A Consensus Algorithm for Common Reference Frame Estimation in Networked Multi-Agent Systems

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Abstract—This paper is concerned with the design of a novel discrete-time consensus algorithm to estimate a common reference frame in a multi-agent system by performing decentralized agreement on a common reference point on a plane without relative pose measurements. The problem of common reference frame estimation on a 2-D plane can be divided into two subproblems: the estimation of a common origin and the estimation of a common direction. This preliminary paper focuses of the first subproblem and points to other approaches presented in the literature to solve the second subproblem. The proposed algorithm is based only on local interactions among the agents which exploit only measurements of relative distances and angles with respect to each other but are unable to estimate their relative attitude during the estimation process. Our approach extends the state of the art by achieving bounded estimation errors in the case of moving agents that simultaneously change their relative and unknown attitude with respect to each other.

I. INTRODUCTION

Algorithms for self-organization in multi-agent systems (MAS) have received significant attention due to the success of decentralized control architectures in tackling cooperative control, agreement in sensor networks, formation control of unmanned vehicles and consensus problems.

Consensus problems involving leader following, attitude synchronization or consensus on relative poses have been investigated in [1], [2] [3], and [4]. These results focus on the case in which a common coordinate system is shared by the agents or there are enough sensing resources to measure not only the relative distance but also the relative pose or attitude between the agents.

The study of the convergence speed of distributed iterative algorithms for the agreement and distributed averaging problem is investigated in [5] assuming that a common coordinate system is known by the agents.

Other works such as [6] deal with control of multi agent system under topological constraints and considered interactions occurring in a global coordinate system. In [7], [8], and [9] networks where the agents have reference frames with identical attitude or relative pose (orientation) are considered.

Other works deal with the network localization problem [10] focusing in the case when a set of anchor nodes is aware of their position with respect to a global coordinate system.

The assumption that the agents share a global coordinate system might be too restrictive in certain circumstances. Indeed, this requires either sophisticated or else. This implies that the network might be too restrictive in certain circumstances. Indeed, this requires either sophisticated sensing equipment such as a computer vision system with common anchors, or access to the Global Position System (GPS), or a compass. In this paper we neither assume that the agents share a common coordinate system nor that they can measure the relative attitude or pose between neighbors directly.

A consensus based algorithm for the estimation of a common reference frame in the plane can be found in [11], [12] where the authors exploit gossiping as communication protocol, i.e., only random, asynchronous and pairwise local state update is allowed. In [13] the authors extend the strategy to estimate the network centroid to a three dimensional space.

In this paper we address the problem of estimating a common reference frame in a multi-agent system by dividing it into two parts. The first part consists in performing distributed agreement on a common reference point to be used in the common coordinate system. The second part consists in performing agreement on a common direction or reference frame orientation between the agents.

For the first part, we propose a PI-like dynamic average consensus algorithm. A characterization of PI algorithms where measurement of relative positions is exploited for solving consensus problems can be found in [14] while in [15] a PI consensus algorithm is used to achieve synchronization in a network of clocks. In general, proportional-integral dynamic average consensus algorithms show an improved tracking performance with respect to the average of constant reference inputs as opposed to other approaches involving only the proportional part. Our approach differs from these works in that the agents are not allowed to compensate for their movements and rotations of their local reference frames by exploiting inertial navigation or else. This implies that the derivative of the agents’ position and angular velocity which is exploited in standard dynamic consensus algorithms cannot be exploited directly.

Algorithms for consensus on the N-torus were proposed in [16] and [17], these algorithms are well suited to perform distributed agreement on a common direction in the plane.

The main contribution of this paper is a distributed algorithm which extends the results in [11], [12], by performing asymptotic distributed agreement of a common reference...
point with bounded error despite that agents are moving and changing their unknown relative pose or attitude with respect to each other. This algorithm paired with an algorithm for distributed agreement on a common direction can be used to estimate a common reference frame in networked multi-agent system moving in $\mathbb{R}^2$.

Our approach differs from previous works in that the local reference frames of the agents can be time-varying but no inertial measurements are involved, only relative distance and angle with respect the pairwise line of sight between neighboring agents. The characterization of the estimation errors of the proposed algorithm is preliminary.

This work is organized as follows. Section II presents the notation and preliminaries. In Section III, the problem statement is formalized. The main result is presented in Section IV, where we propose an algorithm and characterize its convergence properties. A numerical example is shown in Section V, it illustrates the performance of the proposed algorithm. Finally, in Section VI the concluding remarks and future work are discussed.

II. NOTATIONS AND PRELIMINARIES

In this section, some background about graphs, consensus and coordinate transformations is presented.

Let $V = \{1, ..., n\}$ be a set of agents where $v_i$ is the $i$-th agent.

The topology of bidirectional communication channels among the agents is represented by an undirected graph $G = (V, E)$, where $V$ is the set of agents and $E \subset \{V \times V\}$ is the set of edges. An edge $(v_i, v_j) \in E$ exists if there is a communication channel between agent $v_i$ and $v_j$. Self loops $(v_i, v_i)$ are not considered. The set of neighbors of agent $v_i$ is denoted by $N_i = \{v_j : (v_j, v_i) \in E, j = 1, \ldots, n\}$. Let $\delta_i = |N_i|$ be the degree of agent $i$ which represents the total number of its neighbors.

Let matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ be the Laplacian matrix that encodes graph $G$, whose elements are $l_{ij} = -1$ if $(v_i, v_j) \in E$, $l_{ii} = \Delta_i$ and $l_{ij} = 0$ otherwise. The Laplacian matrix has several structural properties. If graph $G$ is undirected, matrix $L$ is symmetric and positive semi-definite, therefore all its eigenvalues are real and non-negative. If the undirected graph $G$ is connected, i.e., there exists a set of consecutive edges which connects any arbitrary pair of agents in the network, then the Laplacian matrix has a single null eigenvalue with corresponding right and left eigenvectors equal to $1_n$, i.e., an $n$-elements vector of ones.

Given a set of agents $V$ and a communication topology $G$, with each agent $v_i$ is associated a state vector $s_i \in \mathbb{R}^d$ associated to each agent. The consensus problem consists in the design of local state update rules among the agents interacting according to the communication topology $G$, so that the state vector associated with each agent converges toward the same value in the whole network, i.e.,

$$\lim_{t \to \infty} s_i(t) = \bar{s}, \quad \forall i \in V. \quad (1)$$

In the convergence analysis of discrete time consensus algorithms it is common to encounter matrices of the form $D = I - \epsilon L$ where $I$ is the $n \times n$ identity matrix and $\epsilon$ is a small constant. If $G$ is undirected and $\epsilon$ is sufficiently small, i.e., $\epsilon < \frac{1}{\Delta}$, then matrix $D$ is a doubly stochastic matrix, i.e., $1^T D = D 1 = 1$. If $G$ is connected then $D$ has a unique unitary eigenvalue to which it corresponds a left and right eigenvector equal to $1_n$.

A. Coordinates transformation for multiple frames

Let $\Sigma_i$ be a local orthonormal reference frame of $\mathbb{R}^2$ where each point in the plane is represented by a vector $p_i = [x_i, y_i]^T$. Let $C(\Sigma_i)$ be a collection of local orthonormal reference frames $\{\Sigma_1(p_{11}, \phi_1), \ldots, \Sigma_n(p_{n1}, \phi_n)\}$ where $p_i$ is the origin of the frame $\Sigma_i$ and $\phi_i$ is the counterclockwise angle from the global and local abscissa axis. Point $p_j$ represented in the local frame $\Sigma_i$ is denoted as $p^i_j$ and can be computed as

$$p^i_j = R(\phi_i)(p_j - p_i), \quad (2)$$

where $R(\phi_i)$ is the rotation matrix defined as

$$R(\phi_i) = \begin{bmatrix} \cos \phi_i & \sin \phi_i \\ -\sin \phi_i & \cos \phi_i \end{bmatrix}. \quad (3)$$

Note that, by eq. (3) it holds $p^i_i = 0_2$ for $i = 1, \ldots, n$ and

$$p^j_j = R(\phi_i)^T p^i_j + p_i. \quad (4)$$

In this paper agents exchange with neighbors local information expressed in different coordinate systems. To do this the generic agent $v_i$ performs a local coordinate transformation with each neighbor separately as computed in [13].

Consider the straight line from $v_i$ to $v_j$. This line is used to build a frame $\Sigma_i^{(ij)}$ where the abscissa is a unitary vector $c_{ij}^1(t)$ in the direction pointing from agent $v_i$ to $v_j$ and the ordinate is the unitary vector $c_{i,j}^2(t)$ orthonormal to $c_{ij}^1(t)$.

Based on the local common frame $\Sigma_i^{(ij)}$, agent $v_i$ and $v_j$, measure their relative distance, and they are able to exchange the value of their current state vector $s_i$, $s_j$ in the coordinate frame of their neighbor. Formally, $c_{ij}^1(t)$ is equal to

$$c_{ij}^1(t) = \frac{p_{ij}^j(t)}{|p_{ij}^j(t)|} = \frac{p_{ij}^j(t)}{d_{ij}^j(t)} \quad (5)$$

where $p_{ij}^j(t)$ is the position of agent $v_j$ measured by agent $v_i$ in its local frame $\Sigma_i$, $d_{ij}^j(t)$ is the distance between agent $v_i$ and its neighbor $v_j$ with respect to frame $\Sigma_i$, and $|\cdot|$ is the Euclidean norm. Note that $d_{ij}^j(t) = d_{ij}^i(t) = |p_{ij}^j(t)| = |p_i(t) - p_j(t)|$. Finally, it holds that

$$c_{i,j}^1(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} c_{ij}^1(t). \quad (6)$$

Agent $v_j$ can compute $s_i^j(t)$ in its own coordinates as follows

$$s_i^j(t) = \begin{bmatrix} d_{ij}^j(t) & -c_{i,j}^1(t) \\ 0 & c_{i,j}^1(t) \end{bmatrix} s_i^j(t) = R(\phi_{ij}(t)) s_i^j(t) + p_{ij}^j, \quad (7)$$

where $R(\phi_{ij}(t))$ is the rotation matrix transforming the coordinates from the local frame $\Sigma_i$ to the frame $[c_{i,j}^1(t), c_{i,j}^2(t)]$, and $\phi_{ij}(t)$ is the counterclockwise angle from the axis $x_i$ to $c_{i,j}^1(t)$ as Fig.1 shows.
In this paper we consider a set of agents $V$ located in a plane and interconnected by a communication network represented by graph $G$. Each agent represents the external environment in its own coordinate system defined by its own local reference frame that moves rigidly with it. In particular we denote with $\Sigma^i(p_i(t), \phi_i(t))$, the local frame of agent $v_i$ at time $t$.

With each agent is associated a state vector $s_i = [s_i^x, s_i^y]^T$ and a direction (orientation) $\theta_i$ representing a 2-dimensional orthonormal reference frame with origin $s_i$ and orientation $\theta_i$ expressed in the global frame $\Sigma^g$.

Each agent actually represents the information about such external reference frame with a state vector expressed in its own local frame $\Sigma^i$, this representation is denoted as $s_i^i$. The relationship between $s_i$ and $s_i^i$ at time $t$ is given by

$$s_i(t) = R(\phi_i(t))^T s_i^i(t) + p_i(t). \tag{8}$$

In the proposed scenario, no agent may obtain or use information about its own absolute position, attitude or pose relative to other agents or with respect to any global reference frame $\Sigma^g$. No agent has knowledge about the coordinate system of other agents. Therefore, we consider agents which do not exploit GPS, compasses, inertial navigation or other technological means to estimate their position or velocities.

Each agent is only allowed to measure the relative position of neighboring agents in its own local reference frame by measuring relative distances and relative angles of sight.

The main objective of this paper is to design a distributed algorithm to establish a consensus on a common reference frame with bounded error in a network. We consider the case in which agents are dynamic and their local reference frames are time-varying with bounded velocities with respect to a global reference frame.

In particular, we assume that the agents have bounded translational velocity and their reference frames rotate with bounded angular speed, namely it holds $|p_i(t+1) - p_i(t)| < \delta P_M$ and $|\phi_i(t+1) - \phi_i(t)| < \delta \phi_M$, $\forall \ t \geq 0$, $i = 1, \ldots, n$, and $\delta P_M, \delta \phi_M \in \mathbb{R}$.

Finally, to achieve our main objective we split the problem of estimating a common reference frame in two parts:

1) Perform consensus on a common origin;
2) Perform consensus on a common direction.

Since agreement on a common direction responds with bounded errors to bounded perturbations to the agents time-varying local reference frame, algorithms taken from the literature can be exploited to solve the second issue. In particular the algorithm proposed in [16] and [17] can be successfully applied to solve the consensus problem on a common direction. For sake of brevity we do not discuss in detail these algorithms in this preliminary work.

In this paper we deal with the first issue by proposing a consensus algorithm able to achieve bounded errors in the estimation of a common origin for the common reference frame despite time-varying local reference frame.

In the proposed scenario, no agent may obtain or use information about its own absolute position, attitude or pose relative to other agents or with respect to any global reference frame $\Sigma^g$. No agent has knowledge about the coordinate system of other agents. Therefore, we consider agents which do not exploit GPS, compasses, inertial navigation or other technological means to estimate their position or velocities.

In the following we consider the case where agents’ reference frame are time-invariant and prove that the algorithm achieves exact estimation of a common point, then we consider the more general case where agents’ reference frame are time-varying with bounded speed and angular velocity and provide a preliminary characterization of the bounded estimation errors.

### A. Time-invariant local reference frames

We now consider the case where $\phi_i(t) = \phi_i$ and $p_i(t) = p_i$ for all $t \geq 0$. Substituting eq. (8) into eq. (9), and considering that $\dot{s}_i^i(t) = R(\phi_i)\dot{s}_i^i(t)$, eq. (9) can be rewritten as

$$R(\phi_i)s_i(t+1) = \left(1 - \varepsilon K_P\right)R(\phi_i)(s_i(t) - p_i) + R(\phi_i)p_i - \varepsilon \sum_{j \in N_i} R(\phi_i)(s_i(t) - p_j) - R(\phi_i)(s_i(t) - p_i). \tag{10}$$
Since $R(\phi_i)^T R(\phi_i) = I$ for $\forall \phi \in [0, 2\pi]$, by simple manipulations, it holds

\[ s_i(t + 1) = (1 - \varepsilon K_P) s_i(t) - \varepsilon \sum_{j \in \mathcal{N}_i} (s_j(t) - s_i(t)) \]
\[ \hat{s}_i(t + 1) = \hat{s}_i(t) - \varepsilon K_I \sum_{j \in \mathcal{N}_i} (\hat{s}_i(t) - \hat{s}_j(t)) \]

Thus, the global system expressed in matrix form is

\[
\begin{bmatrix}
S(t + 1) \\
\hat{S}(t + 1)
\end{bmatrix} =
\begin{bmatrix}
I - \varepsilon K_P I + L_2 & \varepsilon K_I L_2 \\
-I K_I L_2 & I - \varepsilon K_I L_2
\end{bmatrix}
\begin{bmatrix}
S(t) \\
\hat{S}(t)
\end{bmatrix} +
\begin{bmatrix}
\varepsilon K_P \\
0_{n \times n}
\end{bmatrix}
P
\]

\[ S^{tot}(t) = S(t) + \hat{S}(t), \quad (11) \]

where $S(t) = [s_1(t)^T, \ldots, s_n(t)^T]^T$, $\hat{S}(t) = [\hat{s}_1(t)^T, \ldots, \hat{s}_n(t)^T]^T$, $P = [P_1^T, \ldots, P_n^T]^T$ and $S^{tot}(t)$ is a new variable defined as the summation of $S(t)$ and $\hat{S}(t)$.

The next theorem characterizes the convergence properties of the system modeled by (11). Note that in this paper we consider the real eigenvalues of a Laplacian matrix $L$ associated with a connected undirected graph $G$ ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$.

**Theorem 1:** Consider system in eq. (11). Let $G$ be the undirected connected graph describing the communication topology. If $\varepsilon (\lambda_n + K_P) < 2$ and $\varepsilon (K_I \lambda_n) < 2$, where $\lambda_n$ is the largest eigenvalue of the Laplacian matrix associated with $G$, then

\[
\lim_{t \to \infty} S^{tot}(t) = S(t) + \hat{S}(t) = 1_n \otimes \frac{\sum_{i=1}^n p_i}{n}.
\]

**Proof:**

First notice that matrix $L_2 = I_2 \otimes L$ has the same eigenvalues of matrix $L$ but with twice algebraic and geometric multiplicity.

System in eq. (11) is linear and block triangular. Therefore, its eigenvalues correspond to the eigenvalues of the blocks in the diagonal. The eigenvalues of the upper left block $A_1 = I - \varepsilon K_P I + L_2$ are equal to the eigenvalues of the negative definite matrix $-K_P I + L_2$ multiplied by $\varepsilon$ and shifted by 1. Since matrix $A_1$ is symmetric all its eigenvalues are real.

The rightmost eigenvalue is thus $1 - \varepsilon K_P$, while the leftmost eigenvalue is $1 - \varepsilon (\lambda_n + K_P)$. Therefore, if $\varepsilon K_P > 0$ and $\varepsilon (K_I \lambda_n) < 2$ all the eigenvalues of the upper left block $A_1$ of system in eq. (11) are strictly inside the unit circle. Thus, if the agents do not move, i.e., $P(t+1) = P(t) = P$, vector $S(t)$ converges to

\[
\lim_{t \to \infty} S(t) = K_P (K_P I + L_2)^{-1} P.
\]

The eigenvalues of the lower right block $A_4 = I - \varepsilon K_I L_2$ are equal to the eigenvalues of the negative semi-definite matrix $-K_I L_2$ multiplied by $\varepsilon$ and shifted by 1. Since matrix $A_4$ is symmetric all its eigenvalues are real. The rightmost eigenvalue is thus 1 while the leftmost eigenvalue is $1 - \varepsilon (K_I \lambda_n)$. Therefore, if $K_I > 0$ and $\varepsilon (K_I \lambda_n) < 2$ all the eigenvalues of the upper left block $A_1$ of system in eq. (11) are inside the unit circle.

Now, lower right block $A_4$ is doubly stochastic. Since $G$ is connected the unitary eigenvalue of $A_4$ is unique and has algebraic and geometric multiplicity equal to 2 due to the Kronecker product. Since $A_4 1_{2n} = 1_{2n}$, $v_1 = [0_{2n}^T, 1_{2n}^T]^T$ is a right eigenvector of the system matrix in eq. (11) corresponding to the unitary eigenvalue.

No input is fed directly to $\hat{S}(t)$, so the dynamics are autonomous and the system converges to one of its equilibrium points. In particular, at equilibrium, it holds

\[
L_2 \left( S(t) + \hat{S}(t) \right) = 0_{2n}.
\]

Since the only eigenvector corresponding to the null eigenvalue of $L$ is $1_n$, it holds

\[
\lim_{t \to \infty} S(t) + \hat{S}(t) = 1_n \otimes [\alpha_1 \alpha_2]^T, \quad \alpha_1, \alpha_2 \in \mathbb{R}. \quad (12)
\]

Then it holds by eq. (11)

\[
(1_n^T \otimes I_2) S(t+1) = (1_n^T \otimes I_2) S(t) + \varepsilon K_P (1_n^T \otimes I_2) S(t) + \varepsilon K_P (1_n^T \otimes I_2) P.
\]

After a sufficiently long time, as $S(t)$ approaches equilibrium, it holds $\varepsilon K_P (1_n^T \otimes I_2) S(t) = \varepsilon K_P (1_n^T \otimes I_2) P$; therefore,

\[
\lim_{t \to \infty} (1_n^T \otimes I_2) S(t) = (1_n^T \otimes I_2) P. \quad (13)
\]

By eq. (11), since $1_{2n}^T L_2 = 0_{2n}$, it holds $(1_n^T \otimes I_2) \hat{S}(t + 1) = (1_n^T \otimes I_2) \hat{S}(t)$. Furthermore, by definition of $S^{tot}(t)$ in eq. (11) it holds

\[
(1_n^T \otimes I_2) S^{tot}(t) = (1_n^T \otimes I_2) S(t) + (1_n^T \otimes I_2) \hat{S}(t),
\]

since $(1_n^T \otimes I_2) \hat{S}(t) = 0_2$, it holds

\[
\lim_{t \to \infty} (1_n^T \otimes I_2) S^{tot}(t) = (1_n^T \otimes I_2) P.
\]

This implies that $(1_n^T \otimes I_2) (1_n \otimes [\alpha_1 \alpha_2]^T) = (1_n^T \otimes I_2) P$.

Therefore,

\[
\lim_{t \to \infty} S^{tot}(t) = S(t) + \hat{S}(t) = 1_n \otimes \frac{\sum_{i=1}^n p_i}{n},
\]

thus proving the statement. □

**B. Time-varying local reference frames**

Now, we consider the case where the agents reference frames are time-varying. In such case eq. (9) becomes

\[
R(\phi_i(t+1)) (s_i(t+1) - p_i(t+1)) = \]
\[
= (1 - \varepsilon K_P) R(\phi_i(t)) (s_i(t) - p_i(t)) - \varepsilon R(\phi_i(t)) \sum_{j \in \mathcal{N}_i} ((s_j(t) - p_j(t)) - (s_i(t) - p_i(t))),
\]
\[
R(\phi_i(t+1)) \hat{s}_i(t+1) = \]
\[
= R(\phi_i(t)) \hat{s}_i(t) - \varepsilon K_I R(\phi_i(t)) \sum_{j \in \mathcal{N}_i} (\hat{s}_i(t) - \hat{s}_j(t)) - \varepsilon K_I R(\phi_i(t)) \sum_{j \in \mathcal{N}_i} ((s_i(t) - p_i(t)) - (s_j(t) - p_j(t))).
\]

We now denote $R(\phi_i(t+1))^T R(\phi_i(t)) = R(\phi_i(t))^T$. Multiplying both sides of the above equation by $R(\phi_i(t+1))^T$
and denoting $R(\delta\Phi(t)) = \text{diag}(R(\delta\phi_1(t)) \ldots R(\delta\phi_n(t)))$
the networked system evolves according to
\[
\begin{bmatrix}
S(t + 1) \\
\hat{S}(t + 1)
\end{bmatrix} = \begin{bmatrix}
I - \varepsilon K_P I - \varepsilon K_L I \\
-\varepsilon K_L I - \varepsilon K_P I
\end{bmatrix}
\begin{bmatrix}
\mathcal{R}(\delta\Phi(t))^T S(t) \\
\mathcal{R}(\delta\Phi(t))^T \hat{S}(t)
\end{bmatrix}
+ K_P \begin{bmatrix}
I \\
0_{n \times n}
\end{bmatrix} R(\delta\Phi(t))^T P(t)
+ \begin{bmatrix}
I \\
0_{n \times n}
\end{bmatrix} P(t + 1) - R(\delta\Phi(t))P(t).
\]
\[S'^{\text{tot}}(t) = S(t) + \hat{S}(t).\]  
(14)

The following theorem characterizes an upper bound on the estimation error for the case in which local reference frames are time-varying.

**Theorem 2:** Consider system in eq. (14). Let $G$ be the undirected connected graph describing the communication topology. Let $\delta\phi_i(t) \leq \delta\phi_M$ for $i = 1, \ldots, n$ where $\delta\phi_M < (1 - \varepsilon K_P)^{-1} - 1.1$. Let the tuning parameters of the algorithm $\varepsilon$, $K_P$ be such that $\varepsilon(\lambda_n + K_P) < 2$, where $\lambda_n$ is the largest eigenvalue of the Laplacian matrix $\mathcal{L}$ associated with $G$. Then
\[
\lim_{t \to \infty} |s_i(t) - \frac{1}{n} \sum_{i=1}^{n} p_i(t)| \leq B, \forall i = 1, \ldots, n,
\]
with $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and
\[
a = (2\delta\phi_M + \delta\phi_M^2 - 1)(1 - \varepsilon K_P)^2,
\]
\[
b = (2 + \delta\phi_M)(1 - \varepsilon K_P)\Pi,
\]
\[
c = \Pi^2.
\]

**Proof:** See Appendix A.

In this preliminary paper we characterize only the dynamics of $S(t)$. As future work we plan to characterize the dynamics of the augmented state $S'^{\text{tot}}(t)$.

V. NUMERICAL EXAMPLE

This section illustrates the results presented in this paper. In order to do that, a $MAS$ composed by five agents $V = v_1, v_2, v_3, v_4, v_5$ is used. The communication between agents is represented by the following Laplacian matrix:

\[
\mathcal{L} = \begin{bmatrix}
2 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & 0 & -1 \\
0 & -1 & 2 & -1 & 0 \\
-1 & 0 & -1 & 2 & 0 \\
0 & -1 & 0 & 0 & 1
\end{bmatrix}.
\]

The agents’ positions in the plane are $p_1 = [-2, -3]$, $p_2 = [2, -3]$, $p_3 = [4, -0]$, $p_4 = [0, 3]$ and $p_5 = [-4, -0]$ respectively. Their poses are $\phi_1 = 1\text{rad}$, $\phi_2 = 1.5\text{rad}$, $\phi_3 = 2.5\text{rad}$, $\phi_4 = 3\text{rad}$ and $\phi_5 = 4\text{rad}$ respectively. The parameters used in the algorithm (9) are sample time equal to 0.01 ($\tau = 0.01s$), $K_P = 0.05$ and $K_I = 0.5$. All the estimates ($\hat{s}_1, \hat{s}_2, \hat{s}_3$) are initialized to zero. The evolution of each agent’s estimation which evolves according to eq. (9) is presented in Fig. 2. In the scenario proposed in the simulation the reference frames of the agents are time-varying. Agent $v_2$ and $v_3$ start a rotation of $-8^\circ/s$ and $6.5^\circ/s$ respectively at time 3. The rotations end at time 18. Furthermore, agent $v_1$ moves from point $p_1 = [-2, -3]$ at time 35 and ends at point $p_2 = [-4, -4]$ at time 35. Every 25 units of time a reset of the state vector $\hat{S}(t)$ occurs. In Fig. 3 it is shown the evolution of the global centroid estimation error expressed as the norm-2 of the difference between the current estimation vector and the current location of the network centroid. The simulation shown in Fig. 3 confirms that when rotations of the agents’ time-varying reference frames end, the proposed algorithm is able to converge exactly to the current location of the network centroid.

VI. CONCLUSIONS AND FUTURE WORK

A novel approach to the problem of decentralized consensus on a common point in a plane and pose in a multi-agent system in absence of a common reference frame has been presented in this work. The developed algorithm achieves consensus on the network centroid as common origin in the estimated reference frame and can be paired with an attitude agreement algorithm to estimate a common reference frame. The main novelty is to allow agents to move and their local reference frames to rotate while preserving a bounded estimation error.

As future work, first we plan to improve the characterization of the convergence properties of the proposed algorithm.
by providing accurate error bounds to the estimation error, second we plan to characterize error bounds to the estimation of the common reference frame. This will require a characterization of the robustness properties of an algorithm for consensus on a common direction with respect to external disturbances.

References


Appendix

A. Proof of Theorem 2

Due to space limitations we only provide a proof sketch. The extended proof will be available in the extended journal version.

If $\varepsilon (\lambda_n + K_P) < 2$ holds, then the eigenvalues of matrix $I - \varepsilon (K_P I - L_2)$ are strictly inside the unit circle. In the system described by (14), the rotation matrices can be approximated for small rotations $\delta \phi_i(t)$ of the reference frames $\Sigma^i$ during the sampling period as $R(\delta \phi_i(t)) = I + \delta \phi_i(t) R(\pi/2)$, this leads to

$$
\begin{align*}
\begin{bmatrix} S(t+1) \\ \bar{S}(t+1) \end{bmatrix} = & \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} S(t) \\ \bar{S}(t) \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \delta \phi_i(t) R_1 S(t) \\ \delta \phi_i(t) R_1 \bar{S}(t) \end{bmatrix} + \varepsilon K_P \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} P(t) + \varepsilon K_P \bar{S}(t) \end{bmatrix} + \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \delta \phi_i(t) R_1 P(t) \end{bmatrix}.
\end{align*}
\end{equation}
$$

where $\delta \phi_i(t) = \text{diag}(\delta \phi_i(t), \ldots, \delta \phi_N(t)) \otimes I_2$ and $R_1 = R(\pi/2) \otimes I_n$. The dynamics of $S(t)$ can be simplified by considering a disturbance vector $\nu(t) = A_1 \delta \phi_i(t) R_1 S(t) + \varepsilon K_P \nu_P(t) + \varepsilon K_P \delta \phi_i(t) R_1 P(t) + P(t + 1) - P(t) - \delta \phi_i(t) R_1 P(t)$.

Thus, the dynamics of (16) become

$$
S(t + 1) = A_1 S(t) + \nu(t).
$$

We now consider the Lyapunov function $V_1(t) = \|S(t) - \bar{P}\|^2$, where $\bar{P} = I_n \otimes \sum_{i=1}^{n=2} P_i$, is the network centroid at time $t$. Without loss of generality, we consider a time-varying global reference frame $\Sigma^i$ with origin in the network centroid $\bar{P}$, therefore in the following analysis we consider $\bar{P} = 0_{2n}$.

Let $\Delta V_1(t) = V_1(t + 1) - V_1(t)$. It holds

$$
\Delta V_1(t) \leq (1 - \varepsilon K_P) - 2\varepsilon K_P \|S(t)\|^2 + 2(1 - \varepsilon K_P) \|S(t)\|_2 \|\nu(t)\|_2 + \|\nu(t)\|^2_2.
$$

The norm of $\nu(t)$ can be bounded as follows

$$
\|\nu(t)\|^2_2 \leq (1 - \varepsilon K_P)\|\delta \phi_M\|S(t)\|^2_2 + \Pi,
$$

where $\Pi = (\varepsilon K_P + \varepsilon K_P \delta \phi_M + \delta \phi_M) \|P(t)\|^2_2 + \delta \phi_P$, is a bound that depends on the tuning parameters of the algorithm and bounds to angular and translational speed of the agents’ reference frames.

Applying eq. (19) into eq. (18) it holds

$$
\Delta V_1(t) \leq (2(\delta \phi_M + \delta \phi_M^2 + 1)(1 - \varepsilon K_P)^2 - 1)\|S(t)\|^2_2 + 2(1 + \delta \phi_M)\|1 - \varepsilon K_P\|\|S(t)\|^2_2 + \Pi^2.
$$

By denoting

$$
a = (2(\delta \phi_M + \delta \phi_M^2 + 1)(1 - \varepsilon K_P)^2 - 1),
$$

$$
b = 2(1 + \delta \phi_M)\|1 - \varepsilon K_P\| \Pi,
$$

it holds $\Delta V_1(t) \leq a \|S(t)\|^2_2 + b \|S(t)\|^2_2 + c$. Thus, for $\delta \phi_M < (1 - \varepsilon K_P)^{-1} - 1$, $a < 0$ and $V_1(t + 1) - V_1(t) < 0$. This implies that if the estimation error is above the next threshold

$$
\|S(t)\|_2 \geq \frac{-b + \sqrt{b^2 - 4ac}}{2a},
$$

then $V_1(t) = \|S(t) - \bar{P}\|^2$ is decreasing, thus proving the statement. 

\[ \square \]