PERFORMANCE OF INVESTMENT STRATEGIES IN THE ABSENCE OF CORRECT BELIEFS

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Abstract. We study an evolutionary market model with long-lived assets. We show that in the absence of correct beliefs, the strategy which is "closer" to the Kelly rule cannot be driven out of the market. This means that this strategy will either dominate or at least survive. Our techniques are borrowed from the theory of iterated function systems (IFS).

1. Introduction

Evolutionary finance examines the long-run performance of investment strategies. The ideas for evolutionary have a long story in the social science. Survival and extinction questions of investment strategies have been examined in Blume and Easley [9], where discounted sums of expected utilities are maximised by agents. They generalised the pioneering work of Kelly [15], who showed that when the market is composed only of Arrow securities, then the investment principle of *betting one's beliefs* ultimately accumulates all wealth. This principle prescribes dividing wealth among assets according to the probability of their success. Later, the work of Kelly was developed by Brieman [10], and by Cover and his collaborators [1, 6, 7]. In all the above papers the prices in the model were assumed to be exogenous.

In 2002, Evstigneev et al. [11] studied an evolutionary model with short-lived assets. The prices in the of model [11] are derived endogenously. From that time this model has been examined under different assumptions and from different points of view [2, 14] using ideas from the theory of random dynamical systems [3, 8]. Later on, Evstigneev et al. [12, 13] introduced a model with long-lived assets. They showed that the Kelly portfolio rule is evolutionary stable and when all the investors use simple strategies, it dominates the market. Moreover, it has been shown that the Kelly rule forms a unique Nash equilibrium strategy [4]. In all these paper [4, 12, 13] the model of [13] was analysed under the assumption that at least one of the the investors uses the Kelly rule.

In this paper we analyse the model of [13] when all the investors have incorrect beliefs (the exact probability distribution of the states of the world is not available for them). We show that in the absence of correct beliefs, the strategy which is "closer" to the Kelly rule cannot be driven out of the market. This means that this strategy will either dominate or at least survive. Our techniques are borrowed from the theory of iterated function systems (IFS).

1991 *Mathematics Subject Classification.* Primary 91G80, 91G10.
Key words and phrases. Evolutionary Finance, Iterated Function Systems.

In section 2, we recall the model of [13]. In section 3, we recall the notion of IFS and state some results from [5]. In section 4, we show that our market model can be represented by an IFS which satisfies the assumptions of [5]. This section includes the statement and prove of our main result (Theorem 4.4). Finally, in section 5, we illustrate our result by providing a numerical example and a simulation of the relative wealth of the investors.

2. The Model

We will focus on the model introduced in [13]. Let S be a finite set and p be a probability distribution on S such that $p_s > 0$ for all $s \in S$. Let $s_t \in S$, $t = 1, 2, \ldots$, be the *state of the world* at time t.

Assumption 1. We assume that s_t are independent, identically distributed.

In this model there are K *long-lived* assets $k = 1, 2, \ldots, K$ and $I \geq 2$ investors in the market. Every investor i has a portfolio at time t

$$x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right),$$

where $x_{t,k}^i$ is the number of units of asset k in the portfolio x_t^i and

$$\sum_{i=1}^{I} x_{t,k}^i \left(s_t\right) = 1.$$

The dynamics of the market is described by

$$\left(p_t; x_t^1, \ldots, x_t^I\right),$$

where $p_t = \left(p_{t,1}, \ldots, p_{t,K}\right) \in \mathbb{R}_+^K$ is the vector of asset prices.

Assumption 2. We assume that assets pay dividends $d_{t,k}$ which are functions of the state of the world s_t;

$$d_{t,k}\left(s_t\right) = D_k\left(s_t\right).$$

Here the functions $D_k(s)$ are non-negative, do not depend on t explicitly and satisfy

i) $\sum_{k=1}^{K} D_k(s) > 0, \quad s \in S,$
ii) $ED_k\left(s_t\right) > 0, \quad k = 1, \ldots, K.$

An investment strategy $\mu_t^i\left(s_t\right) = \left(\mu_{t,1}^i\left(s_t\right), \ldots, \mu_{t,K}^i\left(s_t\right)\right)$ of investor i is called *simple* if $\mu_{t,k}^i\left(s_t\right) = \mu_k^i$ for all t and s_t, where $\mu^i = \left(\mu_1^i, \ldots, \mu_K^i\right)$ is a fixed non-negative vector such that $0 < \sum_{k=1}^{K} \mu_k^i < 1$. The numbers μ_k^i are the fractions of investor i's budget and these fractions remain constant over time for simple strategies. Moreover, a strategy $\mu^i = \left(\mu_1^i, \ldots, \mu_K^i\right)$ is called *completely mixed* if $\mu_k^i > 0$ for each $k = 1, \ldots, K$. Investment strategies are characterised in terms of *investment proportions*;

$$\lambda^i = \left(\lambda_1^i, \ldots, \lambda_K^i\right), \qquad \lambda_k^i = \frac{\mu_k^i}{\rho},$$

where λ^i is a K-dimensional vector and $\rho \in (0, 1)$ is a number which is equal to the sum $\sum_{k=1}^{K} \mu_k^i$ as long as it does not depend on i and $\lambda_k^i > 0$. Consequently,

$\sum_{k=1}^{K} \lambda_k^i = 1$. Given strategies $\lambda^i = \left(\lambda_1^i, \ldots, \lambda_K^i\right) \in \Delta_+^K$ of investors $i = 1, \ldots, I$ the equation

$$p_{t,k} = \rho \sum_{i=1}^{I} \lambda_k^i \langle d_t + p_t, x_{t-1}^i \rangle, \qquad k = 1, \ldots, K, \tag{2.1}$$

determines the market clearing price $p_{t,k}$ of asset k. The proof that this price exists can be found in [13]. The number of units of asset k in the investor i's portfolio at time t is equal to

$$x_{t,k}^i = \frac{\rho \lambda_k^i \langle d_t + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \qquad k = 1, \ldots, K, \quad i = 1, \ldots, I. \tag{2.2}$$

Wealth w_t^i of investor i at time t is denoted by

$$w_t^i := \langle d_t + p_t, x_{t-1}^i \rangle,$$

which is strictly positive. Total market wealth at time t is equal to

$$W_t = \sum_{i=1}^{I} w_t^i.$$

We are interested in the long-run behaviour of the *relative wealth(market shares)*, $r_t^i := \dfrac{w_t^i}{W_t}$ of the traders. From (2.1) and (2.2), we obtain

$$p_{t+1,k} = \rho \sum_{i=1}^{I} \lambda_k^i \langle d_{t+1} + p_{t+1}, x_t^i \rangle = \rho \sum_{i=1}^{I} \lambda_k^i w_{t+1}^i = \rho \langle \lambda_k, w_{t+1} \rangle,$$

$$x_{t,k}^i = \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle},$$

where $\lambda_k := \left(\lambda_k^1, \ldots, \lambda_k^I\right)$ and $w_t := \left(w_t^1, \ldots, w_t^I\right)$. Finally we have

$$w_{t+1}^i = \sum_{k=1}^{K} \left[\rho \langle \lambda_k, w_{t+1} \rangle + D_k\left(s_{t+1}\right)\right] \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}. \tag{2.3}$$

When we sum up these equations over $i = 1, \ldots, I$, we get

$$W_{t+1} = \sum_{k=1}^{K} \left[\rho \langle \lambda_k, w_{t+1} \rangle + D_k\left(s_{t+1}\right)\right] \frac{\sum_{i=1}^{I} \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}.$$

From last equation we have

$$W_{t+1} = \frac{D\left(s_{t+1}\right)}{1 - \rho}, \tag{2.4}$$

where $D\left(s_{t+1}\right) = \sum_{k=1}^{K} D_k\left(s_{t+1}\right)$ which is strictly positive. From (2.3) and (2.4), we find

$$r_{t+1}^i = \sum_{k=1}^{K} \left[\rho \big\langle \lambda_k, r_{t+1} \big\rangle + (1 - \rho) R_k\left(s_{t+1}\right)\right] \frac{\lambda_k^i r_t^i}{\big\langle \lambda_k, r_t \big\rangle}, \qquad i = 1, \ldots, I, \tag{2.5}$$

where

$$R_k\left(s_{t+1}\right) = \frac{D_k\left(s_{t+1}\right)}{D\left(s_{t+1}\right)}, \qquad k = 1, \ldots, K, \tag{2.6}$$

are the *relative(normalized) payoffs* of the assets. Define

$$\lambda_k^* := ER_k(s), \quad k = 1, \ldots, K, \tag{2.7}$$

where E is the expectation with respect to p. The portfolio rule defined by (2.7) distributes wealth across assets according to expected relative payoffs. In the theory of evolutionary finance there are three possibilities for investor i :

(1) Extinction; i.e., $\lim r_t^i = 0$ a.s.
(2) Survival; i.e., $\limsup r_t^i > 0$ but $\lim r_t^i < 1$ a.s.
(3) Domination; i.e., $\lim r_t^i = 1$ a.s.

3. Iterated Function Systems

To analyse the performance of investment strategies in the absence of "correct beliefs", i.e., in the absence of an investor using the Kelly rule, we invoke the theory of iterated function systems. In this section we define the notion of an iterated function system (IFS) and state some results from [5] for a certain class of IFS. Our ideas are inspired by [5], where techniques from IFS were applied to the model of short-lived assets of Evstigneev et al. [11]. Let τ_s, $s \in S$, be continuous transformations from the unit interval into itself which satisfy the following conditions:

(i) $\tau_s(0) = 0$ and $\tau_s(1) = 1$;
(ii) τ_s are strictly increasing.

Let p be a probability distribution on S such that $p_s > 0$ for all $s \in S$. The collection

$$F = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$$

is called an *iterated function system* (IFS) with probabilities. Let $r_t(s^t)$ denote.

$$r_t\left(s^t\right) := \tau_{s_t} \circ \tau_{s_{t-1}} \circ \ldots \circ \tau_{s_1}(r_0),$$

where $s^t := (s_1, s_2, \ldots, s_t), s_i \in S$. For such IFS, one can easily observe (see Lemma 4.2. [5]) that each constituent map of the IFS can be represented as follows:

$$\tau_s(r) = r^{\beta_s(r)},$$

with $\beta_s(r)$ satisfying;

(1) $\beta_s(r) > 0 \ \ in \ \ (0,1)$;
(2) $(lnr)\,\beta_s(r) \ \ increasing$;
(3) $lim_{r\to 0}(lnr)\,\beta_s(r) = -\infty$;
(4) $lim_{r\to 1}(lnr)\,\beta_s(r) = 0$.

Among other results the following proposition can be found in [5].

Proposition 3.1. *Let* $F = \{\tau_s; p_s\}_{s\in S}$ *be an IFS such that* $\tau_s(r) = r^{\beta_s(r)}$. *Assume that* $0 < b_s \leq \beta_s(r) \leq B_s < \infty$ *for all* $r \in [0,1]$. *If* $E\left(ln\alpha_t \mid s^{t-1}\right) \leq 0$ *a.s., then* $lim_{t\to\infty} r_t(s^t) \neq 0$ *a.s.*

4. Performance of Investment Strategies

In this section, we first verify that the evolution of the relative market wealth (2.5) can be represented by an IFS whose constituent maps satisfy the assumptions of [5]. Then, we use Proposition 3.1 to show that the investor who is closer to the

Kelly rule cannot be driven out of the market (see Theorem 4.4). We have the *relative wealth* of the investors given by

$$r_{t+1}^i = \sum_{k=1}^K \left[\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_k\left(s_{t+1}\right)\right]\frac{\lambda_k^i r_t^i}{\langle\lambda_k, r_t\rangle}, \qquad i = 1,\cdots,I. \tag{4.1}$$

We focus on the case of two investors. From equation (4.1), we obtain

$$r_{t+1} = \sum_{k=1}^K \left\{\rho\left[\overline{\lambda}_k(1-r_{t+1}) + \lambda_k r_{t+1}\right] + (1-\rho)R_k(s_{t+1})\right\}\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}, \tag{4.2}$$

where $\lambda = (\lambda_k)_{k=1}^K$ is the strategy of investor 1 whose relative wealth is r_{t+1} and $\overline{\lambda} = (\overline{\lambda}_k)_{k=1}^K$ is the strategy of investor 2 whose relative wealth is $1 - r_{t+1}$. Now from equation (4.2), we obtain

$$\begin{aligned} r_{t+1}\Big(1 - \sum_{k=1}^K \rho(\lambda_k - \overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}\Big) \\ = \sum_{k=1}^K \Big(\rho\overline{\lambda}_k + (1-\rho)R_k(s_{t+1})\Big)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}. \end{aligned} \tag{4.3}$$

Note that R.H.S. of the equation (4.3) is positive for all t. Then, L.H.S. of this equation is positive for all t. Since $r_{t+1} > 0$ for all t, we have

$$1 - \sum_{k=1}^K \rho(\lambda_k - \overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t} > 0. \tag{4.4}$$

Therefore, we can divide both sides of the equation (4.3) by (4.4) and we obtain

$$r_{t+1} = \frac{\sum_{k=1}^K \Big(\rho\overline{\lambda}_k + (1-\rho)R_k(s_{t+1})\Big)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}}{1 - \sum_{k=1}^K \rho(\lambda_k - \overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}}. \tag{4.5}$$

In conclusion, the above random dynamical system (4.5) can be represented by the IFS $F = \{\tau_s, p_s\}_{s\in S}$ where

$$\tau_s\left(r\right) = \frac{\sum_{k=1}^K \Big(\rho\overline{\lambda}_k + (1-\rho)\,R_k\left(s\right)\Big)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^K \rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} := \frac{A}{B}, \tag{4.6}$$

and $p = (p_s)$ is the distribution on S. Now we start checking that (4.6) satisfies the assumptions of [5].

Lemma 4.1.

(1) $\tau_s\left(0\right) = 0$, $\tau_s\left(1\right) = 1$.
(2) τ_s *is an increasing function which maps* $[0,1]$ *into itself.*
(3) τ_s *is an continuous function on* $[0,1]$.

Proof. (1) $\tau_s(0) = 0$ is obvious.

$$\tau_s(1) = \frac{\sum_{k=1}^{K} \left(\rho\overline{\lambda}_k + (1-\rho) R_k(s)\right) \frac{\lambda_k}{\lambda_k}}{1 - \sum_{k=1}^{K} \rho\left(\lambda_k - \overline{\lambda}_k\right) \frac{\lambda_k}{\lambda_k}}$$

$$= \frac{\rho \sum_{k=1}^{K} \overline{\lambda}_k + (1-\rho) \sum_{k=1}^{K} R_k(s)}{1 - \sum_{k=1}^{K} \rho\left(\lambda_k - \overline{\lambda}_k\right)}$$

$$= \frac{\rho + 1 - \rho}{1} = 1.$$

(2) Let $g_s(r) = \dfrac{\tau_s(r)}{1 - \tau_s(r)}$. Note that $g_s(r)$ is increasing $\Longleftrightarrow$ $\tau_s(r)$ is increasing. Thus, it is enough to show that $g_s(r)$ is increasing.

$$g_s(r) = \frac{\frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}}{1 - \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}}$$

$$= \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^{K}\left(\rho\lambda_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} = \frac{f_s}{h_s}.$$

Observe that

$$f_s(r) = \sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}$$

$$= \sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k}{\overline{\lambda}_k\frac{(1-r)}{r}+\lambda_k}$$

increases as r increases. Moreover,

$$h_s(r) = 1 - \sum_{k=1}^{K}\left(\rho\lambda_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}.$$

Since $\left(\rho\lambda_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}$ increases, $h_s(r)$ decreases. Therefore, $g_s(r)$ increases.

(3) The continuity of τ_s as a function of r is obvious.

□

Lemma 4.2. *Let*

$$\tau(r) = \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}, \qquad r \in [0,1],$$

and

$$\tau(r) = r^{\beta(r)}.$$

Then, for any $r \in [0,1]$, $\ln(\beta(r))$ *is bounded.*

Proof. We have

$$\tau(r) = r^{\beta(r)} = \exp(\ln(r)\beta(r)).$$

Consequently,

$$\beta(r) = \frac{\ln(\tau_r)}{\ln(r)}.$$

The minimum and maximum of $\beta(r)$ can be attained at $r = 0$, $r = 1$ or at a point of local extremum. We apply De L'Hospital rule to find the $\lim_{r\to 0^+}\beta(r)$ and $\lim_{r\to 1^-}\beta(r)$.

$$\begin{aligned}
\lim_{r\to 0^+}\frac{\ln(\tau_r)}{\ln(r)} &= \lim_{r\to 0^+}\frac{\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2}\right\}B}{AB}r \\
&+ \frac{\left\{\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2}\right\}A}{AB}r \\
&= \frac{\sum_{k=1}^{K}(\rho\lambda_k + (1-\rho)R_k)\frac{\lambda_k}{\overline{\lambda}_k}}{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k}{\overline{\lambda}_k}}.
\end{aligned}$$

$$\begin{aligned}
\lim_{r\to 1^-}\frac{\ln(\tau_r)}{\ln(r)} &= \lim_{r\to 1^-}\frac{\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2}\right\}B}{AB}r \\
&+ \frac{\left\{\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2}\right\}A}{AB}r \\
&= \frac{\sum_{k=1}^{K}(\rho\lambda_k + (1-\rho)R_k)\frac{\overline{\lambda}_k}{\lambda_k}}{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)}.
\end{aligned}$$

By solving the following equation

$$\beta^{'}(r) = \frac{1}{\ln(r)}\left(\frac{\tau^{'}(r)}{\tau(r)} - \frac{\beta(r)}{r}\right) = 0, \tag{4.7}$$

we find a local extremum r^* in $(0,1)$ of $\beta(r)$. From the equation (4.7), we get

$$\beta(r) = r\frac{\tau^{'}(r)}{\tau(r)}.$$

We show that

$$\beta(r^*) = r^*\frac{\tau^{'}(r^*)}{\tau(r^*)}$$

is continuous on $[0,1]$. We know that $\tau(r)$ is continuous on $[0,1]$ and $\dfrac{r}{\tau(r)} \neq 0$. Therefore, it remains to show that $\tau^{'}(r^*)$ is continuous on $[0,1]$.

$$\tau^{'}(r^*) = \frac{\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r^*)+\lambda_k r^*\right\}^2}\right\}}{\left\{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r^*}{\overline{\lambda}_k(1-r^*)+\lambda_k r^*}\right\}^2}\times$$
$$\left\{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r^*}{\overline{\lambda}_k(1-r^*)+\lambda_k r^*}\right\}$$
$$+\frac{\left\{\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k\overline{\lambda}_k}{\left\{\overline{\lambda}_k(1-r^*)+\lambda_k r^*\right\}^2}\right\}}{\left\{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r^*}{\overline{\lambda}_k(1-r^*)+\lambda_k r^*}\right\}^2}\times$$
$$\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k+(1-\rho)R_k\right)\frac{\lambda_k r^*}{\overline{\lambda}_k(1-r^*)+\lambda_k r^*}\right\},$$

which is obviously continuous. Therefore, $\beta(r^*)$ is continuous on $[0,1]$. Thus, it attains its maximum and minimum on $[0,1]$. Furthermore, the minimum of $\beta(r^*)$ is strictly positive because

$$\beta(r^*) = \frac{r^*}{\tau(r^*)}\tau^{'}(r^*) > 0.$$

□

Up to this point, we verified that our IFS satisfies the assumptions of [5]. We now state and prove our main result. From now on, we impose the following condition.

Assumption 3. We assume that for $k \in \{1,\dots,K\}$

$$\left\{\begin{array}{rl}\text{either} & \overline{\lambda}_k \le \lambda_k \le \lambda_k^*,\\ \text{or} & \lambda_k^* \le \lambda_k \le \overline{\lambda}_k.\end{array}\right. \tag{4.8}$$

Assumption 3 means that the investment strategy of investor 1 is closer (coordinatewise) than that of investor 2 to the Kelly rule. Before proving our main result (Theorem 4.4) we state and prove the following technical lemma.

Lemma 4.3. *The function*

$$G(r) = \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k+(1-\rho)\lambda_k^*\right)\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} \ge 1, \tag{4.9}$$

for $r \in [0,1]$.

Proof.

$$G(r) \ge 1$$
$$\Longleftrightarrow \sum_{k=1}^{K}\left(\rho\overline{\lambda}_k+(1-\rho)\lambda_k^*\right)\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r} \ge$$
$$1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}$$

$$\iff H\left(r\right) := 1 - \sum_{k=1}^{K} \left\{\rho\left(\lambda_k - \overline{\lambda}_k\right) r + \rho\overline{\lambda}_k + \left(1-\rho\right)\lambda_k^*\right\} \frac{\lambda_k}{\overline{\lambda}_k\left(1-r\right) + \lambda_k r} \leq 0.$$

Since $H\left(1\right) = 0$, it is enough to show that $H\left(r\right)$ is increasing. We have

$$H^{'}\left(r\right) = \sum_{k=1}^{K} \frac{\left(1-\rho\right)\lambda_k^*\lambda_k\left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2}. \tag{4.10}$$

- For $\overline{\lambda}_k \leq \lambda_k \leq \lambda_k^*$, we have

$$\begin{aligned} \frac{\lambda_k^*\lambda_k}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} &\geq \frac{\lambda_k^*\lambda_k}{\left(\max\{\lambda_k, \overline{\lambda}_k\}\right)^2} \\ &\geq \frac{\lambda_k^*\lambda_k}{\left(\lambda_k\right)^2} = \frac{\lambda_k^*}{\lambda_k} \geq 1 \end{aligned}$$

$$\frac{\lambda_k^*\lambda_k\left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} \geq \lambda_k - \overline{\lambda}_k. \tag{4.11}$$

- For $\lambda_k^* \leq \lambda_k \leq \overline{\lambda}_k$, we have

$$\begin{aligned} \frac{\lambda_k^*\lambda_k}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} &\leq \frac{\lambda_k^*\lambda_k}{\left(\min\{\lambda_k, \overline{\lambda}_k\}\right)^2} \\ &= \frac{\lambda_k^*\lambda_k}{\left(\lambda_k\right)^2} \leq 1 \end{aligned}$$

$$\frac{\lambda_k^*\lambda_k\left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} \geq \lambda_k - \overline{\lambda}_k. \tag{4.12}$$

From (4.11) and (4.12), for all k, we have

$$\frac{\lambda_k^*\lambda_k\left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} \geq \lambda_k - \overline{\lambda}_k.$$

Consequently,

$$\sum_{k=1}^{K} \frac{\left(1-\rho\right)\lambda_k^*\lambda_k\left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k\left(1-r\right) + \lambda_k r\right\}^2} \geq 0.$$

So, the function $H\left(r\right)$ is increasing and therefore, the function $H\left(r\right) \leq 0$.

□

Theorem 4.4. *Under assumption* (3) *investor 1 cannot be driven out of the market. He/she will either dominate or at least survive.*

Proof. Let us consider the expression

$$\sum_{s=1}^{L} p_s \ln\left(\beta_s(r)\right) \le \ln\left(\sum_{s=1}^{L} p_s \beta_s(r)\right) = \ln\left(\sum_{s=1}^{L} p_s \frac{\ln(\tau_s(r))}{\ln(r)}\right)$$

$$\le \ln\left(\frac{1}{\ln r}\ln\left(\sum_{s=1}^{L} p_s \tau_s(r)\right)\right)$$

(4.13)

$$= \ln\left(\frac{1}{\ln r}\ln\left(\sum_{s=1}^{L} p_s \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)\,R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right)$$

$$= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(\frac{\sum_{k=1}^{K}\sum_{s=1}^{L} p_s\left[\rho\overline{\lambda}_k + (1-\rho)\,R_k(s)\right]\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right)\right)$$

$$= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(\frac{\sum_{k=1}^{K}\left[\rho\overline{\lambda}_k + (1-\rho)\,\lambda_k^*\right]\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right)\right)$$

$$= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(G(r)\right)\right)\right)$$

$$= \ln\left(1 + \frac{1}{\ln r}\ln\left(G\left(r\right)\right)\right) \le 0.$$

In the last inequality we used the fact that which was proved in Lemma 4.3 $G\left(r\right) \ge 1$. Since the stochastic process s_t is an independent, identically distributed process, we have, by (4.13),

$$E\left(\ln \alpha_t \mid s^{t-1}\right) = \sum_{s=1}^{L} p_s \ln\left(\beta_s\left(r_{t-2}\right)\right) \le 0.$$

Therefore, by Proposition 3.1, $\lim_{t\to\infty} r_t\left(s^t\right) \ne 0$ a.s. This means that investor 1 either dominates or at least survives. □

5. Example

We illustrate our result in Theorem 4.4 by considering a numerical example of the market model with long-lived assets. Assume that we have 3 assets and 4 states of the world, i.e., $K = 3$, $S = \{1, 2, 3, 4\}$. The relative payoff matrix is given by

$$R = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 2/4 & 1/4 \\ 1/5 & 2/5 & 2/5 \\ 1/6 & 2/6 & 3/6 \end{pmatrix}.$$

Suppose that the probability distributions $p_s = \frac{1}{4},\ s = 1, 2, 3, 4$. Then the Kelly rule is given by

$$\lambda_k^* = \left(19/80, 47/120, 89/240\right).$$

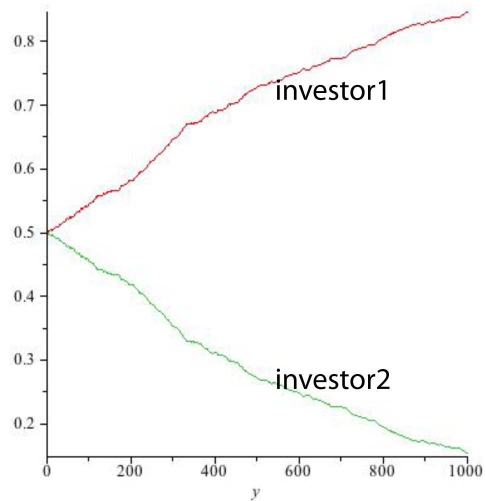

Figure 1. Dynamics of relative wealth share of the example of Section 5

There are two investors in the market and both use simple strategies. Assume, investor 1 uses the strategy

$$\lambda^1 = (5/24, 19/48, 19/48)$$

and investor 2 uses the strategy

$$\lambda^2 = (1/5, 2/5, 2/5)\,.$$

Observe that the strategy of investor 1 is closer than that of investor 2 to the Kelly rule. Thus, Assumption 3 is satisfied for this example. The initial wealth share is given by $r_0 = (1/2, 1/2)$. We have used Maple to simulate the dynamics of the relative market shares of investor 1 and investor 2.
In Figure 4, the red line denotes the relative wealth of investor 1 and the green line denotes the relative wealth of investor 2. The result in Figure 4 coincides with the result of Theorem 4.4; i.e., investor 1 will dominate or at least survive.

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