colon g \mapsto \sigma(s(g))$ on V_χ is an irreducible representation of G_χ. In this way, an irreducible representation of G_{2^n} whose restriction to $\mathrm{M}_n(\mathfrak{o}_1)$ has isotypic vectors for the orbit of χ gives rise to an irreducible representation of G_χ.

Conversely, given an irreducible representation τ of G_χ, $\sigma_\tau((I + \varpi A)s(g)) = \chi(A)\sigma(s(g))$ for $A \in \mathrm{M}_n(\mathfrak{o}_1)$ and $g \in G_\chi$ defines an irreducible representation of the pre-image $\tilde{G}_\chi$ of G_χ in G. The representation of G_{2^n} induced from this representation is irreducible.

Thus the problem of finding all the irreducible representations of G_{2^n} is equivalent to the problem of finding all the irreducible representations of G_χ for all characters χ of $\mathrm{M}_n(\mathfrak{o}_1)$. By the discussion in Section 2.2, the groups G_χ are the same as the centralizer groups of matrices.

Let $A \in \mathrm{M}_n(\mathfrak{o}_1)$. Then, $\mathfrak{o}_1^n$ can be thought of as a $\mathfrak{o}_1[\varpi]$-module where ϖ acts through A. The centralizer of A is the automorphism group of this $\mathfrak{o}_1[\varpi]$-module. For each irreducible monic polynomial $f(\varpi) \in \mathfrak{o}_1[\varpi]$ of degree d which divides the characteristic polynomial of A, the f-primary part of this module is isomorphic to

$$(\mathfrak{o}_1[\varpi]/f(\varpi)^{\lambda_1})^{r_1} \oplus \cdots \oplus (\mathfrak{o}_1[\varpi]/f(\varpi)^{\lambda_l})^{r_l},$$

for some partition λ.

Lemma 6.3. *Let $\mathfrak{O}_1 = \mathfrak{o}_1[\varpi]/f(\varpi)$. The rings $\mathfrak{o}_1[\varpi]/f(\varpi)^k$ and $\mathfrak{O}_1[u]/u^k$ are isomorphic for every $k > 0$.*

Proof. It will be shown by induction that there exists a sequence $\{q_k(\varpi)\}$, in $\mathfrak{o}_1[\varpi]$ such that

(1) $q_1(\varpi) = \varpi$,
(2) $q_{k+1}(\varpi) \equiv q_k(\varpi) \mod f(\varpi)^k$ for all $k > 0$, and,
(3) $f(q_k(\varpi)) \in f(\varpi)^k$ for all $k > 0$.

For $k = 1$ the result is obvious. Suppose that $q_k(\varpi)$ has been constructed. Since $\mathfrak{o}_1$ is a perfect field and $f(\varpi)$ is irreducible, $f'(\varpi)$ is not identically 0. It follows that $f'(\varpi)$ does not divide $f(\varpi)$. Since $q_k(\varpi) \equiv \varpi \mod f(\varpi)$, $f'(q_k(\varpi))$ does not divide $f(\varpi)$. Therefore, the congruence

$$f(q_k(\varpi)) + f(\varpi)^k h(\varpi) f'(q_k(\varpi)) \equiv 0 \mod f(\varpi)^{k+1}$$

can be solved for $h(\varpi)$. Let $h_0(\varpi)$ be a solution. Take $q_{k+1}(\varpi) = q_k(\varpi) + f(\varpi)^k h_0(\varpi)$. The sequence $\{q_k(\varpi)\}$ constructed in this manner has the required properties.

Now note that $\mathfrak{O}_1[u]/u^k \cong \mathfrak{o}_1[\varpi, u]/(f(\varpi), u^k)$. One may define a ring homomorphism

$$\mathfrak{o}_1[\varpi, u]/(f(\varpi), u^k) \to \mathfrak{o}_1[\varpi]/f(\varpi)^k$$

by $\varpi \mapsto q_k(\varpi)$ and $u \mapsto f(\varpi)$. Since $q_k(\varpi) \equiv \varpi \mod f(\varpi)$, ϖ lies in the image of this map, so it is surjective. As vector spaces over $\mathfrak{o}_1$ both rings have dimension kd. Therefore, it is an isomorphism. □

It follows from Lemma 6.3 that the automorphism group of the f-primary part of $\mathfrak{o}_1^n$ is $G_{\lambda,E}$, where E is an unramified extension of F of degree d. The automorphism group of the $\mathfrak{o}_1[\varpi]$-module $\mathfrak{o}_1^n$ is the product of the automorphism groups of its f-primary parts. Therefore, the centralizer of A in G_{1^n} is a product of groups of the form $G_{\lambda,E}$. Considerations of dimension show that $d(\lambda_1 r_1 + \cdots + \lambda_l r_l) \leq n$ for each $G_{\lambda,E}$ that occurs.

Conversely given λ and d satisfying the above inequality, take an irreducible polynomial $f(\varpi) \in \mathfrak{o}_1[\varpi]$ of degree d. Define

$$J_k(f) = \begin{pmatrix} C_f & 0 & 0 & \cdots & 0 & 0 \\ \mathrm{I}_d & C_f & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{I}_d & C_f & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_f & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{I}_d & C_f \end{pmatrix}_{kd \times kd},$$

where C_f is any matrix with characteristic polynomial f. Let

$$A = J_{\lambda_1}(f)^{\oplus r_1} \oplus \cdots \oplus J_{\lambda_l}(f)^{\oplus r_l} \oplus J_k(\varpi - a),$$

where $a \in \mathfrak{o}_1$ is chosen so that $\varpi - a \neq f(\varpi)$ and $k = n - d(\lambda_1 r_1 + \cdots + \lambda_l r_l)$. The centralizer of A contains $G_{\lambda,E}$ as a factor. $\square$

7. The zeta function associated to G_{k^n}

In [Spr75], Springer attaches a zeta function to irreducible representations of $\mathrm{GL}_n(\mathfrak{o}_1)$, and proves that for cuspidal representations it satisfies a functional equation. Later on, Macdonald [Mac80] shows that a functional equation holds for any irreducible representation, provided that it has no 1-component, namely, it is not contained in $i_{n-1,1}(\rho, 1)$ for any representation ρ of $\mathrm{GL}_{n-1}(\mathfrak{o}_1)$. Moreover, Macdonald establishes a bijection between irreducible representations of $\mathrm{GL}_n(\mathfrak{o}_1)$, and equivalence classes of tamely ramified representations of the Weil-Deligne group W'_F, which preserves certain L and ε factors.

In this section we attach a zeta function to any irreducible representation of G_{k^n} and show that it satisfies a functional equation, provided that ρ is not infinitesimally induced. We follow closely [Mac80] and make the necessary adaptations.

Let $\psi\colon \mathfrak{o}_k \to \mathbf{C}^\times$ be an additive character which does not factor through $\mathfrak{o}_{k-1}$. Denote $G = G_{k^n}$ and $M = \mathrm{M}_{k^n} = \mathrm{M}_n(\mathfrak{o}_k)$, and let $\mathbf{C}(M)$ denote complex valued functions on M. For $f \in \mathbf{C}(M)$ define its Fourier transform by

$$\hat{f}(x) = |M|^{-1/2} \sum_{y \in M} f(y) \psi\left(\mathrm{tr}(xy)\right),$$

so that $\hat{\hat{f}}(x) = f(-x)$. Let (ρ, V) be a finite dimensional representation of G. For each $f \in \mathbf{C}(M)$ define the zeta-function

$$\mathcal{Z}(f, \rho) = \sum_{g \in G} f(g)\rho(g) \in \mathrm{End}_{\mathbf{C}}(V).$$

Also, for $x \in M$ let

$$\mathcal{W}(\rho, \psi; x) = |M|^{-1/2} \sum_{g \in G} \psi(\mathrm{tr}(gx))\rho(g).$$

The following lemma is straightforward.

Lemma 7.1.

(a) $\mathcal{Z}(f, \rho) = \sum_{x \in M} \hat{f}(-x)\mathcal{W}(\rho, \psi; x)$.
(b) $\mathcal{W}(\rho, \psi; xg) = \rho(g)^{-1}\mathcal{W}(\rho, \psi; x)$.
(c) $\mathcal{W}(\rho, \psi; gx) = \mathcal{W}(\rho, \psi; x)\rho(g)^{-1}$.

In particular, setting $x = 1$ in parts (b)-(c) of Lemma 7.1 shows that $\mathcal{W}(\rho, \psi; 1)$ commutes with $\rho(g)$ for all $g \in G$. Therefore, if ρ is irreducible, then $\mathcal{W}(\rho, \psi; 1)$ is a scalar multiple of $\rho(1)$. Following [Mac80] we write $\varepsilon(\rho, \psi)\rho(1) = \mathcal{W}(\check{\rho}, \psi; 1)$, where $\check{\rho}$ is the contragredient of ρ, i.e. $\check{\rho}(g) = {}^t\rho(g^{-1})$.

Proposition 7.2. *Let ρ be an irreducible representation of G which is not infinitesimally induced. Then $\mathcal{W}(\rho, \psi; x) = 0$ for all $x \in M \smallsetminus G$.*

Proof. Let $H_x = \{g \in G \mid gx = x\}$. For $g \in H_x$ we have

$$\mathcal{W}(\rho,\psi;x) = \mathcal{W}(\rho,\psi;gx) = \mathcal{W}(\rho,\psi;x)\rho(g^{-1}) = \mathcal{W}(\rho,\psi;x)\rho(e_{H_x}),$$

where $\rho(e_{H_x}) = |H_x|^{-1}\sum_{g\in H_x}\rho(g)$. Hence, it is suffices to show that $\rho(e_{H_x}) = 0$ for $x \in M \smallsetminus G$. Since $\rho(e_{H_x})$ is the idempotent projecting V onto V^{H_x}, it is enough to to show that the latter subspace is null. Let $\mu = (\mu_1,\dots,\mu_n)$ be the divisor type of x. Namely, $0 \le \mu_1 \le \cdots \le \mu_n \le k$, such that acting with G on the right and on the left gives: $gxh = d_\mu = \mathrm{diag}(\varpi^{\mu_1},\dots,\varpi^{\mu_n})$. Then $H_x = gH_{d_\mu}g^{-1}$. Now for any μ we have $H_{d_\mu} \supset H_{d_\nu}$, where $\nu = (0,0,\dots,0,1)$. Therefore, it is enough to show that $V^{H_{d_\nu}} = (0)$. The subgroup H_{d_ν} is given explicitly by

$$H_{d_\nu} = \begin{bmatrix} \mathrm{I}_{n-1} & \varpi^{k-1}\star \\ 0 & 1+\varpi^{k-1}\star \end{bmatrix} = U_{(k^{n-1},k-1)\hookrightarrow k^n} \text{ (see Section 3.2).}$$

It follows that $V^{H_{d_\nu}} = (0)$ if and only if ρ is not infinitesimally induced. □

Theorem 7.3. *For all $f \in \mathbf{C}(M)$ and all irreducible non-infinitesimally induced representations ρ of G we have*

$${}^t\mathcal{Z}(\hat{f},\check{\rho}) = \varepsilon(\rho,\psi)\mathcal{Z}(f,\rho).$$

Proof. If ρ is not infinitesimally induced then nor is $\check{\rho}$, and hence $\mathcal{W}(\check{\rho},\psi;x) = 0$ for all $x \in M \smallsetminus G$. We get

$$\begin{aligned} {}^t\mathcal{Z}(\hat{f},\check{\rho}) &= \sum_{g\in G}\hat{\hat{f}}(-g)\,{}^t\mathcal{W}(\check{\rho},\psi;g) && \text{(by Lemma 7.1(a))} \\ &= \mathcal{W}(\check{\rho},\psi;1)\sum_{g\in G} f(g)\rho(g) = \varepsilon(\rho,\psi)\mathcal{Z}(f,\rho) && \text{(by Lemma 7.1(c)).} \end{aligned}$$

□

The possibility of relating representations of G_{k^n} with some equivalence classes of representations of the Weil-Deligne group W'_F, and consequently extending Macdonald correspondence to higher level, seems very appealing. However, such correspondence, if exists, is expected to be much more involved in view of the complexity of the representation theory of G_{k^n}.

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