A NOTE ON CAREFUL PACKING OF A GRAPH

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Abstract

Let $G$ be a simple graph of order $n$ and size $e(G)$. It is well known that if $e(G) \leq n - 2$, then there is an edge-disjoint placement of two copies of $G$ into $K_n$. We prove that with the same condition on size of $G$ we have actually (with few exceptions) a careful packing of $G$, that is an edge-disjoint placement of two copies of $G$ into $K_n \setminus C_n$.

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1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

For graphs $G$ and $H$ we denote by $G \cup H$ the vertex disjoint union of graphs $G$ and $H$ and $kG$ stands for the disjoint union of $k$ copies of the graph $G$.

Suppose $G_1, \ldots, G_k$ are graphs of order $n$. We say that there is a packing of $G_1, \ldots, G_k$ (into the complete graph $K_n$) if there exist injections $\alpha_i : V(G_i) \rightarrow V(K_n), \ i = 1, \ldots, k,$ such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \rightarrow E(K_n)$ is induced by $\alpha_i$.

A packing of $k$ copies of a graph $G$ will be called a $k$-placement of $G$. A packing of two copies of $G$ i.e. a 2-placement is an embedding of $G$ (in its complement $\overline{G}$). So, an embedding of a graph $G$ is a permutation

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σ on \( V(G) \) such that if an edge \( xy \) belongs to \( E(G) \) then \( \sigma(x)\sigma(y) \) does not belong to \( E(G) \).

A careful packing of a graph \( G \) is a packing of \( C_n \) and two copies of \( G \) into the complete graph. In others words this is an edge-disjoint placement of two copies of \( G \) into \( K_n \setminus C_n \). Geometrically speaking, if we identify the cycle \( C_n \) with a convex \( n \)–gon on the plane, the careful packing of \( G \) means the possibility to draw (edge-disjointly) two copies of \( G \) using only the internal edges.

The following theorem was proved, independently, in [2], [4] and [7].

**Theorem 1.** Let \( G = (V, E) \) be a graph of order \( n \). If \( |E(G)| \leq n - 2 \), then \( G \) can be embedded in its complement \( \overline{G} \).

The example of the star \( K_{1,n-1} \) shows that Theorem 1 cannot be improved by increasing the size of \( G \).

This result have been improved in many ways. For instance, the following theorem completely characterizes those graphs with \( n \) vertices and \( n - 1 \) edges which are embeddable ([5], [6]).

**Theorem 2.** Let \( G = (V, E) \) be a graph of order \( n \). If \( |E(G)| \leq n - 1 \), then either \( G \) is embeddable or \( G \) is isomorphic to one of the following graphs : \( K_{1,n-1} \), \( K_{1,n-4} \cup K_3 \) for \( n \geq 8 \), \( K_1 \cup 2K_3 \), \( K_1 \cup C_4 \), \( K_1 \cup K_3 \) and \( K_2 \cup K_3 \).

**Remark.** For other generalization and improvements of Theorem 2 see for instance [8], [9] or [10]. The general references here are [11] and [1] (see also [12]).

Our purpose is to prove the following

**Theorem 3.** Let \( G \) be a graph of order \( n \), \( n \geq 6 \). If \( e(G) \leq n - 2 \), then there exists a careful packing of \( G \) except for two graphs of order 6: \( K_3 \cup K_2 \cup K_1 \) and \( C_4 \cup 2K_1 \), and for two families of graphs: \( K_{1,n-2} \cup K_1 \) and \( K_{1,n-3} \cup K_2 \).

The proof the theorem is given in the next section.

**Corollary 4.** Let \( G \) be a graph of order \( n \), \( n \geq 3 \). If \( e(G) \leq n - 3 \), then there exists a careful packing of \( G \).
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**Proof.** The corollary is evident for \( n = 3 \) and 4 and easy to verify for \( n = 5 \). For \( n \geq 6 \) it follows from Theorem 3.

We finish this section with some remarks.

Observe first that if we want to pack two copies of a graph \( G \) together with the cycle \( C_n \), then the following necessary condition must hold:

\[
\Delta(G) + \delta(G) \leq n - 3.
\]

For, the vertex \( u \) with \( d(u) = \Delta(G) \) must be placed with another vertex of \( G \) and with a vertex of \( C_n \) of degree 2. Another evident, necessary condition is determined by the number of edges in the complete graph \( K_n \).

We must have \( 2(n - 2) + n \leq \binom{n}{2} \) which implies \( n \geq 6 \).

So, from this point of view, there are only two “small” exceptional graphs in Theorem 3.

Since it is very easy to find a 2-placement for exceptional graphs of Theorem 3, so this theorem is an improvement of Theorem 1. On the other hand, Corollary 4 can also be considered as an improvement of the following theorem of Ore (cf.[3]).

**Theorem 5.** If \( G \) is a simple graph of order \( n \geq 3 \) and \( e(G) > \left( \frac{n-1}{2} \right) + 1 \), then \( G \) is Hamiltonian.

Indeed, restated in terms of packing, Theorem 5 states that if \( G \) is a graph of order \( n \), \( n \geq 3 \), and \( e(G) \leq n - 3 \), then there is a packing of \( G \) into \( K_n \setminus C_n \), whereas Corollary 4 ensures a packing of two copies of \( G \) into \( K_n \setminus C_n \).

2. Proof

We start with some simple observations formulated as lemmas.

**Lemma 6.** Let \( G \) be a graph composed of the cycle \( C_k \) and one vertex, say \( u \), not on the cycle. Denote by \( |N(u,C_k)| \) the number of edges connecting \( u \) with \( C_k \). If \( |N(u,C_k)| > \frac{k}{2} \), then the cycle \( C_k \) can be extended to a cycle of length \( k + 1 \) passing through \( u \).

**Lemma 7.** Let \( G \) be a graph composed of the cycle \( C_k \) and two vertices, say \( u, v \), not on the cycle. If
1. \( uv \in E(G) \),
2. \( |N(u, C_k)| \geq 1, \ |N(v, C_k)| \geq 1 \),
3. \( |N(u, C_k)| + |N(v, C_k)| \geq k + 1 \),

then the cycle \( C_k \) can be extended to a cycle of length \( k+2 \) passing through \( u \) and \( v \).

**Proof.** It is easy to see that at least one of the neighbours of the vertex \( v \) on the cycle \( C_k \) has as its neighbour on the cycle \( C_k \), a vertex connected by an edge with the vertex \( u \). The possibility to extend the cycle \( C_k \) to the cycle \( C_{k+2} \) is now evident.

**Lemma 8.** If the graph \( G \) has an end-vertex, say \( x \), adjacent to the vertex, say \( y \), of degree \( d(y) \geq \frac{n-1}{2} \) and there is a careful packing of \( G' = G \setminus \{x\} \), then there is a careful packing of the graph \( G \).

**Proof.** Observe first that in the careful packing of \( G' \) the image of \( y \) is distinct from \( y \). Indeed, otherwise we would have too many edges adjacent to \( y \) in \( K_{n-1} \) (two edges of \( C_{n-1} \) and at least \( n-2 \) edges belonging to two copies of \( G' \)).

Thus it is easy to extend the packing of \( G' \) (by putting \( x \) on \( x \)) and then to extend \( C_{n-1} \) by applying Lemma 6 to the complement of the graph \( G \).

**Proof of Theorem 3.** In the remainder of this section we adopt the following convention: Given a careful packing of a graph \( G \), we say that an edge \( e \) of \( K_n \) is black or blue if it belongs to the first or second copy of \( G \), respectively, and that an edge \( e \) of \( K_n \) is red if it belongs to the corresponding cycle \( C_n \).

The proof is by induction on \( n \). Without loss of generality we may assume that all the graphs under consideration are of maximum size \( n-2 \). Let us start with small values of \( n \) i.e. \( n = 6 \) and \( n = 7 \). It is easy to see that there are five graphs of order 6 and size 4 which are not exceptional: \( K_1 \cup P_5 \), \( K_1 \cup S'_5 \), \( K_2 \cup P_4 \), \( 2P_3 \) and \( 2K_1 \cup (S_3 + e) \). The careful packings of these graphs are depicted in Figure 1 (the edges of \( C_6 \) are not marked). Observe that they can be used to obtain the careful packings of \( (n, n-2) \)-graphs for \( n = 7 \). We can also use Lemma 8. The details are left to the reader.
Suppose now that the theorem is true for all \( n' < n \) and let \( G \) be an \((n, n-2)\)-graph. Assume also that \( G \) is not one of the exceptional graphs. We shall consider two main cases.

**Case 1.** \( G \) has two independent end-edges.
Denote the independent end-edges of \( G \) by \( uu' \) and \( vv' \), \( u, v \) being the corresponding end-vertices of \( G \). Consider now the graph \( G' = G \setminus \{u, v\} \).
Suppose that there exists a careful packing for \( G' \), say \( \sigma' \). It is easy to extend the bijection \( \sigma' \) to a packing of \( G \). Moreover, since the edge \( uv \) is neither black nor blue, we can consider it as a red one. We assign the red colour also to \( n-4 \) edges connecting \( u \) with \( C_{n-2} \) and to \( n-4 \) edges connecting \( v \) with \( C_{n-2} \). By Lemma 7 (with \( k = n - 2 \)) the careful packing of \( G \) exists. The case where \( G' \) is an exceptional graph will be considered below as **Case 3**.

**Case 2.** \( G \) has not two independent end-edges.
Since \( G \) has at least two tree components, the above condition implies that at least one of them is trivial and the other is a star. Let \( u \) be an isolated vertex of \( G \) and let \( x \) be a vertex defined by
\[
d_G(x) = \min\{d_G(y) : y \in V(G), d_G(y) \geq 2\}
\]
We consider the graph \( G' = G \setminus \{u, x\} \). Suppose that \( G' \) is not one of the exceptional graphs; other cases are considered below as Case 3. Then there exists a careful packing for \( G' \), say \( \sigma' \). It is evident that by putting \( x \) on \( u \) and \( u \) on \( x \) we extend \( \sigma' \) to a packing of \( G \). We may assume that the vertices \( x \) and \( u \) send \( n - 2 - d(x) \) red edges to the red cycle \( C_{n-2} \) contained in \( G' \). We can apply Lemma 7 and obtain a careful packing of \( G \) if \( 2(n - 2 - d_G(x)) \geq n - 1 \). Hence \( n - 3 \geq 2d_G(x) \).

Thus, we may assume that

\[
(*): \quad d_G(x) \geq \frac{n - 2}{2}
\]

So, for \( n \geq 7, d_G(x) \geq 3 \). Consider first the case where \( G \) has two trivial components.

**Case 2 (a)** \( G \) has two isolated vertices, say \( u, v \).

Consider first the case \( n = 8 \). The case by case examination shows that: either \( G \) contains an end-vertex such that we can apply Lemma 8, or \( G \) is such that the graph \( G' = G \setminus \{u, x\} \) is exceptional (see Case 3). So, we may assume that \( n \geq 9 \). Consider now the graph \( G_1 = G \setminus \{u, v, x\} \). If \( G_1 \) is not one of the exceptional graphs, we can apply the induction hypothesis. Let \( \sigma_1 \) be a careful packing of \( G_1 \). Denote by \( y_1 \) a vertex of \( G_1 \) non adjacent to \( x \) (such a vertex exists by the definition of \( x \)). Without loss of generality we may assume that \( y_1 \) is the first vertex on the red cycle \( y_1, y_2, \ldots, y_{n-3} \) corresponding to the careful packing of \( G_1 \). Then the cycle \( xy_1y_2 \ldots y_{n-3}uvx \) can be considered as a red cycle of the careful packing of \( G \), say \( \sigma \), obtained from \( \sigma' \) by putting \( \sigma(x) = v, \sigma(v) = x, \sigma(u) = u \) and \( \sigma(w) = \sigma'(w) \) for \( w \in V(G) \setminus \{u, v, x\} \).

**Case 2 (b)** \( G \) has only one isolated vertex.

Hence \( G \) is of the form \( K_1 \cup K_1,r \cup R \) where \( r \geq 1 \) and the graph \( R \) has no isolated vertices. Moreover, since by Case 1, \( R \) contains no end-vertices we may assume, by (*), that either all vertices of \( R \) are of a degree greater than or equal to \( \frac{n-2}{2} \), or \( R \) is empty. In the first case, for \( n > 6 \), this contradicts the fact that the average degree of \( R \) is equal to 2. In the second case \( G \) is exceptional, a contradiction.

**Case 3.** \( G' \) is one of the exceptional graphs, where \( G' \) denotes one of the graphs defined in Cases 1 or 2 \( (n \geq 8) \).

We shall need some additional notations. Namely, by \( S_p' \) we denote a tree of order \( p \) obtained by subdividing one of the edges of the star \( K_{1,p-2} \) and by \( (K_{1,p-1} + e) \) we denote, as usually, the graph of order \( p \) obtained by adding one edge to the edge-set of the star \( K_{1,p-1} \).
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Without loss of generality we may assume that every other choice of two or three (for $n \geq 9$) vertices in a way described in Cases 1 and 2 leads also to one of the exceptional graphs. Of course, we can proceed as in Case 2 also in the case where the graph $G$ has two independent end-edges.

Recall that $G$ itself is not an exceptional graph.

The case by case examination shows that then $G$ belongs to one of the following families of graphs: $P_3 \cup K_{1,n-4}$, $K_1 \cup S_{n-1}^l$, $2K_1 \cup (K_{1,n-3} + e)$, $K_1 \cup K_3 \cup K_{1,n-5}$, or $n = 8$ and $G$ is isomorphic to $4K_1 \cup K_4$, $2K_1 \cup 2K_3$, $2K_2 \cup C_4$, $K_2 \cup P_3 \cup C_3$ or $3K_1 \cup K_{2,3}$.

Observe that in all graphs belonging to the above mentioned families, except for $K_1 \cup K_3 \cup K_{1,3}$, there is a vertex of a degree greater than or equal to $n - 4$, so we can apply Lemma 8 (since $n \geq 8$).

The careful packings of $4K_1 \cup K_4$, $2K_2 \cup C_4$ or $2K_1 \cup 2K_3$ are very symmetric and easy to find.

The careful packing of $K_2 \cup P_3 \cup C_3$ as well as the careful packing of $K_1 \cup K_3 \cup K_{1,3}$ are depicted in Fig. 2.

Finally, the careful packing of $3K_1 \cup K_{2,3}$ can be easily obtained from the careful packing of $2K_1 \cup K_{1,3}$ into $K_6$.

This completes the proof of the theorem.

References


Figure 2. Careful packing of $K_2 \cup P_3 \cup C_3$ and $K_1 \cup K_{1,3} \cup C_3$


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