Journal of Combinatorial Theory, Series B **72**, 251–256 (1998)
Article No. TB971813

On the Roots of Chromatic Polynomials

Jason I. Brown

Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Canada B3H 3J5
E-mail: brown@mscs.dal.ca

Received November 19, 1996

It is proved that the chromatic polynomial of a connected graph with n vertices and m edges has a root with modulus at least $(m-1)/(n-2)$; this bound is best possible for trees and 2-trees (only). It is also proved that the chromatic polynomial of a graph with few triangles that is not a forest has a nonreal root and that there is a graph with n vertices whose chromatic polynomial has a root with imaginary part greater than $\sqrt{n}/4$. © 1998 Academic Press

1. INTRODUCTION

Let $\pi(G, x)$ and $\chi(G)$ denote respectively the chromatic polynomial and chromatic number of a graph G with n vertices and m edges. A number of papers have considered the location of the roots of the chromatic polynomial of a graph. Birkhoff and Lewis [7] showed that the chromatic polynomial of any plane triangulation has no roots in the intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $[5, \infty)$, and Woodall [22] improved this by showing that in fact there are no roots in (2, 2.546602...) (the latter being the smallest nonintegral real root of the chromatic polynomial of the octahedron). It is well known (see [20]) that no graph has a root of its chromatic polynomial in $(-\infty, 0)$ or $(0, 1)$. Jackson [13] proved that no graph has a root of its chromatic polynomial in the interval $(1, 32/27]$, and Thomassen [19] has shown that Jackson's result is best possible, in that for any $\lambda > 32/27$, there is a graph whose chromatic polynomial has a root arbitrary close to λ.

Regarding the locations of chromatic roots (i.e., roots of chromatic polynomials) in the complex plane, there are relatively few results and many conjectures (cf. [4, 5, 8, 11, 16, 17, 21]). Read and Royle [16] have calculated the chromatic roots of many small graphs, and the structure of these roots is still elusive. The limit points of the chromatic roots of a few special classes of graphs have been determined [2, 3, 16]. In [9] it was proved that all the roots of $\pi(G, x)$ lie within the disk $|z-1| \leqslant m-n+1$;

0095-8956/98 $25.00

this theorem improves on results obtained earlier by Thier [18] and reported in [12].

In the opposite direction, Woodall [21] demonstrated that for fixed a and sufficiently large b, the complete bipartite graph $K_{a,b}$ has a real root close to each integer in the interval $[2, a/2]$, and, hence, there are graphs with real roots far from their chromatic number. We provide a lower bound for the largest modulus of a chromatic root of a connected graph in terms of the numbers of vertices, edges, and triangles.

2. CHROMATIC ROOTS OF LARGE MODULUS

Farrell [11] observed an apparent correlation between the largest real part of a chromatic root of a graph and the number of edges. The next result provides some mathematical basis for this observation. Throughout, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z, respectively.

THEOREM 1. *Let G be a connected graph with $n \geqslant 3$ vertices, m edges, and t triangles, and set*

$$D = (m-1)^2 (n-3)^2 - (n-2)(n-3)[(m-1)(m-2) - 2t], \tag{1}$$

$$B = (m-1)/(n-2) \qquad \text{and} \qquad W = B + \sqrt{D}/(n-2)(n-3) \tag{2}$$

(if $n=3$, then $D=0$ and we take $W=B$). If $D \geqslant 0$, $\pi(G, x)$ has a root whose real part is at least W, and if $D<0$, $\pi(G, x)$ has roots z_1 and z_2 (not necessarily distinct) such that $\Re(z_1) \geqslant B$ and $\Im(z_2) \geqslant \sqrt{-D}/(n-2)(n-3)$.

Proof. First note that if G is a tree or K_3, then $D=0$ and $\chi(G) = B+1$, so that $W=B$ is a root. Thus we can assume $n \geqslant 4$.

We shall need some notation. The chromatic polynomial can be written in the usual form (cf. [6, p. 76–77])

$$\pi(G, x) = \sum_{i=0}^{n-1} (-1)^i b_i x^{n-i},$$

and $b_0 = 1$, $b_1 = m$, and $b_2 = \binom{m}{2} - t$. Clearly $\chi(G) \geqslant 2$. Thus $x(x-1)$ divides $\pi(G, x)$, and hence the chromatic roots of G are 0, 1 and those of

$$g(x) = \frac{\pi(G, s)}{x(x-1)} = x^{n-2} - (m-1)\, x^{n-3} + \left(\binom{m-1}{2} - t\right) x^{n-4} - \cdots.$$

Consider the $(n-4)$th derivative of $g(x)$,

$$g^{(n-4)}(x)=\frac{(n-2)!}{2}\,x^2-(m-1)(n-3)!\,x+\left(\binom{m-1}{2}-t\right)(n-4)!.$$

By (1) and (2), W is one of the roots of this quadratic. A result from the theory of polynomials, due to Lucas (see [15, p. 22]), states that if f is a nonconstant polynomial, then the roots of the derivative f' of f lie in the convex hull of the roots of f. It follows that the roots of $g^{(n-4)}$ must lie in the convex hull of the roots of g, and hence of $\pi(G, x)$. Thus $\pi(G, x)$ must have roots z_1 and z_2 such that $\Re(z_1)\geqslant\Re(W)$ and $\Im(z_2)\geqslant\Im(W)$. The result now follows from the formula for W. ▮

COROLLARY 2. *If G is a connected graph with $n\geqslant 3$ vertices, then $\pi(G, x)$ has a root z whose modulus is at least $(m-1)/(n-2)$. Further, the moduli of all the roots are at most $B=(m-1)/(n-2)$ if and only if G is a tree or a 2-tree (that is, a graph that can built up from the complete graph of order 2 by successively joining a new vertex to both ends of an existing edge).*

Proof. As in the proof of the previous theorem, we can assume that $n\geqslant 4$. The first statement follows directly from Theorem 1. Alternatively, if we work with

$$g^{(n-3)}(x)=(n-2)!\,x-(m-1)(n-3)!$$

instead of $g^{(n-4)}(x)$ in the previous proof, we see that its root, namely B, must lie in the convex hull of the roots of $g(x)$, and hence of $\pi(G, x)$. Now if $g(x)$ has no root of modulus larger than B, it follows that B must be a root of $g, g', ..., g^{(n-3)}$ (for the convex hulls of the roots can only shrink as we differentiate). Thus the monic polynomial $g(x)$ must be $(x-B)^{n-2}$, and hence

$$\pi(G, x)=x(x-1)(x-B)^{n-2}.$$

Now $\pi(G, x)$ is a monic polynomial with integer coefficients, and hence all its rational roots are integers. Thus B must be an integer, which means that $\chi(G)=B+1$ and $\pi(G, x)$ has a root at every nonnegative integer $\leqslant B$. It follows that $B=1$ or 2. If $B=1$, then $m=n-1$ and G is a tree. If $B=2$, then $\pi(G, x)=x(x-1)(x-2)^{n-2}$, and a result of Dmitriev [10] (see also [14, p. 224]) implies that any graph with a polynomial of this form is in fact a 2-tree.

Finally, if G is a tree, then the roots of $\pi(G, x)=x(x-1)^{n-1}$, namely 0 and 1, are at most $B=1$, and if G is a 2-tree, then the roots 0, 1, and 2, of $\pi(G, x)=x(x-1)(x-2)^{n-2}$ are at most $B=2$, as for any 2-tree of order n, $m=2n-3$. ▮

This result offers a simpler proof than Woodall's [21] of the fact that there is no upper bound on the modulus of the chromatic roots of a graph in terms of its chromatic number (as $(m-1)/(n-2)$ can be arbitrarily large for graphs of any fixed chromatic number $k \geqslant 2$).

A similar argument to the proof of Theorem 1 shows that if $\chi(G) \geqslant k \geqslant 2$, then $\pi(G, x)$ has a root whose modulus is at least $(m-\binom{k}{2})/(n-k)$. One can verify that this bound is better than $B=(m-1)/(n-2)$ whenever $B > k-1 \geqslant 2$ (if $B \leqslant k-1$, then of course the root at $k-1$ provides a better bound than B anyway).

3. SOME REMARKS

Trees and 2-trees are *chordal* graphs (i.e., they have no induced cycle of length greater than 3), and so their chromatic polynomials have only real roots (in fact, only integral roots—cf. [17, p. 34]). These graphs have, in general many triangles. In contrast:

COROLLARY 3. *If G is a triangle-free graph that is not a forest, then $\pi(G, x)$ has a nonreal root.*

Proof. Clearly we can assume that G is connected, and hence $m \geqslant n \geqslant 4$. With the notation of Theorem 1, $t=0$ and (1) gives

$$D=(m-1)(n-3)(n-m-1)<0. \tag{3}$$

Hence $g^{(n-4)}$ has a nonreal root, and it follows (by Lucas' theorem) that so does $\pi(G, x)$. ∎

By considering Sturm sequences, one can weaken the requirement for G to be triangle-free. The Sturm sequence of a polynomial p with real coefficients is $p_0, p_1, \ldots$, where $p_0 = p$, $p_1 = p'$, and for $i \geqslant 2$, p_i is the negative of the remainder when p_{i-1} is divided by p_{i-2} (one terminates the sequence when p_i becomes the zero polynomial). It is known (cf. [1, p. 175–176]) that a monic polynomial p has only real roots if and only if all the terms in its Sturm sequence have positive leading coefficient. A calculation on the polynomial g defined in the proof of Theorem 1 shows that g_2 has negative leading coefficient if

$$\frac{2}{n-2}\left(\binom{m}{2}-t\right)-\frac{(m-1)^2\,(n-3)}{(n-2)^2}>0,$$

and this is equivalent to

$$t < \frac{m(m-n)+n-1}{2(n-2)}.$$

Hence,

THEOREM 4. *If G is a connected graph with $n \geqslant 4$ vertices, m edges, and t triangles, then the chromatic polynomial of G has a nonreal root if*

$$t < \frac{m(m-n)+n-1}{2(n-2)}.$$

A number of fascinating questions about chromatic roots emerge. Read and Royle [16] have asked about the smallest real part of a chromatic root. In contrast, let $I(n)$ denote the *largest imaginary* part of a chromatic root among all graphs on n vertices.

THEOREM 5. *For all $n \geqslant 4$, $I(n) > \sqrt{n}/4$.*

Proof. Let $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, a triangle-free graph with n vertices and $m = \lfloor n^2/4 \rfloor$ edges. If n is even then (3) gives

$$-D = \tfrac{1}{16}(n^2-4)(n-3)(n-2)^2 > \tfrac{1}{16}n(n-3)^2\,(n-2)^2$$

since $n^2-4 > n^2-3n$. If n is odd then (3) gives

$$-D = \tfrac{1}{16}(n^2-5)(n-3)^2\,(n-1) > \tfrac{1}{16}n(n-3)^2\,(n-2)^2$$

since $n^2-5 > n(n-2)$ and $n-1 > n-2$. In each case, by Theorem 1, G has a chromatic root with imaginary part at least $\sqrt{-D}/(n-2)(n-3) > \sqrt{n}/4$, and so $I(n) > \sqrt{n}/4$. ∎

The actual value of $I(n)$, and the corresponding extremal graphs, remain unknown.

ACKNOWLEDGMENTS

The author thanks the Natural Science and Engineering Research Council of Canada for financial assistance and the referees for their insightful comments.

REFERENCES

1. E. J. Barbeau, "Polynomials," Springer-Verlag, New York, 1989.
2. S. Beraha and J. Kahane, Is the four-color conjecture almost false?, *J. Combin. Theory Ser. B* **27** (1979), 1–12.
3. S. Beraha, J. Kahane, and N. J. Weiss, Limits of chromatic zeros of some families of graphs, *J. Combin. Theory Ser. B* **28** (1980), 52–65.
4. G. Berman and W. T. Tutte, The golden root of a chromatic polynomial, *J. Combin. Theory Ser. B* **6** (1969), 301–302.
5. N. L. Biggs, R. M. Damerell, and D. A. Sands, Recursive families of graphs, *J. Combin. Theory Ser. B* **12** (1972), 123–131.
6. N. L. Briggs, "Algebraic Graph Theory," Cambridge Univ. Press, Cambridge, 1993.
7. G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–451.
8. F. Brenti, G. F. Royle, and D. G. Wagner, Location of zeros of chromatic and related polynomials of graphs, *Canad. J. Math.* **46** (1994), 55–80.
9. J. I. Brown, Chromatic polynomials and order ideals of monomials, *Discrete Math.*, to appear.
10. I. G. Dmitriev, Characterization of a class of k-trees, *Metody Diskret. Anal.* **38** (1982), 9–18.
11. E. J. Farrell, Chromatic roots—Some observations and conjectures, *Discrete Math.* **29** (1980), 161–167.
12. D. Gernert, A survey of partial proofs for Read's conjecture and some recent results, *Methods Oper. Res.* **49** (1985), 233–238.
13. B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combin. Probab. Comput.* **2** (1993), 325–336.
14. T. R. Jensen and B. Toft, "Graph Colouring Problems," Wiley, New York, 1995.
15. M. Marden, "Geometry of polynomials," Amer. Math. Soc., Providence, 1966.
16. R. C. Read and G. F. Royle, Chromatic roots of families of graphs, *in* "Graph Theory, Combinatorics and Applications" (Y. Alavi *et al.*, Eds.), pp. 1009–1029, Wiley, New York, 1991.
17. R. C. Read and W. T. Tutte, Chromatic polynomials, *in* "Selected Topics in Graph Theory 3" (L. W. Beineke and R. J. Wilson, Eds.), pp. 15–42, Academic Press, New York, 1988.
18. V. Trier, "Graphen and Polynome," Diploma thesis, TU, Munich, 1983.
19. C. Thomassen, The zero-free intervals for chromatic polynomials of graphs, MAT-REPORT 1995-18, Math Inst., Tech. Univ., Denmark, 1995.
20. W. T. Tutte, Chromials, *in* "Lecture Notes in Math." Vol. 411, pp. 243–266, Springer-Verlag, New York/Berlin, 1974.
21. D. R. Woodall, Zeros of chromatic polynomials, *in* "Surveys in Combinatorics: Proc. Sixth British Combinatorial Conference" (P. J. Cameron, Ed.), pp. 199–223, Academic Press, London, 1977.
22. D. R. Woodall, A zero-free interval for chromatic polynomials, *Discrete Math.* **101** (1992), 333–341.