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# On the use of the Wirtinger inequalities for time-delay systems

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**Abstract:** The paper addresses the stability problem of linear time delay system. In the literature, the most popular approach to tackle this problem relies on the use of Lyapunov-Krasovskii functionals. Many results have proposed new functionals and techniques for deriving less and less conservative stability conditions. Nevertheless, all these approaches use the same trick, the well-known Jensen's inequality which generally induces some conservatism difficult to overcome. In light of those observations, we propose to reduce the conservatism of Lyapunov-Krasovskii functionals by introducing new classes of integral inequalities called Wirtinger inequalities. This integral type inequality is firstly shown to encompass Jensen's inequality and is then employed to derive new stability conditions. To this end, a slightly modified Lyapunov functional is proposed. Several examples illustrate the effectiveness of our methodology.

Keywords: Time-delay systems, stability analysis, Lyapunov

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## 1. INTRODUCTION

The last decade has shown an increasing research activity on time-delay systems analysis and control due to both emerging adapted theoretical tools and also practical issues in the engineering field and information technology as well as for biology, economics or ecology (see Sipahi et al. [2011] for a recent survey).

In the case of linear time delay system with a constant delay, three different techniques allow to derive efficient criteria proving the stability of such a system. The first classical methodology relies on the study of the roots of the associated characteristic equation, a quasi-polynomial in  $s$  and  $e^{-hs}$ . Even very effective in practice (see the monograph Gu et al. [2003] and the survey Sipahi et al. [2011]), these technics generally lead to a perfect description of the stability with respect to the delay. Nevertheless, these methods reveal themselves restrictive in the sense that they cannot be extended straightforwardly to the robust case and to the case of time varying delay.

Another important technic comes from the robust analysis framework. In that case, the basic idea is pull out the delay element from a nominal system and to merge it into a unstructured uncertainty. The original time delay system is then modeled as nominal system without delay submitted to a perturbation. The use of classical robustness tools like Small Gain theorem (Niculescu and Chen [1999], Zhang et al. [2001]), IQC approach (Kao and Rantzer [2007]) or quadratic separation approach (Ariba et al. [2010]) allow then to develop effective criteria. The sources of conservatism are then twofold: the way to model the interconnection between the nominal system and the delay uncertainty and the accuracy of the delay covering set (the more precise the uncertainty set is, the less conservative is

the criterion).

The third approach and the most popular remains the use a Lyapunov-Krasovskii functional. In the linear case, its general structure is perfectly known but is numerically difficult to handle (see Gu et al. [2003] for a concise introduction). According to the important literature devoted to this subject (Ariba and Gouaisbaut [2009], Kim [2011], Shao [2009], Sun et al. [2010]), conservative results are then provided often expressed in terms of LMIs if some additional hypothesis are formulated on the Lyapunov functional. The challenge is then to reduce the conservatism of such approaches by either choosing extended state based Lyapunov-Krasovskii functional (Ariba and Gouaisbaut [2009], Kim [2011]) and/or discretized Lyapunov functional (Gu et al. [2003]). Furthermore, apart the choice of an appropriate Lyapunov-Krasovskii  $V$ , an important source of conservatism is the way to bound some cross terms arisen when manipulating the derivative of  $V$ . According to the literature on this subject (see Park et al. [2011], He et al. [2007], Shao [2009] for some recent papers), a common feature of all these techniques is the use of slack variables (Jiang and Han [2006]) and Jensen inequality (Shao [2009], Sun et al. [2010]). At the price of an increasing conservatism, this last inequality allows to get some LMIs which may be solved efficiently with Semi-Definite Programming (SDP) solvers.

In this preliminary work, we aim at reducing the conservatism of Lyapunov-Krasovskii techniques by considering an accurate integral inequality which includes the Jensen's one as a special case : the Wirtinger inequality. This new class of inequalities has been already employed in the

stability of sampled-data systems by Liu and Fridman [2012]. In this paper, its use combined with some special properties of sampled data systems has led to some interesting criteria expressed in terms of LMIs, which are less conservative at least on examples. Following the idea of Liu and Fridman [2012], we firstly propose a new inequality which is shown to be less conservative than previous inequalities often based on Jensen's theorem. The resulting inequality depends not only on  $x(t), x(t-h)$  but also on  $\int_{t-h}^t x(s)ds$ . This last signal is then directly integrated into a new suitable Lyapunov-Krasovskii functional, highlighting so the features of Wirtinger inequality. This first LMI criterion is extended to the case of delay range stability condition (the delay  $h$  is belonging to a prescribed interval  $[h_{min}, h_{max}]$ ). This last condition could then detect some pockets of stability even in case of unstable delay-free systems.

**Notations:** Throughout the paper  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space with vector norm  $|\cdot|$ ,  $\mathbb{R}^{n \times m}$  is the set of all  $n \times m$  real matrices, and the notation  $P > 0$ , for  $P \in \mathbb{R}^{n \times n}$ , means that  $P$  is symmetric and positive definite. The symmetric matrix  $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$  stands for  $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ . The space of functions  $\phi : [a, b] \rightarrow \mathbb{R}^n$ , which are absolutely continuous on  $[a, b]$ , have a finite  $\lim_{\theta \rightarrow b^-} \phi(\theta)$  and have square integrable first order derivatives is denoted by  $W[a, b]$ .

## 2. PRELIMINARIES

Looking at the huge literature devoted to the stability analysis of time-delay systems, the use of Lyapunov-Krasovskii functionals generally leads to some matrix inequalities quite difficult to handle. In order to get numerically tractable inequalities (generally some LMIs to be optimized), researchers have been extensively used the well-known Jensen's inequality which is recalled in the following lemma.

*Lemma 1.* For given symmetric positive definite matrices  $R > 0$  and for any differentiable signal  $\omega$  in  $[a, b] \rightarrow \mathbb{R}^n$ , the following inequality holds:

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \frac{1}{b-a} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \begin{bmatrix} R & -R \\ * & R \end{bmatrix} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \quad (1)$$

In the context of time-delay systems, this inequality has been the core of several important contributions (Gu et al. [2003], He et al. [2007]) although an inherent conservatism which has studied since several years in a Lyapunov-Krasovskii or robust analysis context (see for instance Briat [2011] and Gouaisbaut and Peaucelle [2007]). In the present article, we aim at improving the resulting stability tests by employing a new class of integral inequalities called Wirtinger inequalities, which encompass the Jensen's inequality. In control theory, these types of inequalities have been successfully employed in Liu and Fridman [2012] in the case of sampled data systems and have shown to reduce drastically the conservatism. In this paper Liu and Fridman [2012], the authors used the following inequality

*Lemma 2.* Let  $z \in W[a, b]$  and  $z(a) = 0$ . Then for any  $n \times n$  matrix  $R > 0$ , the following inequality holds

$$\int_a^b z^T(s)Rz(s)ds \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{z}^T(s)R\dot{z}(s)ds \quad (2)$$

There exists also another Wirtinger inequality that can be found in Kammler [2007], which leads to a more precise inequality using a stronger assumption on the function  $z$ . Indeed, the previous inequality can be refined as follows:

*Lemma 3.* Let  $z \in W[a, b]$  and  $z(a) = z(b) = 0$ . Then for any  $n \times n$  matrix  $R > 0$ , the following inequality holds

$$\int_a^b z^T(s)Rz(s)ds \leq \frac{(b-a)^2}{\pi^2} \int_a^b \dot{z}^T(s)R\dot{z}(s)ds \quad (3)$$

In this paper, we aim at understanding how these inequalities can help in the context of assessing stability of time-delay systems. In particular, in the next section, we propose a novel inequality which is proved to be less conservative than the Jensen's one.

## 3. APPLICATION OF THE WIRTINGER'S INEQUALITIES

### 3.1 How to encompass Jensen's inequality?

In the sequel, we present a first new inequality that allows to derive less conservative stability conditions for time delay systems.

*Lemma 4.* For a given symmetric positive definite matrix  $R > 0$  and any differentiable signal  $\omega$  in  $[a, b] \rightarrow \mathbb{R}^n$ , then the following inequality holds:

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \int_a^b \Omega^T(u) \frac{6W_0(R) + \pi^2 W_1(u)}{6(b-a)^2} \Omega(u)du \quad (4)$$

where  $\Omega = [\omega^T(b) \ \omega^T(a) \ \omega^T(u)]^T$  and

$$W_0(R) = \begin{bmatrix} R & -R & 0 \\ * & R & 0 \\ * & * & 0 \end{bmatrix}, \quad W_1(u) = \begin{bmatrix} 2R & R & -6\frac{(u-a)}{(b-a)}R \\ * & 2R & -6\frac{(b-u)}{(b-a)}R \\ * & * & 6R \end{bmatrix}.$$

*Proof :* The proof is based on the second Wirtinger inequality. Consider the function

$$z(u) = (b-u)\omega(a) + (u-a)\omega(b) - (b-a)\omega(u)$$

where  $u \in [a, b]$ . As the function  $z(u)$  satisfies the conditions of Lemma 3, i.e.  $z(a) = z(b) = 0$ , we apply the second Wirtinger inequality to  $z(u)$ . The derivative of  $z$  with respect to  $u$  is given by:

$$\frac{d}{du}z(u) = \omega(b) - \omega(a) - (b-a)\dot{\omega}(u),$$

and the right-hand-side of the Wirtinger inequality can be rewritten as:

$$\begin{aligned} \int_a^b \dot{z}^T(u)R\dot{z}(u)du &= (b-a)^2 \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \\ &\quad - \int_a^b (\omega(b) - \omega(a))^T R (\omega(b) - \omega(a))du. \end{aligned} \quad (5)$$

Consider now the left-hand side of the Wirtinger inequality. Developing this expression leads to:

$$\begin{aligned}
\int_a^b z^T(u)Rz(u)du &= \int_a^b (b-u)^2\omega^T(a)R\omega(a)du \\
&+ \int_a^b (u-a)^2\omega^T(b)R\omega(b)du \\
&+ 2\int_a^b (b-u)(u-a)\omega^T(a)R\omega(b)du \\
&- 2(b-a)\omega^T(b)R\int_a^b (u-a)\omega(u)du \\
&- 2(b-a)\omega^T(a)R\int_a^b (b-u)\omega(u)du \\
&+ (b-a)^2\int_a^b \omega^T(u)R\omega(u)du,
\end{aligned} \tag{6}$$

or equivalently,

$$\begin{aligned}
\int_a^b z^T(u)Rz(u)du &= \frac{(b-a)^3}{3}\omega^T(a)R\omega(a) \\
&+ \frac{(b-a)^3}{3}\omega^T(b)R\omega(b) + 2\frac{(b-a)^3}{6}\omega^T(a)R\omega(b) \\
&- 2(b-a)\omega^T(b)R\int_a^b (u-a)\omega(u)du \\
&- 2(b-a)\omega^T(a)R\int_a^b (b-u)\omega(u)du \\
&+ (b-a)^2\int_a^b \omega^T(u)R\omega(u)du.
\end{aligned} \tag{7}$$

This last equality can be rewritten as

$$\int_a^b z^T(u)Rz(u)du = \frac{(b-a)^2}{6}\int_a^b \Omega^T(u)W_1(u)\Omega(u)du. \tag{8}$$

We conclude the proof by applying the Wirtinger inequality which leads to

$$\begin{aligned}
\frac{\pi^2}{6(b-a)^2}\int_a^b \Omega^T(u)W_1(u)\Omega(u)du &\leq \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \\
&- \frac{1}{(b-a)^2}\int_a^b (\omega(b) - \omega(a))^T R(\omega(b) - \omega(a))du,
\end{aligned} \tag{9}$$

which is equivalent to the inequality proposed in Lemma 4.  $\diamond$

The previous lemma presents an improved version of the Jensen's inequality. Effectively, by splitting the integral given in the right-hand side of the inequality (4), we can highlight the difference with the Jensen's inequality:

$$\begin{aligned}
\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du &\geq \frac{1}{(b-a)}\begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \begin{bmatrix} R & -R \\ * & R \end{bmatrix} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\
&+ \frac{\pi^2}{6(b-a)^2}\int_a^b \Omega^T(u)W_1(u)\Omega(u)du.
\end{aligned} \tag{10}$$

As equation (8) ensure that the last term of the previous inequality is positive, Lemma 4 gives a more accurate results than the Jensen's inequality.

However, a simple inspection of the right-hand side of the inequality (10) shows that this new inequality is difficult to handle since the matrix  $W_1$  is polynomial of the integration variable. It is therefore not directly suitable for assessing the stability of time delay systems via numerical tools like LMIs. In the following, a more practical inequality is derived using a method based on integral manipulations

introduced in the discretization method from Gu et al. [2003].

### 3.2 An appropriate inequality

As mentioned above, we cannot use the Lemma 4 to derive directly a numerically tractable stability criterion. In this subsection, we propose a more suitable formulation by finding a lower bound expressed only in terms of signals  $\omega(b)$ ,  $\omega(a)$  and  $\frac{1}{h}\int_a^b \omega(u)du$ :

*Lemma 5.* For given symmetric positive definite matrices  $R > 0$  and for any differentiable signal  $\omega$  in  $[a, b] \rightarrow \mathbb{R}^n$ , the following inequality holds:

$$\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}\begin{bmatrix} \omega(b) \\ \omega(a) \\ \nu \end{bmatrix}^T W_2(R)\begin{bmatrix} \omega(b) \\ \omega(a) \\ \nu \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}
\nu &= \frac{1}{b-a}\int_a^b \omega(u)du, \\
W_2(R) &= W_0(R) + \frac{\pi^2}{4}\begin{bmatrix} R & R & -2R \\ * & R & -2R \\ * & * & 4R \end{bmatrix}.
\end{aligned}$$

*Proof :* Consider the inequality (10), which is satisfied for any symmetric positive definite matrix  $R$  and any function  $\omega$ . We aim at finding an appropriate lower bound of the right-hand side of this inequality by studying only the last integral of inequality (10),

$$\int_a^b \Omega^T(u)W_1(u)\Omega(u)du,$$

which can be rewritten as follows

$$\mathcal{I} = \int_a^b \begin{bmatrix} \omega(b) \\ \omega(a) \\ \omega(u) \end{bmatrix}^T \begin{bmatrix} W_1^1 & W_1^2(u) \\ * & 6R \end{bmatrix} \begin{bmatrix} \omega(b) \\ \omega(a) \\ \omega(u) \end{bmatrix} du \tag{12}$$

where

$$W_1^1 = \begin{bmatrix} 2R & R \\ * & 2R \end{bmatrix}, \quad W_1^2(u) = -\frac{6}{(b-a)}\begin{bmatrix} (u-a)R \\ (b-u)R \end{bmatrix}$$

In order to find a lower-bound, we apply a similar transformation proposed in Gu et al. [2003] (Proposition 5.21). For any  $\epsilon > 0$ , consider the integral

$$\mathcal{I}^* = \int_a^b \xi^T(u)\begin{bmatrix} \frac{1+\epsilon}{6}R^{-1} & I \\ * & 6R \end{bmatrix}\xi(u)du,$$

where

$$\xi(u) = \begin{bmatrix} W_1^{2T}(u)\begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ \omega(u) \end{bmatrix}.$$

By virtue of the Schur complement, this choice of  $\epsilon$  ensures that

$$\begin{bmatrix} \frac{1+\epsilon}{6}R^{-1} & I \\ * & 6R \end{bmatrix} > 0.$$

The integral  $\mathcal{I}^*$  is then strictly positive for all  $\epsilon > 0$ . Developing the previous integral we get

$$\begin{aligned}\mathcal{I}^* &= \frac{1+\epsilon}{6} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \int_a^b W_1^2(u) R^{-1} W_1^{2T}(u) du \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ &+ 2 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \int_a^b W_1^2(u) \omega(u) du \\ &+ 6 \int_a^b \omega^T(u) R \omega(u) du.\end{aligned}$$

It yields that

$$\begin{aligned}\mathcal{I} &= \mathcal{I}^* + (b-a) \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T W_1^1 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ &- \frac{1+\epsilon}{6} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \int_a^b W_1^2(u) R^{-1} W_1^{2T}(u) du \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix},\end{aligned}$$

Furthermore, simple computations show that

$$\begin{aligned}&\int_a^b W_1^2(u) R^{-1} W_1^{2T}(u) du \\ &= \frac{36}{(b-a)^2} \begin{bmatrix} \int_a^b (u-a)^2 du R & \int_a^b (u-a)(b-u) du R \\ * & \int_a^b (b-u)^2 du R \end{bmatrix} \\ &= \frac{36(b-a)^3}{6(b-a)^2} W_1^1 = 6(b-a) W_1^1,\end{aligned}$$

and consequently,

$$\begin{aligned}\mathcal{I} &= \mathcal{I}^* + (b-a) \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T W_1^1 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ &- (1+\epsilon)(b-a) \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T W_1^1 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ \mathcal{I} &= \mathcal{I}^* - \epsilon(b-a) \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T W_1^1 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}.\end{aligned}$$

Following the same method proposed in Gu et al. [2003], it is possible to apply the Jensen's inequality to the integral  $\mathcal{I}^*$ . This leads to

$$\mathcal{I}^* \geq \frac{1}{b-a} \left( \int_a^b \xi(u) du \right)^T \begin{bmatrix} \frac{1+\epsilon}{6} R^{-1} & I \\ * & 6R \end{bmatrix} \left( \int_a^b \xi(u) du \right)$$

and noting that

$$\int_a^b \xi(u) du = \begin{bmatrix} -3(b-a) \begin{bmatrix} R & R \end{bmatrix} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ \int_a^b \omega(u) du \end{bmatrix},$$

we have

$$\begin{aligned}\mathcal{I}^* &\geq \frac{9(1+\epsilon)(b-a)}{6} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ &- 6 \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \begin{bmatrix} R \\ R \end{bmatrix} \int_a^b \omega(u) du \\ &+ \frac{6}{b-a} \left( \int_a^b \omega(u) du \right)^T R \left( \int_a^b \omega(u) du \right).\end{aligned}$$

Now, introducing the vector

$$\Xi = \frac{1}{b-a} \int_a^b \Omega(u) du = \begin{bmatrix} \omega(b) \\ \omega(a) \\ \frac{1}{b-a} \int_a^b \omega(u) du \end{bmatrix},$$

we have that, for all  $\epsilon > 0$

$$\begin{aligned}\mathcal{I} &- \frac{3(b-a)}{2} \Xi^T \begin{bmatrix} R & R & -2R \\ * & R & -2R \\ * & * & 4R \end{bmatrix} \Xi \\ &\geq \frac{\epsilon(b-a)}{2} \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix}^T \left( 3 \begin{bmatrix} R & R \\ * & R \end{bmatrix} - 2W_1^1 \right) \begin{bmatrix} \omega(b) \\ \omega(a) \end{bmatrix} \\ &\geq -\frac{\epsilon(b-a)}{2} (\omega(b) - \omega(a))^T R (\omega(b) - \omega(a))\end{aligned}$$

Note that the left-hand side of the previous inequality does not depend on the parameter  $\epsilon$  while the right-hand does. The previous inequality holds for all  $\epsilon > 0$ . Then taking  $\epsilon \rightarrow 0$  leads to the following inequality.

$$\mathcal{I} \geq \frac{3(b-a)}{2} \Xi^T \begin{bmatrix} R & R & -2R \\ * & R & -2R \\ * & * & 4R \end{bmatrix} \Xi$$

Replacing  $\mathcal{I}$  by its lower bound into (10) proves the results.  $\diamond$

At this stage, we have provided a useful inequality which has been proved to be less conservative than Jensen's inequality. The next section is devoted to the construction of a new stability criterion.

#### 4. STABILITY ANALYSIS OF SYSTEMS WITH CONSTANT DELAY

##### 4.1 A Lyapunov-Krasovskii functional approach

We present in this sub-section the main result of the article which is based on the use of the Wirtinger's inequality developed in section 3. This approach is based on a modified Lyapunov-Krasovskii functional and allows us to establish the main theorem for the robust delay range stability analysis. Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h) & \forall t \geq 0, \\ x(t) = \phi(t) & \forall t \in [-h_{\max}, 0], \end{cases} \quad (13)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is the initial condition and  $A, A_d \in \mathbb{R}^{n \times n}$  are constant matrices. The delay  $h$  is a positive scalar and satisfies the constraints:

$$h \in [h_{\min}, h_{\max}] \quad (14)$$

where  $h_{\min}, h_{\max}$  are given positive constants. In the following, we aim at assessing the stability of system (13) with delay constraints (14) via the an appropriate Lyapunov-Krasovskii functional. Based on the previous inequality, the following theorem is provided.

*Theorem 6.* For a given constant delay  $h$ , assume that there exist  $n \times n$  matrices  $P = P^T > 0$ ,  $S = S^T > 0$  and  $R = R^T > 0$

$$\Pi_1(h) > 0, \quad \Pi_2(h) < 0 \quad (15)$$

with

$$\Pi_1(h) = \begin{bmatrix} P & Q \\ * & Z + S/h \end{bmatrix}, \quad (16)$$

$$\Pi_2(h) = \Pi_2^0(h) - \frac{1}{h} W_2(R), \quad (17)$$

$$\Pi_2^0 = \begin{bmatrix} \Delta_2^0 & P A_d - Q & h A^T Q + h Z \\ * & -S & h A_d^T Q - h Z \\ * & * & 0 \end{bmatrix} + \begin{bmatrix} A^T \\ A_d^T \\ 0 \end{bmatrix} h R \begin{bmatrix} A^T \\ A_d^T \\ 0 \end{bmatrix}$$

where  $\Delta_2^0 = PA + A^T P + S + Q + Q^T$  and the matrix  $W_2(R)$  is given in Lemma 5, then the system (13) is asymptotically stable for the constant delay  $h$ .

*Proof* : Consider a Lyapunov-Krasovskii functional of the form

$$V(x_t, \dot{x}_t) = \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s)ds \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s)ds \end{bmatrix} + \int_{t-h}^t x(s)Sx(s)ds + \int_{t-h}^t (h-t+s)\dot{x}(s)R\dot{x}(s)ds \quad (18)$$

The previous functional is the one of the simplest type of functionals to derive delay-dependent stability conditions. First of all, following Gu et al. [2003] and using Jensen inequality, a lower-bound for  $V$  can be easily found:

$$V(x_t, \dot{x}_t) \geq \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s)ds \end{bmatrix}^T \Pi_1(h) \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s)ds \end{bmatrix} + \int_{t-h}^t (h-t+s)\dot{x}(s)R\dot{x}(s)ds, \quad (19)$$

and it is clear that the positive definiteness of the matrices  $P$ ,  $S$ ,  $R$  and  $\Pi_1(h)$  implies the positive definiteness of the functional  $V$ . The derivative of the functional along the trajectories of the system (13) leads to

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) &= 2x^T(t)P[Ax(t) + A_d x(t-h)] \\ &+ 2x^T(t)Q(x(t) - x(t-h)) \\ &+ 2[Ax(t) + A_d x(t-h)]^T Q \int_{t-h}^t x(s)ds \\ &+ 2(x(t) - x(t-h))^T Z \int_{t-h}^t x(s)ds \\ &+ x^T(t)Sx(t) - x^T(t-h)Sx(t-h) \\ &+ h[Ax(t) + A_d x(t-h)]^T R[Ax(t) + A_d x(t-h)] \\ &- \int_{-h}^0 \dot{x}^T(t+s)R\dot{x}(t+s)ds. \end{aligned}$$

The previous equality can be reformulated as

$$\dot{V}(x_t, \dot{x}_t) = \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{1}{h} \int_{t-h}^t x(s)ds \end{bmatrix}^T \Pi_2^0(h) \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{1}{h} \int_{t-h}^t x(s)ds \end{bmatrix} - \int_{-h}^0 \dot{x}^T(t+s)R\dot{x}(t+s)ds.$$

According to Lemma 5, the following upper-bound of the derivative of the functional is then obtained:

$$\dot{V}(x_t, \dot{x}_t) = \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{1}{h} \int_{t-h}^t x(s)ds \end{bmatrix}^T \Pi_2(h) \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{1}{h} \int_{t-h}^t x(s)ds \end{bmatrix} \quad (20)$$

where  $\Pi_2(h)$  is defined in (17). Then if the condition (15) from Theorem 6 are satisfied, the solutions of the system (13) are asymptotically stable.  $\diamond$

#### 4.2 Delay range stability criterion

At this stage, the proposed criterion of Theorem 6 can perform the stability analysis only for a given delay  $h$  and not for a delay satisfying the constraints (14)  $h \in [h_{\min}, h_{\max}]$ . Nevertheless, this can be done by modifying

slightly the LMIs conditions as proposed in the following theorem.

*Theorem 7.* For a given constant delay  $h$  satisfying the delays constraints (14)  $h \in [h_{\min}, h_{\max}]$ , assume that there exist  $n \times n$  matrices  $P = P^T > 0$ ,  $S_1 = S_1^T > 0$  and  $R_1 = R_1^T > 0$

$$\tilde{\Pi}_1 > 0, \quad \tilde{\Pi}_2(h_{\min}) < 0, \quad \tilde{\Pi}_2(h_{\max}) < 0, \quad (21)$$

with

$$\tilde{\Pi}_1 = \begin{bmatrix} P & Q \\ * & Z + S_1 \end{bmatrix}, \quad \tilde{\Pi}_2(h) = \begin{bmatrix} \tilde{U} - W_2(R_1) & \tilde{V} \\ * & -R_1 \end{bmatrix},$$

$$\tilde{U} = \begin{bmatrix} \Upsilon_2^0 PA_d - Q & hA^T Q + hZ \\ * & -hS_1 & hA_d^T Q - hZ \\ * & * & 0 \end{bmatrix},$$

$$\Upsilon_2^0 = PA + A^T P + hS_1 + Q + Q^T, \quad \tilde{V} = \begin{bmatrix} hA^T R_1 \\ hA_d^T R_1 \\ 0 \end{bmatrix},$$

then the system (13) is asymptotically stable for the all constant delays  $h \in [h_{\min}, h_{\max}]$ .

*Proof* : Applying Theorem 6 by choosing  $S_1 = \frac{1}{h}S$ ,  $R_1 = \frac{1}{h}R$  yields to  $\tilde{\Pi}_1 > 0$  and  $\Pi_2(h) = \Pi_2^0(h) - W_2(R_1) < 0$ , where the matrix  $W_2(R_1)$  is given in Lemma 5 and where

$$\begin{aligned} \Pi_2^0 &= \begin{bmatrix} \Upsilon_2^0 PA_d - Q & hA^T Q + hZ \\ * & -hS_1 & hA_d^T Q - hZ \\ * & * & 0 \end{bmatrix} \\ &+ \begin{bmatrix} hA^T R_1 \\ hA_d^T R_1 \\ 0 \end{bmatrix} R_1^{-1} [hR_1 A \quad hR_1 A_d \quad 0]. \end{aligned}$$

Using Schur's complement to the inequality  $\Pi_2(h) = \Pi_2^0(h) - W_2(R_1) < 0$  leads to  $\tilde{\Pi}_2(h) < 0$ . Hence, the system (13) is stable for a given delay  $h$  if  $\tilde{\Pi}_1 > 0$  and  $\tilde{\Pi}_2(h) < 0$ . Therefore, simple inspection shows that  $\tilde{\Pi}_2(h)$  is affine and thus convex, in  $h$ . So, the condition  $\tilde{\Pi}_2(h)$  has to be assessed only at the two vertices of the polytope generated by the interval of  $h$  which concludes the proof.  $\diamond$

## 5. EXAMPLES

### 5.1 First example: delay dependent case

Consider the linear time-delay systems (13) with the matrices

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

This system is a well-known delay dependent stable system, that is the delay free system is stable and the maximum allowable delay  $h_{\max} = 6.1721$  can be easily computed by delay sweeping techniques. To demonstrate the effectiveness of our approach, results are compared to the literature and are reported in Table 1. All papers except Kao and Rantzer [2007] use Lyapunov theory in order to derive stability criteria. Many recent papers give the same result since they are intrinsically based on the same Lyapunov functional and use the same bounding cross terms technique i.e. Jensen inequality. Some papers Ariba et al. [2010], Sun et al. [2010], which use an augmented Lyapunov can go further but with a numerically increasing

Theorems	$h_{max}$	number of variables
Gouaisbaut and Peaucelle [2006]	4.472	$1.5n^2 + 1.5n$
He et al. [2007]	4.472	$3n^2 + 3n$
Shao [2009]	4.472	$2.5n^2 + 1.5n$
Sun et al. [2010]	4.472	$3n^2 + 3n$
Kao and Rantzer [2007]	6.1107	$1.5n^2 + 9n + 9$
Ariba et al. [2010]	5.120	$7n^2 + 4n$
Sun et al. [2010]	5.02	$18n^2 + 18n$
Kim [2011]	4.97	$69n^2 + 5n$
Gu et al. [2003] (N=1)	6.059	$5.5n^2 + 2.5n$
Th.6	5.901	$3n^2 + 2n$
Th.6 with $Q = Z = 0$	4.472	$1.5n^2 + 1.5n$

Table 1. Results for Example 1.

Theorems	$h_{min}$	$h_{max}$	number of variables
He et al. [2007]	$\emptyset$	$\emptyset$	$3n^2 + 3n$
Ariba et al. [2010]	0.102	1.424	$7n^2 + 4n$
Gu et al. [2003] (N=1)	0.1006	1.4272	$5.5n^2 + 2.5n$
Th.6	0.1006	1.473	$3n^2 + 2n$
Th.6 with $Q = Z = 0$	$\emptyset$	$\emptyset$	$1.5n^2 + 1.5n$

Table 2. Results for Example 2.

burden, compared to our proposal. Notice that the robust approach Kao and Rantzer [2007] give a very good upper-bound with a similar computational complexity than our result.

Theorem 7 addresses also the stability of systems with *interval delays*, which may be unstable for small delays (or without delays) as it is illustrated with the second example.

### 5.2 Second example: delay range case

Consider the linear time-delay systems (13) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

As  $Re(eig(A + Ad)) = 0.05 > 0$ , the delay free system is unstable and in this case, the results to assess the stability of this system are much more scarce. They are often related to robust analysis Ariba et al. [2010] or discretized Lyapunov-Krasovskii functionals Gu et al. [2003]. The results are reported in Table 2. In this example, Theorem 6 delivers better result than Gu et al. [2003] and Ariba et al. [2010] with a fewer number of variables to be optimized. Notice that with the discretization technique from Gu et al. [2003], increasing  $N$  yields to a better result approaching the analytical bound.

## 6. CONCLUSIONS

In this paper, we have provided a new useful inequality which encompass the Jensen's inequality. In combination with a simple Lyapunov-Krasovskii functional, this inequality leads to new stability criteria for linear time delay system. This new result has been expressed in terms of LMIs and has shown on numerical examples a large improvement of existing results using only a limited number of matrix variables. More generally, this preliminary result can be extended to all existing results which are using the Jensen's inequality, especially the case of time varying delay or sampled data systems.

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