

Memory Formalism for Quantum Control of Dynamic Localization[†]

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An exact evaluation of memory functions to describe dynamic localization and excitation transfer in a small quantum system is provided to clarify the connection between two recently reported phenomena, one in quantum optics and the other in condensed matter physics. The possibility of quantum control on the basis of the calculation, with relevance to systems as widely different as electrons in a crystal, atoms interacting with light, spins in crossed magnetic fields, and Bose condensates falling in a gravitation field, is briefly mentioned. The connection to the interacting electron–phonon system and to nonlinear Schroedinger equations is pointed out.

I. Introduction

Harvey Scher's numerous contributions to transport theory extend from fundamental solutions to puzzles in electron motion in amorphous solids¹ and the formalism of continuous time random walks,² to practical considerations of aquifers,³ material failure,⁴ and random fracture networks in geology.⁵ I had the good fortune of participating in many exciting discussions initiated by Harvey at the University of Rochester and the Xerox Research Labs in the early 70's that led to the now famous Scher–Montroll approach to anomalous transport in disordered systems. This article is dedicated to Harvey on his sixtieth birthday.

Given Harvey's intense involvement with pausing time distributions, which are known to be equivalent to memory functions in generalized master equations,^{6,7} I thought it natural to select for this article a new calculation of a memory function. While simple in the extreme, the calculation addresses an issue under substantial current focus in the overlap area of condensed matter physics and quantum optics and appears to have relevance to electron transport in solids,^{8–10} atom excitation under the influence of light,¹¹ spin evolution under orthogonal magnetic fields,¹² device design in Josephson junction arrays,¹³ and optical lattices¹⁴ as well as Bose condensates.¹⁵ In a remarkable recent experiment on optical lattices, Raizen and collaborators¹⁴ have observed the phenomenon of dynamic localization in solids predicted a decade ago⁸ by Dunlap in collaboration with the present author. This clean verification has been especially responsible for a resurgence of interest in this topic. The following analysis begins with dynamic localization¹⁶ and hopefully sheds some light on aspects of quantum control,¹⁷ a subject attracting a lot of attention in various contexts.

Consider two simple unrelated systems: a two-level atom subjected to optical excitation via an electromagnetic field and an electron subjected in an infinite crystal to an electric field with an oscillating time dependence. Figure 1 shows some striking behavior exhibited by both of them if the time dependence of the external field is sinusoidal. The probability of excitation in the two-level atom depicted in Figure 1A is transferred from one state to the other in spurts represented by two stages, a quiescent plateau and a sudden transfer.¹¹ The

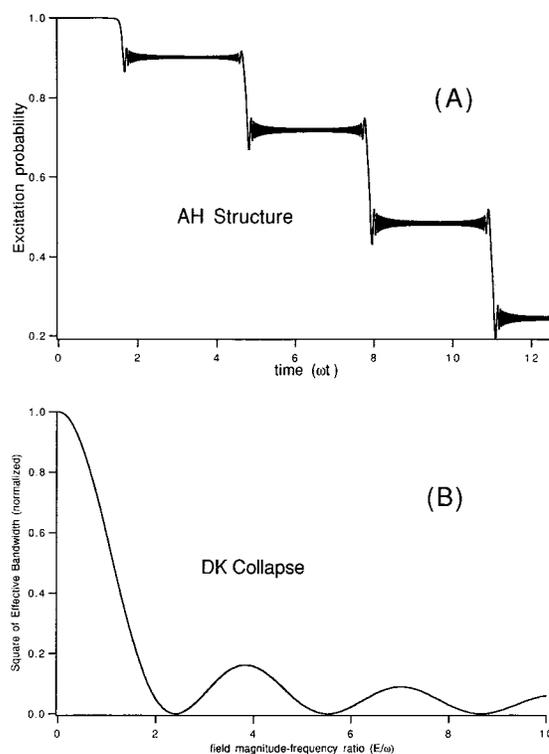


Figure 1. Two phenomena under discussion in the present paper. In (A) we see quiescent plateaus and sudden transfer of excitation probability under a sinusoidal driving field, as in ref 11. In (B) we see a periodic collapse of the effective bandwidth of electrons moving in a crystal under the action of a sinusoidal electric field as in ref 8. The spurts in the transfer in (A) occur at time instants representing the zeros of the applied field. The bandwidth collapse occurs at values of the ratio of the field magnitude to the field frequency, which represent zeros of the Bessel function of order zero. In terms of the system quantities to be described below in section II, parameter values for Figure 1A are V/ω and $E/\omega = 250$, and the abscissa is ωt . In Figure 1B the abscissa is $2E/\omega$.

effective bandwidth of the electron plotted in Figure 1B decreases on the average but collapses (vanishes) repeatedly as the ratio of the field magnitude to the field frequency is increased, the characteristic values at which electron localization sets in (vanishing of bandwidth) being proportional to roots of

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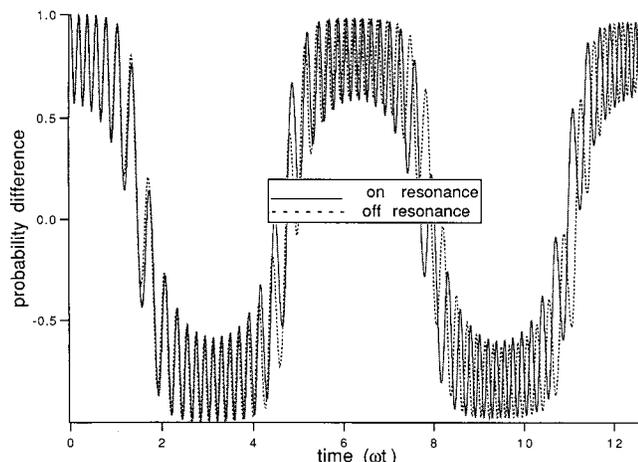


Figure 2. Probability of excitation of the two-state system as in Figure 1A showing lack of dependence of the AH structure on the Bessel root condition. The solid line shows the evolution on resonance (when the condition is satisfied, specifically when the field-frequency ratio equals the tenth root of the Bessel function) as in Figure 2 of ref 11. The dotted line shows the evolution off resonance (when the ratio is as far as possible from the roots, i.e., lies midway between the tenth and eleventh root). Very little change (and no qualitative difference) is apparent. Coordinates are as in our Figure 1A.

the zeroth Bessel function.⁸ Common to these two phenomena, which have been addressed in different communities, optics and condensed matter, respectively, are the sinusoidal time dependence of the applied field, periodic trapping, and the claimed appearance of a Bessel function root condition.^{8,11} The kinship of these two phenomena was addressed in ref 18. Particularly intriguing is the fact that, whereas Agarwal and Harshawarhan¹¹ appear to suggest that the process represented in Figure 1A was related to the Bessel root condition, Raghavan et al.¹⁸ found from their numerical work that it was not. To clarify this statement, let us inspect Figure 2, which represents the process of Figure 1A and demonstrates graphically that probability transfer exhibits essentially identical behavior at the Bessel root condition (for the same parameters as in 11) and *furthest away from the Bessel root condition* (when the relevant field ratio equals a value almost precisely midway between two successive zeros of the Bessel function). The point of Figure 2 is the almost total lack of sensitivity of the structure in Figure 1A (which, after the authors of ref 11, we will call the AH structure in the rest of this paper) to the Bessel root condition. The connection between the AH structure and the DK (dynamic localization) structure predicted in ref 8 has remained unclear despite the simplicity of the system, even after some of its aspects have been understood through the numerical analysis in ref 18. Our interest here lies in clarifying that connection and, additionally, in answering why and when excitation transfer should exhibit the *clean* separation into the plateau and sudden transfer processes seen in Figure 1A. Although one knows that this is surely related to level crossing, it is obvious that the separation does not occur for arbitrary parameter values. Knowing what parameter regimes make the separation possible is particularly important if one is interested in quantum control. We will see that the simple analysis presented below provides satisfactory answers to these questions.

II. System and Essential Results

Let us consider a quantum particle that shuttles back and forth between two states $|1\rangle$ and $|2\rangle$ via interstate matrix elements V , while the applied field \mathcal{E} modifies in a determined fashion the

energies of the two states. This system describes the quantum optics system of Figure 1A directly and the condensed matter system of Figure 1B through the consideration of a very small crystal (consisting of two sites only!). The evolution is described by the Liouville–von Neumann equation

$$i\hbar \frac{d\rho}{dt} = [H, \rho] \quad (1)$$

where the density matrix ρ and the Hamiltonian H are given by

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad H = \begin{pmatrix} E(t) & V(t) \\ V^*(t) & -E(t) \end{pmatrix} \quad (2)$$

In general, V may be complex and time dependent but we will consider it in this paper to be real and constant, in keeping with most applications. We have also obtained results for the general (time dependent and complex) case, which could be of interest to special physical systems and will be discussed elsewhere. Henceforth in this paper \hbar is put equal to 1 for notational convenience.

The main result obtained in the present paper, in terms of which of the questions raised in the Introduction will be discussed below, is an exact (analytic) derivation of an evolution equation obeyed by the probability difference $p(t) = \rho_{11} - \rho_{22}$:

$$\frac{dp(t)}{dt} + 2 \int_0^t ds \mathcal{W}(t,s) p(s) = 0 \quad (3)$$

where the memory function $\mathcal{W}(t,s)$ is known *explicitly* (and simply) in terms of the driving fields:

$$\mathcal{W}(t,s) = 2V^2 \cos\left(\int_s^t dt' 2\mathcal{E}(t')\right) \quad (4)$$

This result involves no approximation. It applies whenever the density matrix is initially diagonal in the representation of the states $|1\rangle$ and $|2\rangle$, as would happen, for instance, if the system were to occupy only one of the two states initially. For more general initial conditions, a term is appended to the right-hand side of (3), as will be shown at the end of the paper. We will see that all questions of interest relating to Figure 1 are answered transparently by a straightforward inspection of (3) and (4). The derivation of (3) and (4) is given in the next section.

III. Exact Derivation of Memory

A convenient starting point for numerical (or analytical) discussions of the evolution is

$$\frac{d}{dt} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -2\mathcal{E} \\ 0 & 2\mathcal{E} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \quad (5)$$

which is, of course, equivalent to (1). The elements p , q , r of the so-called “Bloch vector” are, respectively, the (real) density matrix element combinations $\rho_{11} - \rho_{22}$, $i(\rho_{12} - \rho_{21})$, and $\rho_{12} + \rho_{21}$. Let us use the symbol Λ for the nine-element square matrix in (5). Our interest is in deriving a closed equation for the probability difference $p(t)$. It has been shown by Zwanzig and by Nakajima¹⁹ that, if one defines a linear, time-independent, idempotent projection operator \mathcal{P} , applies \mathcal{P} and $1 - \mathcal{P}$ in turn to an evolution equation such as (1), solves the second equation thus obtained formally for $(1 - \mathcal{P})\rho$, and substitutes the solution in the equation for $\mathcal{P}\rho$, which we will call ρ' , one obtains

$$\frac{d\rho'(t)}{dt} + \mathcal{P}\Lambda(t)\rho'(t) - \mathcal{P}\Lambda(t)\int_0^t ds e^{-\int_s^t dr(1-\mathcal{P})\Lambda(r)}(1-\mathcal{P})\Lambda(s)\rho'(s) = -\mathcal{P}\Lambda(t)e^{-\int_0^t ds(1-\mathcal{P})\Lambda(s)}(1-\mathcal{P})\rho(0) \quad (6)$$

Despite its somewhat formidable appearance, this equation has found many uses ranging from the elucidation of the problem of irreversibility and the validity of Master equations, to the calculation of resistivity and exciton transport parameters. Depending on the context, the projection operator can involve diagonalization, taking a partial trace, integrating over a set of coordinates, and other varied operations.^{19,20} Here let us choose a particular realization of \mathcal{P} to single out the first of the elements of the Bloch vector:

$$\mathcal{P}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} Z_1 \\ 0 \\ 0 \end{pmatrix} \quad (7)$$

apply it to (5) instead of to the von Neumann eq 1 and use its specific properties to obtain an exact result for our system. The relation

$$\Lambda(t)\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} -2VZ_2 \\ 2VZ_1 - 2EZ_3 \\ 2EZ_2 \end{pmatrix} \quad (8)$$

shows that the first term in (6), viz., $\mathcal{P}\Lambda(t)\rho'(t)$ vanishes identically, that

$$(1-\mathcal{P})\Lambda(s)\rho'(s) = 2V\begin{pmatrix} 0 \\ p(s) \\ 0 \end{pmatrix} \quad (9)$$

and that $(1-\mathcal{P})\Lambda(t')$ applied to a vector of the form

$$\begin{pmatrix} 0 \\ Z_2 \\ Z_3 \end{pmatrix}$$

once, produces a c-number times

$$\begin{pmatrix} 0 \\ Z_3 \\ -Z_2 \end{pmatrix}$$

and applied twice produces a c-number times

$$\begin{pmatrix} 0 \\ Z_2 \\ Z_3 \end{pmatrix}$$

thus returning one to the original vector. As a result, the exponential operator in the projection operator expressions in (6) yields a cosine series and a sine series and allows an exact analytic evaluation.²¹ Equation 6 yields

$$\frac{dp(t)}{dt} + 2\int_0^t ds \mathcal{W}(t,s)p(s) = \text{initial value terms} \quad (10)$$

The right hand side vanishes for an initially diagonal density matrix, and eq 3 follows. The memory function appearing in (10) or (3) is not of the faltung type because the Hamiltonian is time dependent and is given specifically by (4), or equivalently by

$$\mathcal{W}(t,s) = 2V^2[\phi_c(t)\phi_c(s) + \phi_s(t)\phi_s(s)]$$

$$\phi_c(t) = \cos\left[\int_0^t ds 2E(s)\right] \quad \phi_s(t) = \sin\left[\int_0^t ds 2E(s)\right] \quad (11)$$

It is unnecessary to specify the lower limit on the time integrals: let us take it to be 0 in the sequel. These exact results will be applied below to understand the structures of Figure 1 and to comment on the possibility of achieving quantum design easily by specifying the driving field.

IV. Understanding the Structures in Figure 1

Let the time dependence of the externally applied field be sinusoidal:

$$E(t) = E \cos \omega t \quad (12)$$

For convenience we will take the spatial distance between the two sites to be 1 and thus refer to the energy E that equals its product with the field magnitude as the field magnitude itself. As in the case of $\hbar = 1$, this allows us to simplify the notation. Through Figures 4 and 5 of their numerical investigation,¹⁸ Raghavan et al. showed that, while dynamic localization is sharply controlled by the Bessel root condition as analyzed by (8), the occurrence of quiescent plateaus and sudden transfers (i.e., the AH structure) is unaffected by changes in the ratio $2E/\omega$ from on-resonance (Bessel roots) to off-resonance values. As explained in the Introduction, Figure 2 of the present paper also shows this insensitivity of the AH structure to the Bessel root condition. Is there then no relation of AH to DK? And is AH not sensitive to values of E/ω at all? Answers to both these questions are found immediately as follows.

For a sinusoidal field, the characteristic functions $\phi_c(t)$ and $\phi_s(t)$ in (11) are given by

$$\phi_c(t) = \cos[(2E/\omega) \sin \omega t] \quad \phi_s(t) = \sin[(2E/\omega) \sin \omega t] \quad (13)$$

If the parameter $2E/\omega$ is large with respect to (w.r.t.) 1, $\phi_c(t)$ and $\phi_s(t)$ oscillate rapidly. Because of these rapid oscillations, one may consider taking $p(s)$ out of the integral in (3) for short times (short w.r.t. the period of the field but long w.r.t. the time of oscillation of the memory). The approximation result, valid for short times, is then

$$\frac{dp(t)}{dt} + 2V^2 \frac{d[h_c^2(t) + h_s^2(t)]}{dt} p(t) = 0 \quad (14)$$

where the characteristic functions $h_c(t)$ and $h_s(t)$ are given by the integrals

$$h_c(t) = \int_0^t ds \cos[(2E/\omega) \sin \omega s] \\ h_s(t) = \int_0^t ds \sin[(2E/\omega) \sin \omega s] \quad (15)$$

Equation 14 can be solved exactly and yields

$$p(t) = e^{-2\int_0^t ds \int_0^s ds' \mathcal{W}(t,s)} = e^{-2V^2[h_c^2(t) + h_s^2(t)]} \quad (16)$$

This short time approximation is plotted in Figure 3 along with the exact solution obtained numerically to a high degree of accuracy from the original density matrix eq 5 for the parameters of Figure 1A: $V/\omega = 2$, $E/\omega = 250$. The short time approximations (16) exhibits the spurts and the plateaus of the exact

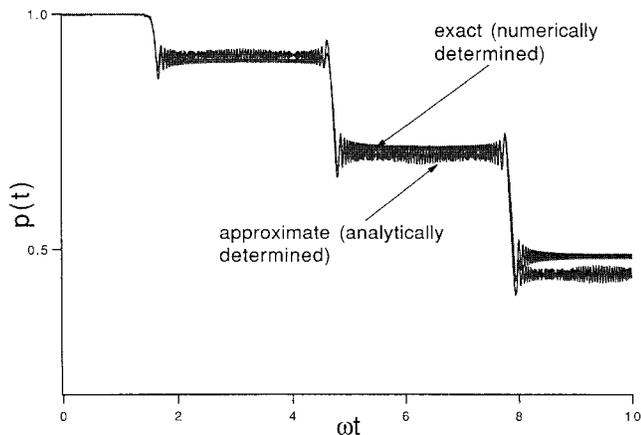


Figure 3. Comparison of the exact solution for the probability difference (obtained numerically) to the analytically obtained short time approximation obtained trivially from the memory function in the present paper. The analytic approximation (16) is seen to provide excellent agreement with the exact solution. Parameter values correspond to Figure 1A: $V/\omega = 2$ and $E/\omega = 250$.

solution simply from corresponding characteristics of the integrals h_c and h_s , and agreement is excellent.

To obtain an indication of the solution for times long w.r.t. the field period, one may replace the memory by its average \overline{W} over a period. While not accurate, this procedure does provide insight into the behavior of the solution. Equation 3 reduces to

$$\frac{d^2 p(t)}{dt^2} + 2\overline{W}p(t) = 0 \quad (17)$$

with the solution

$$p(t) = \cos(\sqrt{2\overline{W}}t) = \cos[2VJ_0(2E/\omega)t] \quad (18)$$

In obtaining the second equality, use has been made of the fact that the averages of ϕ_c and ϕ_s are $J_0(2E/\omega)$ and 0, respectively, which follows from the well-known expansion

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta} \quad (19)$$

The long time approximation¹⁸ follows the overall evolution, neglecting the short time structure. Band collapse at the Bessel root condition is recovered effortlessly since the effective bandwidth $VJ_0(2E/\omega)$ vanishes when the condition is satisfied. The long time replacement of the memory by its average invoked to pass from (3) to (17) neglects the accumulated effect of short time transfer and leads to total band collapse, which is actually not correct for the two-state system (but correct for the infinite chain as in the analysis of (8)). However, the approximate result gives an idea of the tendency of the evolution and gives the correct qualitative description of bandwidth collapse in the *infinite* system.

Immediate analytic insight into the connection of AH and DK structures can be obtained by an examination of the behavior of the characteristic functions ϕ and h . It suffices to look at ϕ_c and h_c as given by (13) and (15), respectively. Both functions are plotted in Figure 4 along with the long time approximation $h_c(t) \approx tJ_0(2E/\omega)$. Parameter values in Figure 4 are $E = 250$ and $\omega = 1$, which satisfies $2E \gg \omega$. The function $\phi_c(t)$ oscillates most rapidly around $\omega t = n\pi$ where $n = 0, 1, 2, \dots$ and least rapidly around $\omega t = (2n + 1)(\pi/2)$. Since $\sin \omega t$ behaves linearly w.r.t. ωt around the former locations and quadratically

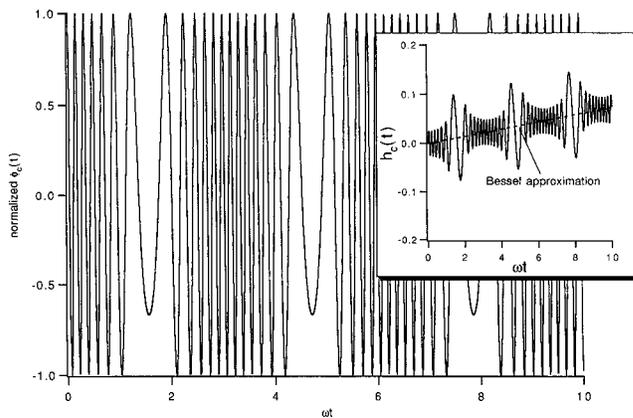


Figure 4. Clear indication of the origin of the AH and DK structures in terms of the memory functions. In the main figure we plot $\phi_c(t) = \cos[(E/\omega) \sin \omega t]$, and in the inset we show its integral $h_c(t)$ from 0 to t along with its long time approximation $tJ_0(E/\omega)$. The parameter value chosen is $E/\omega = 20$. Three time scales are discernible, their reciprocals being the field frequency ω , the field magnitude $2E$, and their geometric mean $\sqrt{E\omega}$. Windows (slow oscillations) open up in $\phi_c(t)$ at locations where the applied field is approximately linear in t . Transfer occurs in these windows, and the characteristic time scale is the reciprocal of $\sqrt{E\omega}$. Midway between these windows, where the driving field is quadratic in t , $\phi_c(t)$ oscillates most rapidly, the characteristic time scale being the reciprocal of E . The integral of $\phi_c(t)$ shows the same behavior as the probability difference itself and, on the average, follows a linear form with slope proportional to $J_0(E/\omega)$. Bandwidth collapse in the infinite system corresponds to values of $2E/\omega$ that equal Bessel roots.

around the latter locations, $\phi_c(t)$ evolves with a characteristic time that varies as $1/2E$ in the rapid regions to $1/\sqrt{E\omega}$ in the slow regions. It is in the clearly discernible windows visible in Figure 4 at the slow regions that the sudden transfer occurs. The time scale separation is clean only when $2E \gg \omega$, as can be seen easily by replotting Figure 4 for a value of $2E/\omega$, which is not too large. No AH structure appears in such cases.

The function $h_c(t)$ oscillates rapidly elsewhere but itself undergoes transfers in the slow windows, as seen in Figure 4. Expansions of the cosine of the sine given above make this clear. The long time expression $h_c(t) \approx tJ_0(2E/\omega)$, which approximates so well the actual $h_c(t)$ in Figure 4, on the average, corresponds, obviously, to the DK structure. We thus see at once that the two questions posed in the Introduction are answered as follows. While the AH structure (quiescent plateaus and sudden spurts) has little to do with the Bessel root condition, it does indeed bear a kinship to the DK structure in that the two are manifestations of the evolution in extreme time limits. Furthermore, both structures are controlled by the ratio $2E/\omega$. The AH structure appears when the ratio is large, with consequent time scale separation in the ϕ and h functions, whereas the DK structure arises when the ratio equals Bessel roots, with consequent collapse of the bandwidth. Incidentally, for the AH structure to be visible, it is not merely sufficient for $2E$ and ω to be disparate in value. In the limit that $2E/\omega$ is *small*, the cosine of the cosine remains near the value 1 and rapid oscillations over the time scale of the field period do not occur: the time scales $1/2E$ and $1/\sqrt{E\omega}$ fall, in this case, well outside the field period.

The memory functions of the problem, and consequently the evolution of the probability difference, display four time scales: the first is the period of the applied field ($1/\omega$), the second is controlled by the magnitude of the applied field ($1/2E$), the third is essentially the geometrical mean of these two ($1/\sqrt{E\omega}$), and the fourth is the bandwidth renormalized by the

Bessel function ($VJ_0(2E/\omega)$). Considering only the situation $E \gg \sqrt{E\omega} \gg \omega \gg VJ_0(2E/\omega)$, for which the structures discussed here are particularly visible, we see that internal oscillations in the probability transfer occur on the first time scale, transfer spurts occur on the second, repetition of the spurts occurs on the third, and an overall transfer occurs on the fourth time scale. There is a fifth, relatively unimportant, time scale, viz., $1/V$, over which the probability transfer occurs at very short times.

The availability of the analytic expressions (10) and (4) suggests that quantum control could be achieved by designing the time dependence of the appropriate driving fields. The applied fields discussed above have been sinusoidal and, therefore, characterized by a single time constant, the period. Interesting features of the probability evolution emerge if the applied field has multiple time constants. An examination of Figure 1A shows that quantities characterizing excitation transfer in the small quantum system can be looked upon as being the extent of the transfer during the spurts, the length of the quiescent plateau, and the amplitude of the oscillations within the plateau. All these can be controlled more or less independently by choosing an appropriate time dependence and strength of the driving field. Because the extremely low temperatures achieved in optical lattices and Bose condensates minimize the effects of undesirable scattering, the motion of atoms in optical lattices and, indeed, of entire condensates between traps can be seriously regarded as realizations of the system analyzed here. Manipulation of the electromagnetic fields forming the optical lattice or the condensate traps in order to achieve almost any desired time dependence of the driving field $\mathcal{E}(t)$ seems to be practically possible.¹⁴ In addition to the optical lattice or optical trap systems, magnetic macromolecules in giant-spin materials could provide interesting examples of systems for quantum control. As shown by Raghavan et al.,¹² both the interstate transfer matrix element V and the state energy \mathcal{E} turn out to be time dependent and determined by applied magnetic fields in mutually orthogonal directions. Considerable control is possible through independent variation in the time dependence of the applied fields, and roots of Bessel functions of orders other than 0 can be made, at will, to become significant in the evolution. Control in electron transfer in solids, for which the DK investigation was first undertaken,⁸ is more difficult (but by no means impossible⁹) because of the effect of imperfections but can indeed be attempted in superlattices which allow manipulation of the effective lattice constant and, thus, of the ratio E/ω . Device considerations introduced by Dunlap et al.¹³ would additionally suggest examples of design in frequency-to-voltage converters. The applied field can also be regarded to be provided, at least in part, by the displacement of the lattice with which the electron is interacting, although control of that part would be relatively difficult. This is the subject of the next section.

V. Relation to Electron–Phonon System and the Discrete Nonlinear Schroedinger Equation

There are remarkable overlaps of the evolution of the dynamic localization¹⁶ system studied here with interacting electron–phonon systems and the discrete nonlinear Schroedinger equation (DNLSE). In the so-called nonlinear dimer described by the DNLSE,^{22,23} the “externally applied field” $\mathcal{E}(t)$ is the displacement of a harmonic oscillator interacting so strongly with the two-state system that it is slaved by the evolution of the two-state system. The field $\mathcal{E}(t)$ is then proportional to the probability difference $p(t)$ rather than externally determined. In place of (12), one has

$$\mathcal{E}(t) = \chi p(t) \quad (20)$$

where χ measures the strength of the interaction (nonlinearity). Despite the nonlinearity of the evolution matrix Λ

$$\Lambda = \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -\chi p \\ 0 & \chi p & 0 \end{pmatrix} \quad (21)$$

in the resulting form of (5), the memory function can be determined and has the (nonlinear) form

$$\mathcal{W}(t,s) = 2V^2 \cos[\chi(\int_0^t dt' p(t'))] = 2V^2 \cos \chi[Q(t) - Q(s)] \quad (22)$$

The quantity $Q(t)$ is defined^{22,24} as the integral of the probability difference:

$$Q(t) = \int^t dt' p(t') \quad (23)$$

The memory equation (3) then undergoes an interesting reduction to the physical pendulum equation obeyed by $Q(t)$,

$$\frac{d^2 Q(t)}{dt^2} + \left(\frac{4V^2}{\chi}\right) \sin \chi Q(t) = 0 \quad (24)$$

whose solution can be written exactly in terms of Jacobian elliptic functions. Evolution of $p(t)$ exhibits self-trapping via the $cn-dn$ transition, as related elsewhere,^{22,23} and describes polaronic transfer.

This nonlinear dimer involving polaronic transfer on the one hand, and the dynamic localization dimer under sinusoidal driving fields treated in the body of the present paper on the other, can be viewed as extreme limits of a third system. The latter is the simplest interacting electron–phonon system in the approximation wherein phonons are treated classically but the electron is treated quantum mechanically. Known as the semiclassical dimer, this system evolves under the action of a matrix Λ dependent on $\mathcal{E}(t)$ as in (5), rather than its nonlinear counterpart (21) in which $p(t)$ appears explicitly, but with $\mathcal{E}(t)$ given by

$$\frac{d^2 \mathcal{E}(t)}{dt^2} + \omega^2 \mathcal{E}(t) = \omega^2 \chi p(t) \quad (25)$$

The limit in which $\mathcal{E}(t)$ is independent of $p(t)$ (which would occur if χ were to vanish) describes the dynamic localization problem. The limit in which $\mathcal{E}(t)$ is slaved by $p(t)$, i.e., the situation in which the second time derivative in (25) can be neglected as the result of a time disparity argument,^{22,23} leads to the discrete nonlinear Schroedinger equation and the phenomenon of self-trapping.

Refraining from making the classical approximation in the description of the phonons leaves one with the (fully) quantum dimer, known in some circles as the spin–boson system. Its reduction to the semiclassical dimer or the nonlinear (DNLSE) dimer has been the subject of much debate and investigation. Recent work^{25,26} has cast doubts²⁷ on the validity of the DNLSE description as well as the semiclassical description but has shown²⁸ that a certain memory function representation²⁹ provides, in many relevant parameter ranges, an excellent description of the fully quantum dimer. That memory function description involves a weak coupling approximation in the transfer matrix element V and is based on (3), but with a memory of the convolution type:

$$\mathcal{W}(t,s) = \mathcal{W}(t-s) = 2V^2 e^{-2g^2(1-\cos\omega(t-s))} \cos[2g^2 \sin(\omega(t-s))] \quad (26)$$

Here, g is the dimensionless coupling constant that describes the electron–phonon interaction, $g^2\omega$ being a measure of the polaronic depression of the energy. It is quite remarkable that, as in the dynamic localization problem, this approximated memory function in the fully quantum problem is characterized by four time constants, whose reciprocals, for large coupling ($g > 1$), are the phonon frequency ω , the product $g\omega$, the polaronic depression $g^2\omega$, and the tunneling frequency $V \exp(-g^2)$. The correspondence of the first three with the three respective quantities ω , $\sqrt{E\omega}$, and $2E$ in the dynamic localization problem is complete if one makes the identification $g^2 \rightarrow E/\omega$. The suppression factor for the bandwidth in each case is a function of the corresponding quantities g^2 and $2E/\omega$. The function is an exponential in the former and a Bessel function in the latter case. The *shape* of the spin–boson memory is, however, different from the dynamic localization memory. The exponential factor produces decays and the memory is characterized by periodic occurrences of what has been called “silent runs”.

There exists yet another physical system that unifies the dynamic localization system and the nonlinear dimer in a manner different from the semiclassical or the full quantum dimer. The evolution equation is

$$\frac{d}{dt} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} 0 & -2V & 0 \\ 2V & 0 & -2\mathcal{E} - \chi p \\ 0 & 2\mathcal{E} + \chi p & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \quad (27)$$

where the field \mathcal{E} is given by (12). It represents a system in which the strong electron–phonon interaction and the slaving assumption have already produced a nonlinear dimer to which an independent sinusoidal field is applied externally. Its study constitutes the generalization to the nonlinear domain of the DK analysis and has been carried out by Bishop and collaborators¹⁰ numerically for arbitrarily large systems. Questions have been sometimes raised as to why the numerical investigation of this system shows the Bessel root condition to continue to apply unmodified in spite of the nonlinearity. In the case of the dimer this can be understood simply in terms of the present calculation. The exact memory function resulting from (27) is

$$\mathcal{W}(t,s) = 2V^2 \cos[\chi(Q(t) - Q(s)) + \int_0^t dt' 2\mathcal{E}(t')] \quad (28)$$

The cosine may be expanded and for times long with respect to the period of the driving field \mathcal{E} , the characteristic functions ϕ may be replaced by their averages over a period as in the long time analysis of the linear dynamic localization system presented in the earlier section. Instead of (24) or (17) one now gets

$$\frac{d^2 Q(t)}{dt^2} + \left(\frac{4V^2}{\chi}\right) J_0^2(2E/\omega) \sin \chi Q(t) = 0 \quad (29)$$

which shows that bandwidth collapse is governed by the Bessel root condition despite the nonlinearity. This is in keeping with the results reported by Bishop and collaborators.¹⁰ It is also simple to analyze²⁸ for short times and conclude that the AH structure is largely unaffected.

VI. Remarks

The contribution of the present analysis to the understanding of the interesting structures exhibited in small quantum systems

under driving fields typified in Figure 1 is based on the exact derivation of a useful evolution equation, viz., (3). The advantage of the explanations presented lies in their being based on simple features of analytically known memory functions, such as the characteristic windows that open up in $\phi_c(t)$ and $\phi_s(t)$ for large values of E/ω . The variety of connections mentioned between the system considered and the DNLS, with or without external driving fields, and the spin–boson system, with or without the semiclassical approximation, are also interesting. It is hoped that the simple analysis of this paper will be particularly worthwhile in quantum design.

It might be useful to mention in passing that, instead of the integrodifferential equation that has been derived here for the probability difference, one can also derive a second-order differential equation with time dependent coefficients for the complex amplitude at each of the two states and analyze that equation through exact reductions to integral form or approximately via WKB methods in appropriate time and parameter regimes. The advantage of the equation for p provided here is that it allows a direct description of the most physical of the system quantities. The full generalization of (3) for arbitrary initial conditions $p(0)$, $q(0)$, $r(0)$, and for time dependent V is

$$\frac{dp(t)}{dt} + 2 \int_0^t ds \mathcal{W}(t,s) p(s) = I_q(t) q(0) + I_r(t) r(0) \quad (30)$$

where the memory function $\mathcal{W}(t,s)$ is

$$\mathcal{W}(t,s) = 2V(t) V(s) \cos\left(\int_s^t dt' 2\mathcal{E}(t')\right) \quad (31)$$

and the effect of the off-diagonal nature of the initial density matrix appears to arise from the initial values of $q(0)$, $r(0)$, and the explicit driving terms on the right hand side

$$I_q(t) = 2V(t) \cos\left(\int_s^t dt' 2\mathcal{E}(t')\right) \\ I_r(t) = 2V(t) \sin\left(\int_s^t dt' 2\mathcal{E}(t')\right) \quad (32)$$

Closed equations for $q(t)$ and $r(t)$ can also be written through slight modifications of the projection operators.

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- (16) Perhaps a comment about terminology may be in order here, given the festive context of this article and Harvey's well-known sense of humor. David Dunlap claims that I originally chose the term "dynamic" over "dynamical" to hint at him and me as being the duo of Batman and Robin (I do not know in what order). I am not certain of this claim but "dynamic localization" was coined to keep it separate from "static (Anderson) localization" and also from "self-trapping localization" which occurs in polaron contexts. The introduction of "dynamical localization" in recent papers in the close neighborhood of the literature of this subject has caused some amusing confusion.
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- (21) The convention in the projection operator literature, e.g., refs 19 and 20, is that the necessary time ordering operation is implied in the

exponential operator in (6). To avoid possible pitfalls in the calculation, one must keep in mind that the operators $(1 - \mathcal{P})\Lambda(t)$ commute at different times provided they operate, as they do in the exponential in (6), only on vectors of the type

$$\begin{pmatrix} 0 \\ Z_2 \\ Z_3 \end{pmatrix}$$

It also helps to note that the n th order term in the expansion of the memory turns out to be $(1/n!)[(\int_0^t dt' \mathcal{E}(t')]^n$.

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