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Iterative Solutions to Classical Second-Order Ordinary Differential Equations

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Abstract

An elementary scheme is detailed for introducing certain basic concepts in the solution of (especially) the basic second-order ordinary differential equations of classical mathematical physics. The method proposed, an integration/iteration process, allows the development of (generally Frobenius) power series, as well as exposing the rudiments of the Green function approach to solving linear ordinary differential equations. The method assumes only a background knowledge compatible with most introductory calculus courses.

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Keywords Second-order ordinary linear differential equations, iterative solutions, Green functions, computer algebra systems

1. Introduction

There is a group of second-order linear ordinary differential equations (ODE) that play a prominent role throughout the realm of Mathematical Physics [1], [4].

- Hermite's equation

$$y'' - 2xy' + \lambda y = 0 \tag{1}$$

- Chebyshev's equation

$$(1-x^2)y''-xy'+n^2y=0 \tag{2}$$

- Legendre's equation

$$(1-x^2)y''-2xy'+n(n+1)y=0 \tag{3}$$

- Gauss' hypergeometric equation

$$x(1-x)y''+(c-(a+b+1)x)y'-aby=0 \tag{4}$$

- Kummer's equation

$$xy''+(c-x)y'-ay=0 \tag{5}$$

- Laguerre's equation

$$xy''+(1-x)y'+\lambda y=0 \tag{6}$$

Our aim in this note is to motivate the search for power series solutions to equations (1) to (6), by integrating equations (1) to (6) and then enforcing an iteration scheme of solution (which usually carries the name of Picard [7]) on the resultant integral equations. We will show that equations (1), (2) and (3) can be solved immediately by direct integration and iteration, to obtain, first, power series solutions and then, with the appropriate restrictions, polynomial solutions. Having regular singular points, equations (4), (5) and (6) (which are related [1]), require a slight adjustment to the integration method, however, before we again obtain series solutions via direct integration and iteration. As a bonus, we find that the concept of a Green function emerges naturally from the analysis also. Of course, this idea/method is not original and is developed, from the theoretical point of view in, for example, Dettman [2, 3], to whom the reader is referred to for further details (though our approach involves a slight twist on that of Dettman [2])

As Dettman's approach [2, 3] is mostly theoretical (dealing, as it does, with the existence and uniqueness problem of determining series solutions to second-order linear ODE) the present paper can be looked upon as being complimentary to Dettman's work as well as providing worked examples of the iteration process. With the, now, universal availability of computer algebra systems the worked examples presented below are automatically tutorial examples for undergraduate courses on differential equations also. Further, the importance of computer algebra systems in mathematics education is now well established, by custom and practice, and it is hoped that this expository paper will add to the possible uses of such systems, particularly in the teaching of series solutions to ODE.

2. The Basic Method

First, we note that equations (1), (2) and (3) may all be re-written in the standard form

$$y'' = R(x, y, y', y'') \tag{7}$$

Integrating (7), repeatedly, from the origin, we obtain the implicit solution to equation (7) as

$$y(x) = a_0 + a_1 x + \int_0^x \left(\int_0^t R(u, y, y', y'')du \right) dt \tag{8}$$

Next, the result of integrating (8) by parts is the implicit solution to (7) in the form

$$y(x) = a_0 + a_1 x + \int_0^x (x - t) R(t, y, y', y'')dt \tag{9}$$

Interestingly, the implicit solution (9) is in the general form of a Green function solution to the equation (7). With the appropriate choice of starting function (and choice of the arbitrary constants a_0 and a_1), we may use (9) to generate iterative schemes to obtain series solutions to (7) of the form

$$y_{m+1}(x) = a_0 + a_1 x + \int_0^x (x - t) R(t, y_m, y'_m, y''_m)dt, \quad m = 1,2,3,\ldots \tag{10}$$

Consider Hermite's equation (1), with λ a constant parameter, which we rewrite as

$$y'' = 2xy' - \lambda y \tag{11}$$

so that equation (11) is in the standard form of equation (7), with

$$R(x, y, y', y'') = 2xy' - \lambda y \tag{12}$$

From (10) an (12), we obtain an implicit iterative solution scheme for (1) of the form

$$y_{m+1} = a_0 + a_1 x + \int_0^x (x - t)(2ty'_m - \lambda y_m)dt \tag{13}$$

We obtain two particular iteration schemes, giving rise to two independent particular solutions to (1), by choosing

1. $a_0 = 1$ and $a_1 = 0$, so that for $m = 1,2,3,\ldots$

$$y_{m+1} = 1 + \int_0^x (x-t)(2ty'_m - \lambda y_m)dt \tag{14}$$

If we set $y_1 = 1$, then we have an iteration scheme giving rise to the first of the particular solutions of (1), $y_1(x)$ say.

2. $a_0 = 0$ and $a_1 = 1$, so that for $m = 1,2,3,\ldots$

$$y_{m+1} = x + \int_0^x (x-t)(2ty'_m - \lambda y_m)dt \tag{15}$$

If we set $y_1 = x$, then we have an iteration scheme giving rise to the first of the particular solutions of (1), $y_2(x)$ say. After a few iterations we find that (with $a_0 = 1$ and $a_1 = 0$)

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6!}x^6 + \frac{\lambda(\lambda-4)(\lambda-8)(\lambda-12)}{8!}x^8 \mp \cdots \tag{16a}$$

and (with $a_0 = 0$ and $a_1 = 1$)

$$y_2(x) = x - \frac{(\lambda-2)}{3!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!}x^7 + \frac{(\lambda-2)(\lambda-6)(\lambda-10)(\lambda-14)}{9!}x^9 \mp \cdots \tag{16b}$$

We see then, that if $\lambda = 2n, \quad n = 0,1,2,3,\ldots$, then we get polynomial solutions to the Hermite equation – Hermite polynomials of course.

Chebyshev's equation, equation (2) provides our second example. If we rewrite (2) as

$$y'' = x^2y'' + xy' - n^2y \tag{17}$$

so that equation (17) is in the standard form of equation (7), with

$$R(x, y, y', y'') = x^2y'' + xy' - n^2y \tag{18}$$

From (10) and (18), we obtain an implicit iterative solution scheme for (2) of the form

$$y_{m+1} = a_0 + a_1 x + \int_0^x (x-t)(t^2 y''_m + t y'_m - n^2 y_m) dt \tag{19}$$

As before, we may split this into two iteration schemes

1. $$y_{m+1} = 1 + \int_0^x (x-t)(t^2 y''_m + t y'_m - n^2 y_m) dt \tag{20}$$

2. $$y_{m+1} = x + \int_0^x (x-t)(t^2 y''_m + t y'_m - n^2 y_m) dt \tag{21}$$

for $m = 1,2,3,\ldots$.

After a few iterations we find that our first particular solution is $y_1(x)$, where

$$y_1(x) = 1 - \frac{n^2}{2!} x^2 + \frac{n^2(n-2)(n+2)}{4!} x^4 - \frac{n^2(n-2)(n-4)(n+2)(n+4)}{6!} x^6 \pm \tag{22a}$$

and our second particular solution is $y_2(x)$, where

$$y_2(x) = x - \frac{(n-1)(n+1)}{3!} x^3 + \frac{(n-1)(n-3)(n+1)(n+3)}{5!} x^5 - \frac{(n-1)(n-3)(n-5)(n+1)(n+3)(n+5)}{7!} x^7 \tag{22b}$$

We see then, that if $\pm n = 0,1,2,3,\ldots$, then we get polynomial solutions to the Chebyshev equation – Chebyshev polynomials in this case.

In our third example we consider the Legendre equation. If we rewrite (3) as

$$y'' = x^2 y'' + 2xy' - n(n+1)y \tag{23}$$

so that equation (17) is in the standard form of equation (7), with

$$R(x, y, y', y'') = x^2 y'' + 2xy' - n(n+1)y \tag{24}$$

From (10) and (24), we obtain an implicit iterative solution scheme for (3) of the form

$$y_{m+1} = ax + b + \int_0^x (x-t)(t^2 y''_m + 2t y'_m - n(n+1) y_m) dt \tag{25}$$

As is now usual, we split (25) into two iteration schemes, for $m = 1,2,3,\ldots$.

1. $$y_{m+1} = 1 + \int_0^x (x-t)(t^2 y''_m + 2t y'_m - n(n+1) y_m) dt \qquad (26)$$

2. $$y_{m+1} = x + \int_0^x (x-t)(t^2 y''_m + 2t y'_m - n(n+1) y_m) dt \qquad (27)$$

After a few iterations we find that our first particular solution is $y_1(x)$, where

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!} x^6 \pm \qquad (28a)$$

and our second particular solution is $y_2(x)$, where

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} x^7 \qquad (28b)$$

We see then, that if $n = 0,1,2,3,\ldots$, then we get polynomial solutions to the Legendre equation – Legendre polynomials this time.

3. The Basic Method – A Slight Extension

Equations (4), (5) and (6) are solved by a minor extension of the previous method, involving (still) direct integration and iteration. By 'eyeballing' (4), (5) and (6), we see that they may all be re-written in the standard form (see, also, [2])

$$y'' + \frac{\gamma}{x} y' = R(x, y, y', y'') \qquad (29)$$

with $\gamma = c$ for the hypergeometric and the confluent hypergeometric equations, while $\gamma = 1$ for Laguerre's equation. Equation (29) can be considered, formally, to be a first-order linear ODE in y' with integrating factor x^γ and find that (29) becomes

$$(x^\gamma y')' = x^\gamma R(x, y, y', y'') \qquad (30)$$

Integrating (30) repeatedly, we obtain the implicit solution to equation (29) as

$$y(x) = a_0 + a_1 x^{1-\gamma} + \int_0^x \left(t^{-\gamma} \int_0^t u^{\gamma} R(u, y, y', y'') du \right) dt \quad (31)$$

Next, we may put the implicit solution (31) to equation (29) in a close relation to the previous case, by an integration by parts. The result of integrating (31) by parts is the implicit solution to (29) in the form

$$y(x) = a_0 + a_1 x^{1-\gamma} + \int_0^x \frac{t^{\gamma}}{1-\gamma} (x^{1-\gamma} - t^{1-\gamma}) R(t, y, y', y'') dt \quad (32)$$

Again, the implicit solution (32) is in the general form of a Green function solution to the equation (29). With the appropriate choice of starting function (and a_0 and a_1), we may use (32) to generate iterative schemes to obtain series solutions to (29) from

$$y_{m+1}(x) = a_0 + a_1 x^{1-\gamma} + \int_0^x \frac{t^{\gamma}}{1-\gamma} (x^{1-\gamma} - t^{1-\gamma}) R(t, y_m, y'_m, y''_m) dt \quad (33)$$

for $m = 1,2,3,\ldots$. From (33), we see that there is a possibility of non-integral powers through our iteration scheme and, more generally, Frobenius power series [1].

In our first example, we consider Gauss' hypergeometric equation (4). If we rewrite (4) as

$$y'' + \frac{c}{x} y' = xy'' + (a+b+1)y' + \frac{ab}{x} y \quad (34)$$

with a, b and c constant parameters, then we see equation (34) is in the standard form of (29), with $\gamma = c$ and

$$R(x, y, y', y'') = xy'' + (a+b+1)y' + \frac{ab}{x} y \quad (35)$$

so that the implicit iterative solution scheme solution to (34) is of the form (33), or

$$y_{m+1}(x) = a_0 + a_1 x^{1-\gamma} + \int_0^x \frac{t^{c}}{1-c} (x^{1-c} - t^{1-c}) [t y''_m(t) + (a+b+1) y'_m(t) + \frac{ab}{t} y_m(t)] dt \quad (36)$$

To obtain the *hypergeometric series solution* to (4), we let $a_0 = 1$ and $a_1 = 0$ in (36) and generate the specific recurrence relation

$$y_{m+1}(x) = 1 + \int_0^x \frac{t^c}{1-c}(x^{1-c} - t^{1-c})[ty''_m(t) + (a+b+1)y'_m(t) + \frac{ab}{t} y_m(t)]dt \tag{37}$$

for $m = 1,2,3,\ldots$. Using the starting function $y_1 = 1$, the first few iterations lead to

$$y(x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)}{c(c+1)(c+2)(c+3)}\frac{x^4}{4!} + \cdots \tag{38}$$

and we recognise the leading terms of the *hypergeometric series*.

We may obtain, also, a second solution to the hypergeometric equation by the current method, if we only restrict the parameter c. If, in (36), we let $a_0 = 1$ and $a_1 = 0$ then, for $m = 1,2,3,\ldots$, we can write down the recurrence relation

$$y_{m+1}(x) = x^{1-c} + \int_0^x \frac{t^c}{1-c}(x^{1-c} - t^{1-c})[ty''_m(t) + (a+b+1)y'_m(t) + \frac{ab}{t} y_m(t)]dt \tag{39}$$

Using the starting function $y_1(x) = x^{1-c}$, the recurrence relation (39) generates after a few iterations the first few terms of a series solution to (4) of the form

$$y(x) = x^{-c}\left(x - \frac{(a-c+1)(b-c+1)}{(c-2)}x^2 + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!(c-2)(c-3)}x^3 - \frac{(a-c+1)(a-c+2)(a-c+3)(b-c+1)(b-c+2)(b-c+3)}{6!(c-2)(c-3)(c-4)}x^4 \pm \cdots \right) \tag{40}$$

and c may not be a positive integer.

We note in passing, that the power series (40) is actually a Frobenius power series. The possibility of such power series arising from our more general iterative scheme (33) has been noted already.

Kummer's differential equation (5), known also as the confluent hypergeometric equation, can be rewritten in the standard form (29) as (see, also [6])

$$y'' + \frac{c}{x}y' = y' + \frac{a}{x}y \tag{41}$$

with a and c constant parameters. We see equation (31) is in the standard form of (29), with $\gamma = c$ and

$$R(x, y, y', y'') = y' + \frac{a}{x} y \tag{42}$$

and the implicit iterative solution scheme solution to (34) is of the form (33), that is

$$y_{m+1}(x) = a_0 + a_1 x^{1-\gamma} + \int_0^x \frac{t^c}{1-c}(x^{1-c} - t^{1-c})\left(y'_m + \frac{a}{t} y_m \right) dt \tag{43}$$

If we let $a_0 = 1$ and $a_1 = 0$ in (43) and, for $m = 1,2,3,\ldots$, generate the recurrence relation

$$y_{m+1}(x) = 1 + \int_0^x \frac{t^c}{1-c}(x^{1-c} - t^{1-c})\left(y'_m + \frac{a}{t} y_m \right) dt \tag{44}$$

then, using the starting function $y_1 = 1$, we get the first few terms of the well-known confluent hypergeometric series as

$$y(x) = 1 + \frac{a}{c} x + \frac{a(a+1)}{c(c+1)} \frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)} \frac{x^3}{3!} + \frac{a(a+1)(a+2)(a+3)}{c(c+1)(c+2)(c+3)} \frac{x^4}{4!} + \frac{a(a+1)(a+2)(a+3)(a+4)}{c(c+1)(c+2)(c+3)(c+4)} \frac{x^5}{5!} + \cdots \tag{45}$$

provided c is never zero or a negative integer.

On the other hand, If we let $a_0 = 0$ and $a_1 = 1$ in (43) and, for $m = 1,2,3,\ldots$, generate the recurrence relation

$$y_{m+1}(x) = x^{1-c} + \int_0^x \frac{t^c}{1-c}(x^{1-c} - t^{1-c})\left(y'_m + \frac{a}{t} y_m \right) dt \tag{46}$$

then, using the starting function $y_1 = x^{1-c}$, we get the first few terms of the *second confluent hypergeometric series* as

$$y(x) = x^{-c}\left(x - \frac{(a-c+1)}{(c-2)}x^2 + \frac{(a-c+1)(a-c+2)}{(c-2)(c-3)}\frac{x^3}{2!} - \frac{(a-c+1)(a-c+2)(a-c+3)}{(c-2)(c-3)(c-4)}\frac{x^4}{3!} \pm \cdots \right) \tag{47}$$

provided c is never a positive integer greater than one. Again, (47) presents the first few terms of a Frobenius power series. Apparently, polynomial solutions to Kummer's equation exist when $c = 1$ and a is a non-negative integer.

Finally, for Laguerre's equation, (6), we write

$$y'' + \frac{1}{x}y' = y' - \frac{\lambda}{x}y \tag{48}$$

with λ constant. Equation (48) is in the standard form of (29), with $\gamma = 1$ and

$$R(x, y, y', y'') = y' - \frac{\lambda}{x}y \tag{49}$$

In this case, it appears that the general formula (33) must break down. However, if we take the limit as $\gamma \to 1$ in (33), then the resulting formula reads

$$y(x) = a + \int_0^x t \ln\left|\frac{x}{t}\right| R(t, y, y', y'')dt \tag{50}$$

where $a = a_0 + a_1$ and we see that, with (49) in mind, the method will yield a single particular solution to Laguerre's equation, through the iteration routine ($a = 1$)

$$y_{m+1}(x) = 1 + \int_0^x t \ln\left|\frac{x}{t}\right|\left(y'_m - \frac{\lambda}{x}y_m \right)dt \tag{51}$$

for $m = 1,2,3,\ldots$. Obviously, this result, (51), can be obtained by direct integration of (48). With the usual starting function $y_1 = 1$, the iteration routine (51) gives us

$$y(x) = 1 - \lambda x + \frac{\lambda(\lambda-1)}{2^2}x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{2^2 3^2}x^3 + \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{2^2 3^2 4^2}x^4 - \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{2^2 3^2 4^2 5^2}x^5 \pm \cdots \tag{52}$$

Once more, we see that if the parameter λ is a non-negative integer, then we may expect polynomial solutions to Laguerre's equation – Laguerre's polynomials.

If we consider the solutions of Kummer's equation and Laguerre's equation then, if we set $c = 1$ in (47), we find that (47) reduces to

$$y(x) = 1 + ax + \frac{a(a+1)}{2^2}x^2 + \frac{a(a+1)(a+2)}{2^2 3^2}x^3 + \cdots \qquad (53)$$

and we get the first few terms of (52), with $\lambda = -a$, as we would expect from an examination of the two differential equations, (41) and (48).

4. Conclusions

Requiring only a knowledge of basic integration, differentiation and the idea of an iterative solution to an equation (all of which are standard to introductory calculus courses), we see that the elementary method proposed above leads naturally to the idea of searching for power series solutions to linear ODE. In addition, the idea of a Green function solution to a linear ODE emerges in an equally natural fashion from the general method. Finally, it is a simple matter to apply this integration/iteration process to first-order linear ODE, other types of second-order linear ODE (especially those with constant coefficients) and even to certain non-linear ODE [7],the critical point being, as mentioned in the introduction, the universal availability of computer algebra systems to enable students to perform the calculations (swiftly and accurately).

The basic method presented here, again as mentioned in the introduction, is not new and other examples of its application can be found elsewhere, under varying circumstances for

different types of second-order ODE [7]. It is possible to further generalize this methodology, in fact to higher-order linear ODE and the theory behind this is presented by Fabrey [5], who considers, also, the inhomogeneous ODE, although Fabrey limits his discussion to the uniqueness of such solutions of higher-order ODE. Finally, if a second solution is not obtained immediately by the iteration, then in certain circumstances it may be possible to apply the Wronskian method [8] to obtain a second solution. The main point is, that with two linearly independent solutions, $\mathrm{y}_1(\mathrm{x})$ and $\mathrm{y}_2(\mathrm{x})$ say, to the general second-order linear ODE (with coefficients $a(x)$, $b(x)$ and $c(x)$)

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0 \qquad (54)$$

then the Wronskian, $W(x)$ of equation (54) satisfies the relation

$$W(x) \equiv \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(0)e^{-\int \frac{b(x)}{a(x)}dx} \tag{55}$$

with $W(0)$ a constant, which, in turn leads to a first order equation for one solution, given the other solution to (54). For full details, the interested reader is referred to [8].

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