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BOUNDEDNESS OF MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN WEIGHTED L^p SPACES

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ABSTRACT. Given a weight $w(x) > 0$ in $\mathbf{R}^n$, necessary and sufficient conditions are found for the boundedness of the Hardy-Littlewood maximal function and singular integral operators from $L^p(w)$ to some other weighted L^p space. The dual question is also considered and partially answered.

1. Introduction. Weighted L^p inequalities for the Hardy-Littlewood maximal function as well as for some singular integral operators are known to hold if and only if the weight function w belongs to Muckenhoupt's class A_p **[8, 2]**. In **[9]**, the following question was raised: Find conditions on $w(x)$ so that these operators are bounded from $L^p(w)$ to some other weighted space $L^p(u)$. For the conjugate function operator on the torus $\mathbf{T}$, P. Koosis **[6]** has found that a necessary and sufficient condition in the L^2 case is $w^{-1} \in L^1(\mathbf{T})$. Here we shall extend this result to L^p, where the condition becomes $w^{-p'/p} \in L^1(\mathbf{T})$, and is the same for the conjugate function as for the Hardy-Littlewood maximal operator. Moreover, all this can be extended to $\mathbf{R}^n$, where, in the L^p case, the weight w must verify $w^{-p'/p} \in L^1_{\text{loc}}$ with an additional condition limiting the growth at infinity of $w^{-p'/p}$.

2. Boundedness of the maximal function. Let M denote the Hardy-Littlewood maximal operator in $\mathbf{R}^n$

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|$$

where Q is always a cube in $\mathbf{R}^n$ and $|\cdot|$ denotes Lebesgue measure. We shall consider in particular the cubes $Q_R = \{x \in \mathbf{R}^n\colon \max_{1 \leqslant i \leqslant n} |x_i| \leqslant R\}$. $L^0(\mathbf{R}^n)$ will be the space of all measurable functions in $\mathbf{R}^n$ provided with the topology of local convergence in measure, i.e. $\lim_j f_j = 0$ (in L^0) iff $\lim_j |\{x \in Q\colon |f_j(x)| > \lambda\}| = 0$ for every cube Q and every $\lambda > 0$. We recall that a pair (v, w) of positive measurable functions in $\mathbf{R}^n$ belongs to the class A_p, $1 < p < \infty$, when

$$\sup_Q |Q|^{-1} \left(\int_Q v\right)^{1/p} \left(\int_Q w^{-p'/p}\right)^{1/p'} < \infty$$

and this condition is equivalent to the fact that M be bounded from $L^p(w)$ to weak-$L^p(v)$ (see **[9]**).

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THEOREM A. *Given* $w(x) > 0$ *in* $\mathbf{R}^n$ *and* $1 < p < \infty$, *the following conditions are equivalent*:

(a) *There exists* $u(x) > 0$ *in* $\mathbf{R}^n$ *such that* M *is bounded from* $L^p(w)$ *to* $L^p(u)$.

(b) $w^{-p'/p} \in L^1_{\mathrm{loc}}$ *and* $\limsup_{R\to\infty} |Q_R|^{-p'} \int_{Q_R} w(x)^{-p'/p}\,dx < \infty$.

(c) *There exists* $v(x) > 0$ *in* $\mathbf{R}^n$ *such that* $(v, w) \in A_p$.

(d) $Mf(x) < \infty$ *a.e. for every* $f \in L^p(w)$.

(e) M *is a continuous operator from* $L^p(w)$ *to* $L^0(\mathbf{R}^n)$.

PROOF. It is obvious that (c) implies (b). If (b) holds, for every $f \in L^p(w)$ we have by Hölder's inequality

$$\limsup_{R\to\infty} |Q_R|^{-1} \int_{Q_R} |f| \leqslant \|f\|_{L^p(w)} \limsup_{R\to\infty} |Q_R|^{-1} \left(\int_{Q_R} w^{-p'/p} \right)^{1/p'} < \infty$$

and this is equivalent to $Mf(x) < \infty$ a.e. Thus (d) is obtained from (b). An application of the Banach principle (see [4]) proves that (d) implies (e). Since M is the maximal operator corresponding to a family of positive operators, it is a consequence of Nikishin's theorem (see [5]) that (e) implies (c). Therefore, (b), (c), (d) and (e) are equivalent, and (a) implies that $(u, w) \in A_p$. We only have to prove that (a) follows from (b), which is the main point of the theorem. For each fixed cube Q, we shall prove that there exists $u_Q(x) > 0$ on Q such that

$$\int_Q (Mf)^p u_Q \leqslant \int_{\mathbf{R}^n} |f|^p w \qquad (f \in L^p(w)). \tag{1}$$

Once this is done, it suffices to take a partition of $\mathbf{R}^n$ into a sequence (Q_j) of disjoint cubes, and then (a) is verified with $u(x) = \sum_j 2^{-j} u_{Q_j}(x) \chi_{Q_j}(x)$.

In order to prove (1), we take $R > 1$ such that $Q \subset Q_R$, and decompose each $f \in L^p(w)$ as $f = f' + f''$, where $f'' = f\chi_{Q_{2R}}$ and $f' = f - f''$. Then, an elementary geometric argument shows that, for every $x \in Q$

$$\begin{aligned} Mf'(x) &\leqslant \sup_{h>R} |Q_h|^{-1} \int_{Q_{2h}} |f'| \\ &\leqslant \sup_{h>R} 2^n \|f'\|_{L^p(w)} |Q_{2h}|^{-1} \left(\int_{Q_{2h}} w^{-p'/p} \right)^{1/p'} \leqslant C_R \|f'\|_{L^p(w)} \end{aligned}$$

so that we obtain

$$\int_Q (Mf')^p \leqslant |Q| C_R^p \int_{\mathbf{R}^n} |f'|^p w \qquad (f \in L^p(w)), \tag{2}$$

On the other hand, given scalars (α_j) and functions (f_j) such that $\operatorname{supp}(f_j) \subset Q_{2R}$, $\int |f_j|^p w \leqslant 1$, we use the l^p-valued extension of the weak type (1, 1) inequality for the maximal operator due to Fefferman and Stein [3] to get

$$\begin{aligned} \left| \left\{ x : \left(\sum_j |\alpha_j M f_j(x)|^p \right)^{1/p} > \lambda \right\} \right| &\leqslant A_p \lambda^{-1} \int_{Q_{2R}} \left(\sum_j |\alpha_j f_j|^p \right)^{1/p} dx \\ &\leqslant A_p \lambda^{-1} \left(\int \sum_j |\alpha_j f_j(x)|^p w(x)\,dx \right)^{1/p} \left(\int_{Q_{2R}} w^{-p'/p} \right)^{1/p'} \leqslant A_{R,p} \lambda^{-1} \left(\sum_j |\alpha_j|^p \right)^{1/p}. \end{aligned}$$

If $q < 1$, Kolmogorov's inequality relating the L^q norm with the weak-L^1 norm implies

$$\int_Q \Big(\sum_j |\alpha_j Mf_j(x)|^p \Big)^{q/p} dx \leq \frac{A_{R,p}}{1-q} |Q|^{1-q} \Big(\sum_j |\alpha_j|^p \Big)^{q/p}.$$

According to Maurey's factorization theorem [7, Theorem 2], there exists a measurable function g such that

$$\int_Q |g|^r < \infty, \qquad \int_Q \left| \frac{Mh(x)}{g(x)} \right|^p dx \leq 1 \qquad \left(\frac{1}{r} = \frac{1}{q} - \frac{1}{p} \right)$$

for every function h supported in Q_{2R} with $\|h\|_{L^p(w)} \leq 1$. Thus

$$(3) \qquad \int_Q (Mf''(x))^p |g(x)|^{-p} \, dx \leq \int_{\mathbf{R}^n} |f''|^p \qquad (f \in L^p(w)).$$

From (2) and (3) we obtain (1) with $u_Q(x) = 2^{1-p} \inf(|g(x)|^{-p}, |Q|^{-1} C_R^{-p})$. Since $u_Q^{-r/p} \in L^1(Q)$, and $r/p = q/(p-q)$ increases to p'/p as $q \to 1$, the last assertion of the theorem also follows.

The dual question, i.e., finding conditions on $u(x)$ so that M is bounded from some $L^p(w)$ to $L^p(u)$, was also raised in [9]. A partial answer is contained in the following.

THEOREM B. *Given $u(x) > 0$ in $\mathbf{R}^n$ and $1 < p < \infty$, in order that there exists $w(x) < \infty$ a.e. such that M is bounded from $L^p(w)$ to $L^p(u)$, it is*

(i) *necessary that $u \in L^1_{\text{loc}}$ and* $\limsup_{R\to\infty} |Q_R|^{-1} (\int_{Q_R} u)^{1/p} < \infty$,

(ii) *sufficient that $u \in L^1_{\text{loc}}$ and, for some $q < p$,* $\limsup_{R\to\infty} |Q_R|^{-1} (\int_{Q_R} u)^{1/q} < \infty$.

PROOF. If M is bounded from $L^p(w)$ to $L^p(u)$, the pair (u, w) belongs to A_p, and part (i) follows easily. The proof of (ii) depends on the following fact which will be obtained as a by-product of the results for singular integral operators:

[$*$] *If $u \in L^1_{\text{loc}}$ and* $\limsup_{R\to\infty} |Q_R|^{-1} (\int_{Q_R} u)^{1/q} < \infty$, *for every $r > q \geq 1$ there exists $w(x) > 0$ such that $(u, w) \in A_r$.*

Using [$*$] with $q < r < p$ we see that M is bounded from $L^r(w)$ to weak-$L^r(u)$, and since it is bounded on L^∞ we only have to interpolate by the Marcinkiewicz theorem.

3. Boundedness of singular integrals. By a singular integral operator (s.i.o.) in $\mathbf{R}^n$ we shall mean an operator of the form

$$Tf(x) = K * f(x) = \text{p.v.} \int K(y) f(x-y) \, dy$$

with the kernel K satisfying the conditions

$$(4) \qquad |\hat{K}(x)| \leq B,$$

$$(5) \qquad |K(x)| \leq B \|x\|^{-n},$$

$$(6) \qquad |K(x-y) - K(x)| \leq B \|y\| / \|x\|^{n+1} \quad \text{when } \|y\| < \|x\|/2,$$

where $\|\cdot\|$ stands for Euclidean norm in $\mathbf{R}^n$. The least constant B for which (4), (5) and (6) hold will be denoted by $B(T)$. The simplest examples of such operators are the Riesz transforms

$$R_j f = K_j * f, \qquad K_j(x) = c_n x_j / \|x\|^{n+1} \qquad (j = 1, 2, \ldots, n)$$

where $c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2)$ (see [11]).

THEOREM C. *Given $w(x) > 0$ in $\mathbf{R}^n$ and $1 < p < \infty$, the following conditions are equivalent*:

(a) *There exists $u(x) > 0$ in $\mathbf{R}^n$ such that, for every singular integral operator of the type described above*

$$\int |Tf(x)|^p u(x)\, dx \leqslant B(T) \int |f(x)|^p w(x)\, dx \qquad (f \in L^p(w)).$$

(b) $w^{-p'/p} \in L^1_{\mathrm{loc}}$ *and* $\int_{\mathbf{R}^n} w(x)^{-p'/p}(1 + \|x\|)^{-np'}\, dx < \infty$.

(c) *The Riesz transforms are continuous operators from $L^p(w)$ to $L^0(\mathbf{R}^n)$.*

Moreover, if any of these conditions hold and $s < p'/p$, $u(x)$ can be obtained in (a) *such that $u^{-s} \in L^1_{\mathrm{loc}}$.*

PROOF. (a) *implies* (c). This is obvious because convergence in $L^p(u)$ (with $u(x) > 0$ everywhere in $\mathbf{R}^n$) implies local convergence in measure. In fact, if $f_j \to 0$ (in $L^p(u)$) and m_u denotes the measure $dm_u(x) = u(x)\, dx$,

$$m_u(\{x: |f_j(x)| > \lambda\}) \leqslant \lambda^{-p} \|f_j\|^p_{L^p(u)} \to 0$$

for every $\lambda > 0$, and since $u^{-1} \in L^1(Q, m_u)$ for every cube Q

$$|\{x \in Q: |f_j(x)| > \lambda\}| = \int_{\{|f_j| > \lambda\} \cap Q} u^{-1}\, dm_u \to 0.$$

(c) *implies* (b). Since $R = \sum_{j=1}^n R_j$ is continuous in measure in $L^p(w)$, if we fix our attention on the unit ball $B = \{x \in R^n: \|x\| \leqslant 1\}$, there exists $\lambda_0 > 0$ big enough so that

(7) $$|\{x \in B: |Rg(x)| > \lambda_0 \|g\|_{L^p(w)}\}| < 2^{-n}|B|$$

for all $g \in L^p(w)$. Let $P = \{x \in \mathbf{R}^n: x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_n \geqslant 0\}$ be the first "quadrant" in $\mathbf{R}^n$. If $f \in L^p(w)$ and $x \in (-P) \cap B$,

$$\begin{aligned} |R(|f|\chi_P)(x)| &= \sum_{j=1}^n c_n \int_P |f(y)|(x_j - y_j)\|x - y\|^{-n-1}\, dy \\ &= c_n \int_P |f(y)| \left(\sum_{j=1}^n |x_j - y_j| \right) \|x - y\|^{-n-1}\, dy \geqslant c_n \int_P |f(y)| \|x - y\|^{-n}\, dy \\ &\geqslant c_n \int_P |f(y)|(1 + \|y\|)^{-n}\, dy. \end{aligned}$$

Since $|(-P) \cap B| = 2^{-n}|B|$, (7) implies

(8) $$c_n \int_P |f(y)|(1 + \|y\|)^{-n}\, dy \leqslant \lambda_0 \|f\chi_P\|_{L^p(w)}.$$

By the same argument, (8) holds if we replace P by any other of the 2^n "quadrants" in $\mathbf{R}^n$. Therefore

$$\int_{\mathbf{R}^n}|f(y)|(1+\|y\|)^{-n}\,dy < 2^{n/p'}c_n^{-1}\lambda_0\|f\|_{L^p(w)}.$$

By writing the integrand as $|f(y)|(1+\|y\|)^{-n}w(y)^{-1}w(y)\,dy$, we see that the function $w(y)^{-1}(1+\|y\|)^{-n}$ belongs to $L^{p'}(w)$. This proves (b).

(b) *implies* (a). Fix a cube Q and take $R > 1$ such that $Q \subset \{x: \|x\| \leq R\}$. As in Theorem A, it will suffice to find a constant $C > 0$ and a function $v(x) > 0$ on Q with $v^{-s} \in L^1(Q)$ (where $s < p'/p$ is given) such that, for every s.i.o. T with $B(T) \leq 1$, the following inequalities hold:

$$\text{(9)}\qquad \int_Q |Tf(x)|^p v(x)\,dx \leq \|f\|^p_{L^p(w)} \quad \text{when } \operatorname{supp}(f) \subset \{x: \|x\| \leq 2R\},$$

$$\text{(10)}\qquad \sup_{x\in Q}|Tf(x)| \leq C\|f\|_{L^p(w)} \quad \text{when } \operatorname{supp}(f) \subset \{x: \|x\| > 2R\}.$$

To prove (10) we take $C = \frac{3}{2}(\int_{\|y\|>1} w(y)^{-p'/p}\|y\|^{-np'}\,dy)^{1/p'}$. Then, if f is supported in $\{x: \|x\| \geq 2R\}$ and $B(T) \leq 1$, we have by (5) and (6)

$$\begin{aligned}\sup_{x\in Q}|Tf(x)| &\leq \sup_{x\in Q}\int_{\|y\|>2R}|f(y)|\,|K(x-y)|\,dy\\ &\leq \sup_{x\in Q}\int_{\|y\|>2R}|f(y)|(\|x\|\,\|y\|^{-n-1}+|K(y)|)\,dy\\ &\leq \frac{3}{2}\int_{\|y\|>2R}|f(y)|\,\|y\|^{-n}\,dy \leq C\|f\|_{L^p(w)}.\end{aligned}$$

The proof of (9) depends on the vector valued inequalites for singular integrals due to Benedek, Calderón and Panzone [1]. Given a sequence of s.i.o. $(T_j)_1^\infty$ with $B(T_j) \leq 1$, the operator $\tilde{T}$ defined on l^p-valued functions by $\tilde{T}(f_1, f_2, \ldots, f_j, \cdots) = (T_1f_1, T_2f_2, \ldots, T_jf_j, \ldots)$ satisfies the hypothesis of [1, Theorem 1], and therefore, it is of weak type (1, 1), i.e.

$$\left|\left\{x\in\mathbf{R}^n: \left(\sum_j |T_jf_j(x)|^p\right)^{1/p} > \lambda\right\}\right| < A_p\lambda^{-1}\left\|\left(\sum_j |f_j|^p\right)^{1/p}\right\|_1$$

with A_p depending only on p (and not on the particular sequence of operators (T_j), provided that $B(T_j) \leq 1$). By the same argument as Theorem A (e.g. Kolmogorov's inequality and Maurey's Theorem 2 of [7]) we obtain a function $g \in L^r(Q)$, with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and $q < 1$ arbitrarily close to 1, such that $\int_Q |h(x)/g(x)|^p\,dx < 1$ for any function h in the family

$$\mathcal{F} = \{Tf \mid T \text{ s.i.o. with } B(T) \leq 1, \|f\|_{L^p(w)} < 1, \operatorname{supp}(f) \subset \{x: \|x\| < 2R\}\}.$$

This proves (9) with $v(x) = |g(x)|^{-p}$, and taking q so that $r/p = q/(p-q) = s$, it follows that $v^{-s} \in L^1(Q)$.

Since every s.i.o. T is selfadjoint, T is bounded from $L^p(w)$ to $L^p(u)$ if and only if it is bounded (with the same norm) from $L^{p'}(u^{-p'/p})$ to $L^{p'}(w^{-p'/p})$ (see [10] for the simple proof of this fact). Therefore, Theorem C already gives us the complete answer of the dual question for s.i.o.

THEOREM D. *Given* $u(x) > 0$ *in* $\mathbf{R}^n$ *and* $1 < p < \infty$, *the following conditions are equivalent*:

(a) *There exists* $w(x) < \infty$ *a.e. such that, for every singular integral operator of the type considered here*

$$\int |Tf(x)|^p u(x)\, dx \leqslant B(T) \int |f(x)|^p w(x)\, dx \qquad (f \in L^p(w)).$$

(b) $u \in L^1_{\text{loc}}$ *and* $\int_{\mathbf{R}^n} u(x)(1 + \|x\|)^{-np}\, dx < \infty$.

(c) *There exists* $w(x) < \infty$ *a.e. such that the Riesz transforms are bounded from* $L^p(w)$ *to* $L^p(u)$.

Moreover, given $s < 1$, $w(x)$ *in* (a) *and* (c) *can be obtained such that* $w^s \in L^1_{\text{loc}}$.

At this point, the fact needed in the proof of Theorem B is easy to obtain.

PROOF OF [*]. We assume that $u \in L^1_{\text{loc}}$, $u > 0$ and $h(t) = \int_{\|x\| \leqslant t} u \leqslant Ct^{nq}$ ($t > 1$). If $q < r$, by using polar coordinates and integration by parts

$$\begin{aligned}\int_{\mathbf{R}^n} u(x)(1 + \|x\|)^{-nr}\, dx &= \int_0^\infty (1 + t)^{-nr} t^{n-1}\, dt \int_{\|x'\|=1} u(tx')\, d\sigma(x') \\ &= \int_0^\infty h'(t)(1 + t)^{-nr}\, dt = nr \int_0^\infty h(t)(1 + t)^{-nr-1}\, dt < \infty.\end{aligned}$$

By Theorem D, there exists $w(x) < \infty$ a.e. such that the Riesz transforms are bounded from $L^r(w)$ to $L^r(u)$, and this implies $(u, w) \in A_r$ (see [2, 9]).

The proofs of Theorems A, B, C, D work also in the periodic case (and are even simpler because there is no limitation at infinity for the weights). In particular, for the torus $\mathbf{T} \cong [0, 1)$, if we denote by $\tilde{f}$ the conjugate function of $f \in L^1(\mathbf{T})$, we ask for weights $u(x)$, $w(x)$ such that

$$\int_{\mathbf{T}} |\tilde{f}|^p u \leqslant \int_{\mathbf{T}} |f|^p w \quad (f \text{ trigonometric polynomial}) \tag{11}$$

COROLLARY. (i) *Given* $w(x) > 0$ *in* $\mathbf{T}$ *and* $1 < p < \infty$, (11) *holds for some* $u(x) > 0$ *if and only if* $w^{-p'/p} \in L^1(\mathbf{T})$. *In this case, and if* $s < p'/p$ *is given, u can be found such that* $u^{-s} \in L^1(\mathbf{T})$.

(ii) *Given* $u(x) > 0$ *in* $\mathbf{T}$ *and* $1 < p < \infty$, (11) *holds for some* $w(x) < \infty$ *a.e. if and only if* $u \in L^1(\mathbf{T})$. *In this case, and if* $s < 1$ *is given, w can be found such that* $w^s \in L^1(\mathbf{T})$.

For $p = 2$, (i) has been proved by P. Koosis [6], who obtains $u(x)$ such that $\log u \in L^1(\mathbf{T})$. The corollary is also true for the inequality (11) with Mf (maximal function of $f \in L^1(T)$) instead of $\tilde{f}$ (part (ii) is well known in this case; see [3, Lemma 1]).

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ADDED IN PROOF. L. Carleson and P. Jones have obtained essentially the same results of Theorems A and C by a somewhat different method (Mittag-Leffler Institute, Report No. 2, 1981).

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