Maslov class of lagrangian embedding to Kahler manifold

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Introduction

The development of symplectic geometry during the last twenty years of the previous century and the first years of the present one is originated mostly on the tendency to transport or reformulate numerous results, derived in classical mechanics for symplectic vector spaces, to the case of compact symplectic manifolds. Although various problems of special interest are formulated for the compact case only (f.e., the Arnold conjecture), it is natural to move from simple cases to complicated ones, from the flat case to the compact one. It is specifically important for such problems as the problem of quantization of classical mechanical systems, since the physical background of the problem requires the compatibility of the results for the flat and the compact cases. Namely (see, f.e., [21]), a quantization scheme can be acknowledged as a working one if its reduction to the case of the symplectic vector space agrees with the canonical quantization, a dogma of modern theoretical physics.

One of the essential ingredients of the symplectic geometry of the vector space is the Maslov class, induced by lagrangian embedding. Recall (see [1]), if (V, ω) — a symplectic vector space with a constant symplectic form ω and $S \subset V$ is a lagrangian submanifold, smoothly embedded[1] to V, then this embedding induces a class $m_S \in H^1(S, \mathbb{Z})$, which is integer valued and

[1] in the present paper we consider the case of smooth embeddings only, although our constructions with some corrections can be generalized to the case of lagrangian immersions

it implies its most important property — the invariance under continuous deformations preserving the lagrangian condition. It follows that the class m_S is stable with respect to the hamiltonian deformations of (V, ω) and is equivariant under the symplectomorphisms of our vector space. Most important applications of the Maslov class are the application for the minimality problem, solved in [12], and the realization of the class as a correction term in the quantization method of Karasev - Maslov which is often called the quasi classical quantization, see [11]. Specifically the second application attracted our attention to the possibility of generalizations of the Maslov class to the case of lagrangian embeddings satisfied some special conditions with respect to the Levi - Civita connection to a Kahler manifold, to exploit it in constructions of algebraic lagrangian geometry (see [7]) and ALG(a) - quantization (see [17]).

The starting point of the present work was the attentive reading of the paper [15], were the Maslov class is related to the abelian connection on the restricted to a lagrangian submanifold anticanoncial line bundle induced by the Levi - Civita connection. In other words, the present paper is just the explanation of three lines of the paper [15]. Usually one deduces a generalized Maslov class from the mean curvature of a lagrangian submanifold (see [13], [2]) which is a real 1 -form, and in this circumstance the minimality problem is solved automatically since a lagrangian submanifold has the minimal riemannian volume in the class of local deformations if and only if its mean curvature vanishes (see [3], [13]). However even if the mean curvature of a lagrangian submanifold is a closed form it is rather difficult to show that its cohomology class is integer valued (but it is done in several cases, see [13]). In the papers [15] and [16] in the special situation of the Kahler - Einstein manifolds some universal Maslov class is defined via connections, and this automatically implies that it is integer valued, but now it is necessary to relate this class to the mean curvature. The main technical result of the present paper is exactly to establish the desired relationship in the general case. Namely, Proposition 9 states and proves that for any oriented lagrangian submanifold $S \subset M$ of a Kahler manifold M the mean curvature α_H up to scaling coincides with the 1 - form, which represents the determinant Levi - Civita connection on the anticanonical line bundle with respect to the canonical flat structure. Since the curvature of the determinant Levi - Civita connection multiplied by $\frac{1}{2\pi i}$ coincides with the Ricci form ρ of the

Kahler manifold, from this fact one gets the following identity

$$d\alpha_H = \rho|_S,$$

which is already known from [4].

The construction presented in section 1 endows a lagrangian submanifold of a Kahler manifold, such that the restriction to it of the determinant Levi - Civita connection is trivial and admits covariant constant sections, with a map to S^1, which is called *the phase*; in the case when the Maslov class is trivial the phase is represented by a smooth function

$$\phi_S : S \to U(1),$$

which is defined up to a constant phase scaling (the logarithmic derivative of the "total" phase gives a closed 1- form, whose cohomology class coincides with the Maslov class m_S). For the case of the Kahler - Einstein manifolds when the Ricci tensor is proportional to the symplectic form,

$$\rho = k\omega,$$

it makes possible to define some half weighting rules for Bohr - Sommerfeld cycles, which are studied in [7]. It is the main result of the paper, and we hope that it will be exploited in algebraic lagrangian geometry, constructed by A. Tyurin and A. Gorodentsev in [7].

At the time he was a student of V.I. Arnold the author could not present the adequate diploma work and now he hopes that this paper can be regarded as such a work at the same time expressing author's gratitude to the former scientific advisor. The present work couldn't be done without help and remarks of A. Gorodentsev. The author would like to express his deep gratitude to D. Orlov, D. Auroux and S. Nemirovskiy. During the work on this paper the author was supported by the Max Planck Institute for Mathematics (Bonn, Germany) and European Center for Nuclear Research (Geneva, Switzerland) and I would like to cordial thank all staff of both the institutions. The work was partially supported by RFBR (grants NN 05 - 01 - 01086, 05 - 01- 00455).

1 The phase and the Maslov class

The definition of the Maslov class in the classical case — for a lagrangian embedding to a symplectic vector space — can be generalized to the case of any Kahler manifold for lagrangian submanifolds which possess special properties with respect to the determinant Levi - Civita connection. At the same time in the most generic case the Maslov class is not invariant under hamiltonian deformations, however restricting the construction to the class of Kahler - Einstein manifolds the Maslov class is an invariant of isodrastic deformations, and for even more specified case — the case of Calabi - Yau manifolds — it is invariant for all lagrangian deformations.

The basic construction is as follows. Let (M, ω, I) be any Kahler manifold which is understood as a symplectic manifold (M, ω) equipped with an integrable complex structure I compatible with ω. Equivalently the manifold can be understood as a symplectic manifold equipped with a compatible riemannian metric G with holonomy $U(n)$, where $\dim_{\mathbb{R}} M = 2n$. These representations are equivalent. In absolutely general non integrable case a riemannian metric induces the Levi - Civita connection on the tangent bundle TM, and in the integrable case this connection commutes with the operator I. This means that the Levi - Civita connection reduces to a hermitian connection on the holomorphic tangent bundle $T^{1,0}M$. Its determinant $\det T^{1,0}M$ is a complex line bundle which is called the anticanonical bundle and is denoted as K_M^{-1}; it carries a canonical hermitian structure and an abelian connection A_{LC}, induced by the total Levi - Civita connection. To avoid confusions in the rest of the paper we will call A_{LC} on K_M^{-1} the determinant Levi - Civita connection (since it really is the determinant of the total Levi - Civita connection). The curvature of this connection, $F(A_{LC})$, up to scaling coincides with the Ricci form ρ of the Kahler structure on M (all details can be found in [8]).

Further, let $S \subset M$ be a smooth oriented lagrangian submanifold: it means that $\dim S = \frac{1}{2} \dim M = n$ and

$$\omega|_S \equiv 0.$$

Then it's not hard to see that the restriction of the anticanonical bundle to S has the form

$$K^{-1}|_S = \det TS \otimes \mathbb{C}. \tag{1}$$

In particular, it implies that the restriction of the (anti) canonical class to any oriented lagrangian submanifold is topologically trivial. This is true as well in the case of non integrable complex structures. Moreover, the compatibility condition for ω and I implies that one has even more generic fact:

Proposition 1 *The holomorphic tangent bundle $T^{1,0}M$ being restricted to S has the form*

$$T^{1,0}M|_S = TS \otimes \mathbb{C};$$

moreover, the hermitian structure on $TM|_S$, defined by the hermitian triple (ω, I, G), is the complexification of the special orthogonal structure, defined by the riemannian metric G on TS.

Indeed, the proof[2] just requires to apply the Darboux - Weinstein theorem, [20], which describes neighborhoods of lagrangian submanifolds. The restriction of the real tangent bundle $TM|_S$ can be filled by the deformation family of TS under the action of operators

$$\cos\phi \cdot \mathrm{Id} + \sin\phi \cdot \mathrm{I}, \quad 0 \leq \phi < \pi.$$

We will use Proposition 1 in section 3, and now it is sufficient for our aims to consider its reduction to the determinants. Namely, the identification (1) is completed by the considerations of the canonical hermitian structure on $K^{-1}|_S$ and the special orthogonal structure on $\det TS$ which are compatible. This means that the space of hermitian connections $\mathcal{A}_h(K^{-1}|_S)$ containes two distinguished elements: the determinant Levi - Civita connection and the trivial connection, defined by the complexification of the determinant part of the restricted to S Levi - Civita connection. Indeed, let's restrict the metric G to S and consider a trivialization of $\det TS$, defined by an appropriate polivector field of constant length (it is dual to a volume form $d\mu$). Simultaneously this trivialization in view of the isomorphism (1) defines a trivialization of $K^{-1}|_S$, and the corresponding trivial connection is denoted as A_0.

Now, suppose that an oriented lagrangian embedding $S \subset M$ satisfies a strong condition — the determinant connection A_{LC} is *flat* and has *trivial periods* on $H_1(S, \mathbb{Z})$. It's well known (see, f.e., [5]) that the abelian connection space has very simple structure; in particular in our case the connections A_{LC} and A_0 are *gauge equivalent*. They are related by a gauge transformation

$$g_S = g(A_{LC}, A_0) \in \mathrm{Map}(S, U(1))/U(1) = \mathrm{Map}(S, S^1). \tag{2}$$

From this one gets two definitions for an oriented lagrangian embedding with trivial restriction of the determinant Levi - Civita connection:

[2] suggested to the author by Dennis Auroux, and I want to express the cordial thanks for this hint

Definition 1 The gauge transformation g_S (2) is called the phase of the lagrangian submanifold $S \subset M$.

Definition 2 The Maslov class of the lagrangian embedding $S \subset M$ to the Kahler manifold (M, ω, I) is the integer valued cohomology class on S

$$m_S = g_S^* h,$$

(see (2)), where $h \in H^1(S^1, \mathbb{Z})$ is the generator.

It is convenient to represent the phase (at least locally in the case of nontrivial Maslov class) in the form

$$e^{i\phi_S} : S \to U(1),$$

instead of the gauge transformation (2), that is described by a real function ϕ_S, defined up to an additive constant. By misuse of language we will sometimes call the function by the same name "phase" if it does not lead to confusion.

In general case, the logarithmic derivative of $e^{i\phi_S}$ gives a closed 1- form whose cohomology class coincides with the Maslov class $m_S \in H^1(S, \mathbb{Z})$.

One finishes the construction of the Maslov class for a lagrangian embedding with trivial restriction of the determinant Levi - Civita connection with the following

Proposition 2 *The definitions of the phase g_S and the Maslov class m_S for a lagrangian embedding with trivial restriction of the determinant Levi - Civita connection are correct.*

Indeed, the construction doesn't need any additional data and hence is universal.

In general for a specified problem the phase can be defined for any lagrangian embedding however for this one needs to introduce some special rules and thus one loses the universality. For example, one can define an abelian connection A^0_{LC}, which is the result of the orthogonal projection of the determinant Levi - Civita connection A_{LC} to the gauge orbit $\mathcal{G}_h(A_0) \subset \mathcal{A}_h(K^{-1}|_S)$ (all details of the theory of connections and curvatures as well as the Hodge theory of harmonic forms can be found in [5]).

Consider the curvature of the connection A_{LC}:

$$F(A_{LC}) = F_{LC} \in i\Omega^2_S.$$

From the Chern - Weyl theory we know that the curvature form F_{LC}, multiplied by $\frac{i}{2\pi}$, is a closed real form which represents the cohomology class $c_1(K^{-1}|_S) \in H^2(S,\mathbb{Z})$. But this class is trivial (see (1)), thus $\frac{i}{2\pi}F_{LC}$ is an exact form. Choose the minimal (with respect to the riemannian metric) correction term Δ_1 to connection A_{LC} such that $A_{LC} - \frac{i}{2\pi}\Delta_1$ is flat. According to the Hodge theory such a form exists and is unique: from the set of forms which satisfy

$$d\Delta_1 = \frac{i}{2\pi}F_{LC},$$

we choose the correction term imposing additionally that $\Delta_1 \perp \ker d$. Further, the connection

$$A^1_{LC} = A_{LC} - \frac{i}{2\pi}\Delta_1$$

is already flat, and one can consider its character on the fundamental group $\pi_1(S)$. But the connections we study are abelian hence the character of the connection A^1_{LC} on the commutant of the group $\pi_1(S)$ is trivial, and it descends to the periods of the connection on the lattice

$$H_1(S,\mathbb{Z}) = \pi_1(S)/[\pi_1(S),\pi_1(S)].$$

The periods of the connection A^1_{LC} on $H_1(S,\mathbb{Z})$ are uniquely defined by the corresponding class from $H^1(S,\mathbb{R})$, and there are such harmonic forms $\Delta_2 \in \mathcal{H}^1_S$, that connection

$$A^0_{LC} = A_{LC} - \frac{i}{2\pi}\Delta_1 - \frac{i}{2\pi}\Delta_2$$

has trivial periods on the lattice $H_1(S,\mathbb{Z})$ and hence the trivial character on $\pi_1(S)$. In some cases one can choose unique correction form, distinguished by the minimality condition with respect to the riemannian norm, and such a minimal form is unique only if the periods of the connection A^1_{LC} on the primitive elements of $H_1(S,\mathbb{Z})$ don't have half integer values. If this condition holds then we have uniquely defined correction term Δ_2, correctly defined trivial connection A^0_{LC} and again we define the gauge transformation g_S, which transport A^0_{LC} to A_0. The exception is given by some "boundary" case when at least one period is half integer:

$$\mathrm{Mon} A^1_{LC}(\gamma) = -1, \quad \gamma \in H_1(S,\mathbb{Z}),$$

In this case it is necessary either to define half integer Maslov classes or to introduce some additional data to choose Δ_2. If all the periods are not half integer then the correction term Δ_2 is uniquely defined and the projection A^0_{LC} is correctly defined.

Anyway, for an oriented lagrangian embedding $S \subset M$ to a Kahler manifold (M, ω, I), we can decompose the connection A_{LC} into three parts:

— the part which responses to the curvature F_{LC} (that is to the restriction of the Ricci form ρ), denoted as Δ_1,

— the part which corresponds to the periods of A^1_{LC}, that is a point on the Jacobian

$$H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}),$$

denoted as Δ_2,

— the phase, a gauge transformation, denoted as g_S.

Let's emphasize again that the choice of Δ_2 is not universal and can be done universally just in certain specified cases. For example, if we consider lagrangian embeddings with bounded periods that is the embeddings for which the periods of the flat part A^1_{LC} on the primitive elements of the lattice $H_1(S, \mathbb{Z})$ strictly less than $\frac{1}{2}$, then for the embeddings the corrections terms Δ_1, Δ_2 are correctly defined as well as the phase g_S is, and one has

Proposition 3 *The difference of connections A_{LC} and A_0 in the affine space $\mathcal{A}_h(K^{-1}|_S)$ equals to*

$$A_{LC} - A_0 = \frac{i}{2\pi}(\Delta_1 + \Delta_2 + d\ln g_s). \tag{3}$$

Topologically the universal Maslov class is the correction term for the topological type of the covariantly constant, with respect to the determinant connection, section of the anticanonical line bundle restricted to a lagrangian submanifold. Suppose that a lagrangian embedding $S \subset M$ is oriented and suppose that the restriction of the determinant connection A_{LC} to this submanifold possesses a covariant constant section $\tilde{S} \subset P = S \times S^1$, where P is the principal $U(1)$ - bundle, associated to the restriction of the anticanonical line bundle $K^{-1}|_S$. Then one has a simple remark

Proposition 4 *The homology class of the submanifold $\tilde{S} \subset P$ has the form*

$$[\tilde{S}] = [S] \oplus P.D.(m_S) \otimes P.D.(h) \in H_n(P, \mathbb{Z}) =$$
$$\mathbb{Z} \otimes H_n(S, \mathbb{Z}) \oplus H_{n-1}(S, \mathbb{Z}) \otimes H_1(S, \mathbb{Z}),$$

where the group $H_n(P, \mathbb{Z})$ is decomposed according to the Kunneth formula, and $P.D.(\alpha)$ denotes the homology class, Poincare dual to the cohomology class α.

It follows, in particular, that in the case of trivial A_{LC}, non degenerated highest $(n, 0)$ - form θ defines a subset $S_1 \subset S$ via condition

$$S_1 = \{p \in S | \operatorname{Im} \theta_p = 0\},$$

and then the homology class $[S_1] \in H_{n-1}(S, \mathbb{Z})$ is Poincare dual to the universal Maslov class:

$$P.D.(m_S) = [S_1].$$

The geometrical sense of the phase g_S is quite clear as well: the dual phase transformation $\bar{g}_S$ sends the highest real form $d\mu \in \Gamma(\det T^*S)$ to a form of type $(n, 0)$. Recall, that the action $\bar{g}_S$ on $d\mu$ is not just the multiplication by the complex valued function $e^{-i\phi_S}$, but indeed a gauge transformation in the corresponding $U(1)$ - bundle. Therefore, in general we have

Proposition 5 *For a lagrangian embedding $S \subset M$ to a Kahler manifold M with trivial restriction of the determinant Levi - Civita connection the phase g_S induces a real linear map*

$$g_S^* : \Gamma(\det T^*S) \to \Omega_M^{n,0}|_S,$$

such that $g_S^*(f\alpha) = f g_S^*(\alpha)$ for a real function $f \in C^\infty(S, \mathbb{R})$.

We complete the section with one more definition.

Definition 3 A lagrangian submanifold $S \subset M$ of a Kahler manifold (M, ω, I) is called special lagrangian if the restriction to it of the determinant Levi - Civita connection is trivial and the phase g_S is constant.

2 Examples. Stability with respect to deformations.

Example 1. Let $(M = V, \omega, I)$ be a symplectic vector space with a constant symplectic form ω and a constant complex structure I. This means that an identification of the fibers of the tangent bundle $TV \equiv V \times V$ is fixed and thus a trivialization of TV is fixed, defined up to gauge $SO(n)$ - transformation and it corresponds to a trivial connection without torsion and this connection is exactly the Levi - Civita connection of the metric G, which completes the pair (ω, I) to the corresponding Kahler triple. The classical construction from [1] of the universal Maslov class of an oriented lagrangian embedding $S \subset M$ in this cases uses the Gauss map

$$\gamma : S \to \mathrm{Gr}^{\uparrow}_{Lag} \times V$$

into the grassmannization of oriented lagrangian subspaces, and since there is the standard map

$$\det : \mathrm{Gr}^{\uparrow}_{Lag} = \frac{U(n)}{SO(n)} \to U(1),$$

then the combination $\det \cdot \gamma : S \to U(1)$ gives the phase of the lagrangian submanifold S and the Maslov class $m_S \in H^1(S, \mathbb{Z})$ on it. It' s not hard to see that the classical construction is a reduction of the construction given in the previous section for the flat case.

Indeed, the Levi - Civita connection of metric G, trivializes the tangent bundle, induces the determinant connection A_{LC} on the anticanonical line bundle K_M^{-1}, such that there exists a global covariantly constant with respect to A_{LC} section $\theta^* \in \Gamma(K^{-1})$, dual to a highest holomorphic form $\theta \in \Omega_V^{n,0}$. Since the determinant Levi - Civita connection A_{LC} is flat and globally trivial then after restriction to any oriented lagrangian submanifold $S \subset M$ it is automatically contained by the gauge class of the trivial connection A_0. Moreover, the phase g_S in this case is related in a simple manner[3] to the gauge transformation which relates two trivializations: the restriction $\bar{\theta}|_S$ and non degenerated polivector field $\tau \in \Gamma(\det TS)$, dual to $\pm d\mu \in \Gamma(\det T^*S)$, where $d\mu$ is the volume form of the riemannian metric $G|_S$ in an orientation (since g_S is defined as a $U(1)$ - function up to constant $e^{i \cdot c}$ then the choice of orientation doesn't impact on the answer — due to this fact our construction doesn't depend on the orientation!). The last gauge transformation can be easily computed as follows: the highest antiholomorphic form $\bar{\theta}$ after restriction to S can be evaluated to the highest polivector field τ of the unit length, and since ω and I are compatible it follows that for any point $p \in S$ the result $\bar{\theta}_p(\tau)_p \in U(1)$. Thus we get the map

$$\bar{\theta}(\tau) : S \to U(1),$$

[3] two trivializations are related by an element from $\mathrm{Map}(S, U(1))$ while two connection — by an element from $\mathrm{Map}(S, U(1))/U(1)$, thus to pass from the first one to the second one needs just to add "up to constant $e^{i \cdot c}$", see [5]

which in principle depends on the choice of orientation and precisely coincides with the composition $\det \cdot\gamma$ from [1] in the oriented case. The phase g_S can be derived from $\bar{\theta}(\tau)$ if one forgets about the group structure on $U(1)$, but under this the classes $(\det \cdot\gamma)^* h, g_S^* h \in H^1(S, \mathbb{Z})$, where $h \in H^1(U(1) = S^1, \mathbb{Z})$ is the generator, obviously coincide.

In the non orientable case for the definition of the Maslov class one uses the following trick: instead of the map det one considers the map $\det^2$ (see [1]). It's easy to see that in terms of the construction of section 1 it corresponds just to passage to the second power of the anticanonical line bundle which in this case is identified to the complexification of the trivial bundle $(\det TS)^2$; under this shift to the squares the connections A_{LC} and A_0 are doubled, where the connection $2 \cdot A_0$ is trivial now and the gauge transformation g_S^2 which we finally get, on the one hand, is compatible with the composition $\det^2 \cdot\gamma$, and on the other hand is the square of the gauge transformation g_S, therefore the classical construction from [1] gives double Maslov class on the version of section 1.

Example 2. It is natural to pass from the classical flat case, presented in Example 1, to the case when for a Kahler manifold (M, ω, I) the determinant Levi - Civita connection on the anticanonical line bundle $K_M^{-1} \to M$ admits a covariantly constant section. In this case the Ricci form of the Kahler structure (which equals up to constant to the curvature of the determinant connection) vanishes identically, and such a manifold is called the Calabi - Yau manifold[4]. Then for any orientable lagrangian submanifold $S \subset M$ the restriction to it of the determinant Levi - Civita connection A_{LC} is a flat connection with trivial periods and thus automatically $A_{LC} \in \mathcal{G}_h(A_0)$. Therefore in this case the correction terms Δ_1, Δ_2 from (3) are trivial, and for any submanifold we get the phase g_S comparing A_{LC} and A_0. As it was done in Example 1, the phase g_S can be computed from the comparing of the trivializing sections for A_{LC} and A_0, and the last two are dual to the highest holomorphic from θ (defined up to $U(1)$) and the volume form $d\mu$ (defined up to sign) on S respectively. Again one can, fixing θ, evaluate the highest holomorphic form on the highest polivector field τ of unit length, getting the map

$$\bar{\theta}(\tau) : S \to U(1),$$

[4]more precise, one has to require the simply connectedness of M, but according to a specific tradition, comes from the string theory, one calls "Calabi - Yau" any Kahler manifold with trivial canonical class

which coincides up to $U(1)$ to the phase g_S.

The space of all lagrangian submanifolds of M contains a distinguished subset of such submanifolds $S \subset M$, that the image of the phase map g_S consists of a single point:

$$\gamma_S(S) = p \in S^1.$$

These lagrangian submanifolds are called *special* lagrangian; they play the important role in the studying of the mirror symmetry effect, see [9], [16], [19]. From this one sees that Definition 3 of section 1 agrees with the standard terminology.

It's not hard to see that in the Calabi - Yau case the Maslov class, defined in the previous section, is an invariant of continuous lagrangian deformations:

Proposition 6 *Let $\phi_t : S \to M$ be a lagrangian deformation of a smooth lagrangian submanifold $S = S_0$ in a Calabi - Yau manifold (M, ω, I). Then the Maslov class m_S is an invariant of this deformation:*

$$m_S = \phi_t^* m_{S_t} \in H^1(S, \mathbb{Z}).$$

Indeed, since for all t the restriction to S_t of A_{LC} belongs to the orbit $\mathcal{G}_h(A_0)$, for all elements of the deformation the phase g_{S_t} is correctly defined and the topological type of

$$g_{S_t} : S_t \to S^1$$

doesn't depend on the deformation. But the topological type is precisely the Maslov class m_{S_t}.

Example 3. The next one in the hierarchy is the case of Kahler - Einstein manifolds, that is the manifolds for which

$$\omega = k\rho,$$

where ρ is the Ricci form of the Kahler metric (this case includes the case of Calabi - Yau manifolds for which $k = 0$). For the manifolds one has the definition of the Maslov *index* for *Bohr - Sommerfeld* lagrangian submanifolds, given by Fukai in [6]. It's not hard to see that this Maslov index can be lifted to a *class* from $H^1(S, \mathbb{Z})$, which is the Maslov class presented in section 1.

For a Kahler - Einstein manifold (M, ω, I) the Ricci form ρ is proportional to the symplectic (= Kahler) form, hence for any lagrangian submanifold $S \subset M$ the determinant Levi - Civita connection A_{LC} is flat being restricted to S, but in this case it can be non trivial. The connection S_{LC} has trivial periods on Bohr - Sommerfeld lagrangian submanifolds only by the definition of these ones (see [7]). Consequently, for a Bohr - Sommerfeld lagrangian submanifold $S \subset M$ the construction of section 1 does work and hence we get the phase $g_S : S \to S^1$. The corresponding class $g_S^* h, h \in H^1(S^1, \mathbb{Z})$, gives the value of the Maslov index from [6], being evaluated on the elements of $\pi_1(S)$.

However in the case of Kahler - Einstein manifolds as well as the Maslov class is correctly defined for Bohr - Sommerfeld lagrangian submanifolds, this class is invariant not for all lagrangian deformations but for a subclass consists of *isodrastic* deformations (see [7]). A lagrangian deformation is called isodrastic if it preserves the periods of the connection. In the Kahler - Einstein case for any lagrangian embedding $S \subset M$ the correction term Δ_1 from (3) of the difference $A_{LC} - A_0$ is trivial, but the correction term Δ_2 from (3), responsible for the periods, is trivial only for Bohr - Sommerfeld lagrangian submanifolds. If a deformation is not isodrastic then along it the term Δ_2 can vary on the Jacobian

$$J_S = H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}),$$

in half integer point of the Jacobian the phase is not defined at all and going around the torus one gets the changing by a unit of the Maslov class. But under isodrastic deformations the correction term Δ_2 doesn't change and it follows that the Maslov class is invariant. From this we have

Proposition 7 *For an isodrastic deformation $\phi_t : S \to M$ of a Bohr - Sommerfeld lagrangian submanifold $S = S_0$ the Maslov class is an invariant of the deformation*

$$m_S = \phi_t^* m_{S_t} \in H^1(S, \mathbb{Z}).$$

Isodrastic deformations are generated by strictly hamiltonian vector fields and hence they are often called hamiltonian but we use this more sonorous term to avoid confusions which can happen by the following reason. A vector field is called hamiltonian if it preserves the symplectic form. But this vector field induces the deformation which can be non isodrastic if the field is not generated by a function that is not strictly hamiltonian.

At the end, in the general situation a lagrangian deformation can be collected by lagrangian embeddings with bounded periods (for each element of such a deformation the monodromy of the connection A^1_{LC} is equal to -1 for no primitive elements from $\pi_1(S)$), and for the elements of this deformation

— the correction terms Δ_2 are correctly defined;

— the phases ϕ_{S_t} are correctly defined;

— the Maslov class is correctly defined and is an invariant of the deformation family.

It can be formulate, f.e., as follows:

Proposition 8 *Let $\phi_t : S \to M$ be a lagrangian deformation of $S = S_0$ with bounded periods. Then the Maslov class m_{S_t} is correctly defined via the correction connection A^0_{LC} and is an invariant of the deformation:*

$$m_S = \phi_t^*(m_{S_t}) \in H^1(S, \mathbb{Z}).$$

The last proposition is just an illustration how the construction of section 1 can be exploited in more general situation than Bohr - Sommerfeld lagrangian embeddings to Kahler - Einstein manifolds. However the last case is of our main interest in view of possible applications to algebraic lagrangian geometry.

3 Minimality

The problem of minimality for riemannian volume of lagrangian submanifolds and the possibility of the deformation to a minimal lagrangian submanifold are solved long ago for the case of symplectic vector spaces, see [12]. The compact case has been studied in [3] and [13], [14] — first, Bryant proved that a lagrangian submanifold $S \subset M$ is minimal only if the restriction of the Ricci form ρ to S is trivial and the determinant Levi - Civita connection A_{LC} admits a covariantly constant section that is its periods are trivial; then Oh reduced the minimality problem to the consideration of the Hodge decomposition of the *mean curvature* of the lagrangian submanifold.

Recall some definitions from riemannian geometry. Let $S \subset M$ be an embedding to a riemannian manifold. Then it is defined the *second quadratic form*

$$\mathrm{II} : TS \to N \otimes T^*S,$$

where N is the normal bundle. Namely, sections of TS are differentiated as sections of $TM|_S$ with respect to the total Levi - Civita connection ∇_{LC}

on the "big" tangent bundle $TM|_S$, and then the result is projected to the normal component in $TM|_S \otimes T^*S$. It's not hard to see that this composition is a tensor — not a differential operator — and this tensor is called the second quadratic form. At the same time the total Levi - Civita connection ∇_{LC} on $TM|_S$ can be recovered from the reduced Levi - Civita connection ∇^S_{LC}, induced by the restriction of the riemannian metric G to S, and the second quadratic form II.

Further, in our situation $S \subset M$ is a lagrangian embedding to a symplectic manifold (M, ω), and the choice of a compatible riemannian metric G attaches to S certain real 1- form α_H, called the *mean curvature*, by the following rules: the trace of the second quadratic form

$$\mathrm{tr} \quad \mathrm{II}$$

is a section of the normal bundle and since for a lagrangian embedding the normal bundle N is isomorphic to T^*S, see [20], then the trace is represented by a section of T^*S, that is by a 1 -form.

According to a Hodge theory, any form α in presence of a metric is decomposed into three parts

$$\alpha = \alpha_1 + \alpha_0 + \alpha_{-1},$$

where $\alpha_{\pm 1}$ is (co) exact and α_0 is harmonic, and this decomposition is unique (see [5]). We have

$$\alpha_1 + \alpha_0 \in \ker d, \quad \alpha_0 + \alpha_{-1} \in \ker d^*$$

and

$$\alpha_0 \in \ker d \cap \ker d^* = \mathcal{H}^1,$$

where the last intersection is exactly the space of harmonic forms.

It's known (see [3], [13]), that a lagrangian submanifold S_0 is minimal with respect to local lagrangian deformations (L - minimal) if and only if its mean curvature α_H is trivial. In [13] one proposes the notion of H- minimality — the minimality with respect to isodrastic deformations — and one shows that a lagrangian embedding is H - minimal if and only if $d^*\alpha_H = 0$. Moreover, Oh proved that for a lagrangian embedding to a Calabi - Yau manifold the mean curvature is always closed and represents an integer cohomology class.

The constructions of section 1 are related to the minimality problem by the following identity.

Proposition 9 *Let $S \subset M$ be an orientable lagrangian embedding to a Kahler manifold (M, ω, I). Then the connections A_{LC}, A_0, defined in section 1, and 1 - form α_H are related by identity:*

$$2\pi i \alpha_H = (A_{LC} - A_0). \tag{4}$$

As a corollary we get, in particular, the coincidence of the restricted Ricci form and the differential of the mean curvature,

$$\rho|_S = d\alpha_H,$$

which has already been established in [4], since

$$\rho|_S = -\frac{i}{2\pi} F_{LC} = \frac{1}{2\pi i} d(A_{LC} - A_0).$$

Let us stress that such an identity is possible in the case of integrable complex structures only.

The proof of Proposition 9 is based on the reconstruction of the total Levi - Civita connection ∇_{LC} on $TM|_S$ from the reduced connection ∇^S_{LC} on TS and the second quadratic form II.

Indeed, let's represent $TM|_S$ as the complexification of TS and consider the holomorphic tangent bundle $T^{1,0}M|_S$, which is a $U(n)$- bundle. Then it carries two $U(n)$ - connections: the total Levi - Civita connection ∇_{LC} (by the intergability condition the holonomy of metric G is $U(n)$) and the connection, which comes after the complexification from the reduced Levi - Civita connection and we will denote this one by the same symbol ∇^S_{LC} to simplify the explanation. Any two $U(n)$- connections over a $U(n)$ - bundle are differ by a 1- form with values in the Lie algebra $u(n)$. It's not hard to see that for our $U(n)$ - connections ∇_{LC} and ∇^S_{LC} this 1- form is given by the second quadratic form being considered as a 1- form with values in symmetric endomorphisms of TS. It is very simple to convert a symmetric endomorphism of TS to the corresponding skew symmetric endomorphism of the complexification $TS \otimes U(1)$, and thus one sees that

$$\nabla_{LC} - \nabla^S_{LC} = 2\pi i \mathrm{II}. \tag{4'}$$

Further, by the definition of our connections A_{LC} and A_0 on the restriction on the anticanonical line bundle $K^{-1}|_S = \det T^{1,0}M|S$ one gets, reducing the situation in (4') to the determinants, that the identity holds

$$A_{LC} - A_0 = 2\pi i \mathrm{tr}\mathrm{II},$$

and thus taking into account the definition of the mean curvature it follows the statement of Proposition 9.

Thus for any lagrangian embedding $S \subset M$ to a Kahler manifold (M, ω, I) there is the same 1-form, the mean curvature α_H, which can be in view of identity (4) decomposed into components in two ways: as a 1 - form in the Hodge theory and as a connection form on a trivial bundle in the theory of connections:

$$\begin{array}{l}\Delta_1 + \Delta_2 + d \ln g_S = \alpha_H = \\ (\alpha_H)_1 + (\alpha_H)_h + (\alpha_H)_{-1}.\end{array}$$

The components of the decompositions are related as follows:

$$\begin{array}{l}\Delta_2 + d \ln g_S = (\alpha_H)_1 + (\alpha_H)_h \in \ker d \\ \Delta_1 = (\alpha_H)_{-1} \in \operatorname{Im} d^*,\end{array}$$

and on the real cohomology level

$$[(\alpha_H)_h] = [\Delta_2] + m_S \in H^1(S, \mathbb{R}).$$

Therefore in the generic case the Maslov class is the "integer part" of the cohomology class which is presented by the harmonic part of the mean curvature.

The identity (4) implies a number of corollaries.

Corollary 1 *A lagrangian submanifold $S \subset M$ is L - minimal if and only if the restriction to it of the anitcanonical line bundle with the determinant Levi - Civita connection admits a covariantly constant section and the phase g_S is constant.*

Indeed, the existence of a covariantly constant section is equivalent to $\Delta_1 = \Delta_2 = 0$ in the decomposition (3), and the rest component $d \ln g_S$ is trivial if and only if the phase is constant. In other words a minimal lagrangian submanifold must be special lagrangian.

Corollary 2 *A lagrangian submanifold $S \subset M$ with trivial restriction of the determinant Levi - Civita connection and trivial Maslov class is H- minimal if and only if it is L - minimal or special lagrangian.*

Really according to [13], the H - minimality is equivalent to the condition $d^* \alpha_H = 0$; the components Δ_1, Δ_2 from (3) always lie in the kernel of d^*, and if the Maslov class m_S is trivial then the component $d \ln g_S$ is an exact form and it lies in the kernel of d^* if and only if it vanishes.

Finally, the propositions of section 2 together with the identity (4) show that

Corollary 3 *1. A lagrangian submanifold $S \subset M$ of a Calabi - Yau manifold M with nontrivial Maslov class can not be transported to a minimal lagrangian submanifold by lagrangian deformations.*

2. A Bohr - Sommerfeld lagrangian submanifold $S \subset M$ of a Kahler - Einstein manifold M with nontrivial Maslov class can not be transported to a minimal lagrangian submanifold by isodrastic lagrangian deformations.

General case requires more detailed considerations and since the applications we need are contained by the case of Kahler - Einstein manifolds we stop here with the presented relationships.

4 Applications

The definition of the phase g_S of a lagrangian embedding to a Kahler manifold given at section 1 of the present paper makes it possible to develop some approaches to the problems of mirror symmetry and geometric quantization (see [15], [19], [7]).

Special lagrangian geometry. There is an approach to the mirror symmetry problem which is based on the studies of the moduli spaces of special lagrangian submanifolds of a Calabi - Yau manifold (see [16], [19]). Special lagrangian submanifolds are mirror dual to stable holomorphic bundles on the mirror partner, but the theory of the moduli spaces of special lagrangian submanifolds of Calabi - Yau manifolds, presented in [10], is not completely finished yet, and the main difficulties come from the singularities which limiting special lagrangian submanifold can carry. A.N. Tyurin some time ago proposed to study even more restricted class — the class of Bohr - Sommerfeld special lagrangian submanifolds whose moduli space has virtual dimension zero. The same question has sense for the case of Kahler - Einstein manifolds as well and we consider this case in this section.

If (M, ω, I) is a Kahler - Einstein manifold then for each orientable lagrangian submanifold S the restriction of the determinant Levi - Civita connection A_{LC} is a flat connection. By the definition of Bohr - Sommerfeld lagrangian submanifolds, see [7],

$$\mathcal{M}_{SpBS} = \mathcal{M}_{BS} \cap \mathcal{M}_{SpLag}$$

consists of lagrangian submanifolds with trivial period part Δ_2, for which the phase g_S and the Maslov class m_S are correctly defined and moreover the first one is constant and the last one is trivial. Codimension of $\mathcal{M}_{BS}$ in the space of all lagrangian submanifolds equals to $b_1(S)$ (see [7]); due to Corollary 2 the dimension $\mathcal{M}_{SpLag}$ coincides with the dimension of the space of H- minimal lagrangian submanifolds which is computed by Oh in general case in the paper [14] and is equal to $b_1(S)$. The H - minimality is quite reasonable to investigate in connection with the Bohr - Sommerfeld condition since locally the moduli space $\mathcal{M}_{BS}$ is exactly generated by isodrastic deformations, see [17]. Therefore one could expect that the dimension of $\mathcal{M}_{SpBS}$ is zero:

$$\mathcal{M}_{SpBS} = \{p_1, ..., p_d\}$$

and the number of the points, the space $\mathcal{M}_{SpBS}$ consists of, is a symplectic invariant. However in the same paper Oh presented an example when the real dimension of the space of H- minimal lagrangian submanifold is greater than the virtual one. Briefly recall the example and discuss why this jumping takes place.

Let (M, ω, I) be projective line $\mathbb{CP}^1$ endowed with the standard Fubini - Study metric. Then (see the toy example from the last section of [7]) a smooth loop $\gamma \subset M$ (which is always lagrangian due to the dimensional reason) is Bohr - Sommerfeld if and only if it divides the surface of $S^1 = \mathbb{CP}^1$ into two parts with the same area. On the other hand, a smooth loop γ is minimal (H - minimal) if and only if it is a big circle (a circle). Therefore the local dimension of $\mathcal{M}_{\min}$ equals to 2, while the virtual dimension equals to $b_1(S) = 1$. Respectively, the intersection

$$\mathcal{M}_{BS} \cap \mathcal{M}_{\min} = \mathcal{M}_{SpBS}$$

has dimension 2, while we expect it equals to zero. In [14] one explains this fact by the existence of a special symmetry of the Fubini - Study metric; if we deform this structure to such a structure that the sphere is transformed to a ball for the american football then for this Kahler structure the space $\mathcal{M}_{\min}$ is already 1 dimensional and the intersection

$$\mathcal{M}_{BS} \cap \mathcal{M}_{\min} = \mathcal{M}_{SpLag}$$

in this case consists of the following two components — a single point (which corresponds to equator) and a 1 -dimensional component (which corresponds to meridians). Deforming the Kahler structure to even more generic one we would get the required zero dimensional space $\mathcal{M}_{SpLag}$.

The cause of the dimensional jumping of the space $\mathcal{M}_{SpLag}$ is hidden in the existence of some infinitesimal symmetries of the Kahler structure which correspond to *quasisymbols* on (M, ω, I), that is the smooth real functions whose hamiltonian vector fields preserve whole the Kahler structure (all details see in [17]). For the Fubini - Study Kahler structure on S^2 the quasisymbol space has dimension 2; for the "american ball" the dimension of the quasisymbol space equals to 1 and in this case the equator is invariant with respect to their action while the meridians are not and it follows that we have two components of different dimensions.

Another example, presented in [18]: one takes as (M, ω, I) an elliptic curve and since an elliptic curve doesn't admit quasisymbols the space $\mathcal{M}_{SpLag}$ is zero dimensional (and finite). Quasisymbols exist neither for a "real" Calabi - Yau manifold (it follows from the simply connectedness) no for an abelian variety. It was conjected that for a Kahler - Einstein manifold (M, ω, I), which doesn't admit quasisymbols, the space $\mathcal{M}_{SpLaG}$ is zero dimensional and the number of points, which form $\mathcal{M}_{SpLag}$, is a symplectic invariant. In general case the dimension of $\mathcal{M}_{SpLag}$ coincides with the dimension of the quasisymbol space, which acts on $\mathcal{M}_{SpLag}$, and the number of connected components of $\mathcal{M}_{SpLag}$ is a symplectic invariant.

To prove the conjecture it suffices to express the phase for the deformation of a Bohr - Sommerfeld special submanifold $S_0 \in \mathcal{M}_{SpLag}$, induced by a smooth function $f \in C^\infty(S_0, \mathbb{R})$ (all details of the correspondence between functions and isodrastic deformations see in [7]). Since S_0 has trivial Maslov class, being special lagrangian, any isodrastic deformation of S_0 has trivial Maslov class too and hence the phase of S_t is presented by the form

$$e^{i\phi_S}, \tag{5}$$

where ϕ_S is a smooth function, defined up to an additive constant (precisely as f is, see [7]). A naive suggestion that $f = \phi$, of course, is false — the functions are related by a differential equation with the principal term, equals to the Laplace operator. The precise formula, which relates the phase with the deformation, would lead not only to checking of the conjecture, presented above, but as well to the definition of some half weighting rules for Bohr - Sommerfeld lagrangian submanifolds.

Weighting and half weighting rules.

The moduli space of half weighted Bohr - Sommerfeld lagrangian cycles introduced in [7], plays an important role in quantization of classical mechanical systems, see [17]. Consider a Kahler - Einstein manifold (M, ω, I) and suppose that $[\omega] \in H^2(M, \mathbb{Z})$ — that is the prequantization condition holds. Then (see [7]) one chooses prequantization data (L, a), and in our case when

$$[\omega] = k \cdot K_M$$

the prequantization bundle can be chosen together with an isomorphism, identified the hermitian structures on L and K_M. Then the prequantization connection $a \in sA_h(L)$ can be chosen such that $a_K \in \mathcal{A}_h(L^k) = \mathcal{A}_h(K_M)$ is contained by the same gauge class as the connection $\tilde{A}^*_{LC}$ is, where $\tilde{A}_{LC}$ is the determinant Levi - Civita connection on whole M. Let's fix a positive volume $r > 0$ and consider the moduli space of half weighted Bohr - Sommerfeld lagrangian cycles $\mathcal{B}^{hw,r}_{BS}$ of the fixed volume (the definition see in [7]). Let's take the component $\mathcal{B}^{hw,r}_{BS,0} \subset \mathcal{B}^{hw,r}_{BS}$, which consists of lagrangian submanifolds with trivial Maslov class. Then we have a bi - section

$$\mathcal{B}_{BS,0} \to \mathcal{B}^{hw,r}_{BS,0},$$

defined by the phase half weighting rule

$$\theta^2_S = (\phi_S + c) d\mu,$$

where ϕ_S is defined in (5), $d\mu$ is the volume form induced by the riemannian metric G on S, and constant $c \in \mathbb{R}$ is defined by the condition

$$\int_S (\phi_S + c) d\mu = r.$$

It's easy to see that under this rule minimal lagrangian submanifolds are distinguished by the fact that for each minimal one the square of the canonical half weight θ^2_S is proportional to the riemannian volume form.

At the end we remark that it is possible to give an interpretation for the Maslov class, presented in section 1 of this paper, in non integrable case as well. But we postpone the discussion of this question until the time when some reasonable geometric or topological applications will be found hoping that this time will come soon.

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