On the varieties of representations and characters of a family of one-relator groups. Their irreducible components

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Abstract

Let us consider the group $G = \langle x, y \mid x^m = y^n \rangle$ with m and n nonzero integers. In this paper, we study the variety of representations $R(G)$ and the character variety $X(G)$ in $SL(2,\mathbb{C})$ of the group G, obtaining by elementary methods an explicit primary decomposition of the ideal corresponding to $X(G)$ in the coordinates $X = t_x$, $Y = t_y$ and $Z = t_{xy}$. As an easy consequence, a formula for computing the number of irreducible components of $X(G)$ as a function of m and n is given. Finally we compute the number of irreducible components of $R(G)$ and show that the projection $t : R(G) \longrightarrow X(G)$ does not preserve the number of irreducible components.

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Introduction

Given a finitely generated group G, the set $R(G)$ of its representations over $SL(2,\mathbb{C})$ can be endowed with the structure of an affine algebraic variety (see [11]), the same holds for the set $X(G)$ of characters of representations over $SL(2,\mathbb{C})$ (see [1]). Since different presentations of a group G give rise to isomorphic representation and character varieties; the study of geometric invariants of $R(G)$ and $X(G)$ like the dimension or the number of irreducible components is of interest in combinatorial group theory (see [9, 10, 14] for instance). The varieties of representations and characters have also many applications in 3-dimensional geometry and topology (see [6, 5, 16] for instance).

In [2, Theor. 3.2.] an explicit set of polynomials defining the character variety of a finitely presented group was given. Nevertheless this family of polynomials is not always satisfactory in order to give a geometrical description of the character variety. In this work, using elementary algebraic and arithmetic methods, we give an explicit primary

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decomposition of the ideal corresponding to the character variety of the group $G_{m,n} = \langle x, y \mid x^m = y^n \rangle$ with $m, n \neq 0$, thus obtaining an easy geometrical description of it. This easier description allows us to compute geometrical invariants such as the number of irreducible components and to study its relation with the variety of representations. The representation variety of $G_{m,n}$ was already studied in [10] where its dimension and the number of irreducible components of maximal dimension were computed. We obtain here those results by a different approach and complete them by computing the number of irreducible components also for smaller dimension.

Observe that if $\gcd(m, n) = 1$ then $G_{m,n}$ is precisely the fundamental group of the exterior of the (m, n)-torus knot $K_{m,n}$, thus we have obtained the character variety for any torus knot. As an application we use our work to study the relation between the character variety of a torus knot $K_{m,n}$ and that of its mirror image $K_{m,-n}$.

The paper is organized as follows. In Section 1, we recall the construction of the character variety of a finitely presented group. In Section 2, we introduce some families of polynomials and give some technical results about them which are very useful in subsequent sections. Section 3 is devoted to present the most important result of this paper, giving a complete description of the ideal associated with $X(G)$. In Section 4, we use our main result to explicitly compute the number of irreducible components of $X(G)$ and in Section 5 we compute the number of irreducible components of the variety of representations $R(G)$ studying their relation with those of $X(G)$. Finally in Section 6 we give some brief comments concerning the abelian component of $X(G)$, the relationship between the character variety of the exterior of a torus knot and that of its mirror image and an even simpler description of $X(G)$ in the case of the torus knots $K_{m,2}$.

1 The character variety of a finitely presented group

Let G be a group, a *representation* $\rho : G \longrightarrow SL(2, \mathbb{C})$ is just a group homomorphism. We say that two representations ρ and ρ' are equivalent if there exists $P \in SL(2, \mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise we say ρ is *irreducible*. The following proposition presents some useful characterizations of reducibility.

Proposition 1.1. *(see [1, Lemma 1.2.1. and Prop. 1.5.5.])*

1) *Let $\rho : G \longrightarrow SL(2, \mathbb{C})$ be a representation. The following conditions are equivalent:*

(a) ρ is reducible.

(b) $\rho(G)$ is, up to conjugation, a subgroup of upper triangular matrices.

(c) $\operatorname{tr}\rho(g) = 2$ for all g in the commutator $G' = [G, G]$.

2) *If G is generated by two elements g and h, then $\rho : G \longrightarrow SL(2, \mathbb{C})$ is reducible if and only if $\operatorname{tr}\rho([g, h]) = 2$.*

Now, let us consider a finitely presented group $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_s \rangle$ and let $\rho : G \longrightarrow SL(2, \mathbb{C})$ be a representation. It is clear that ρ is completely determined by the k-tuple $(\rho(x_1), \dots, \rho(x_k))$ and thus we can identify

$$R(G) = \{(\rho(x_1), \dots, \rho(x_k)) \mid \rho \text{ is a representation of } G\} \subseteq \mathbb{C}^{4k}$$

with the set of all representations of G into $SL(2, \mathbb{C})$, which is therefore (see [1]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : G \longrightarrow SL(2, \mathbb{C})$ its character $\chi_\rho : G \longrightarrow \mathbb{C}$ is defined by $\chi_\rho(g) = \operatorname{tr} \rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is irreducible [1, Prop. 1.5.2.]. Now choose any $g \in G$ and define $t_g : R(G) \longrightarrow \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$. It is easily seen that $T = \{t_g \mid g \in G\}$ is a finitely generated ring ([1, Prop. 1.4.1.]) and, moreover, it can be shown using the well-known identities

$$\begin{aligned} \operatorname{tr} A &= \operatorname{tr} A^{-1} \\ \operatorname{tr} AB &= \operatorname{tr} BA \\ \operatorname{tr} AB &= \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1} \end{aligned} \tag{1}$$

which hold in $SL(2, \mathbb{C})$ (see [2, Cor. 4.1.2.]) that T is generated by the set:

$$\{t_{x_i}, t_{x_i x_j}, t_{x_i x_j x_h} \mid 1 \leq i < j < h \leq k\}.$$

Now choose $\gamma_1, \dots, \gamma_\nu \in G$ such that $T = \langle t_{\gamma_i} \mid 1 \leq i \leq \nu \rangle$ and define the map $t : R(G) \longrightarrow \mathbb{C}^\nu$ by $t(\rho) = (t_{\gamma_1}(\rho), \dots, t_{\gamma_\nu}(\rho))$. Observe that $\nu \leq \frac{k(k^2+5)}{6}$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety which is well defined up to canonical isomorphism [1, Cor. 1.4.5.] and is called the *character variety* of the group G in $SL(2, \mathbb{C})$. Note that $X(G)$ can be identified with the set of all characters χ_ρ of representations $\rho \in R(G)$.

For every $0 \leq j \leq k$ and for every $1 \leq i \leq s$ we have that $p_{ij} = t_{r_i x_j} - t_{x_j}$ is a polynomial with rational coefficients in the variables $\{t_{x_{i_1} \dots x_{i_m}} \mid m \leq 3\}$, (see [2, Cor. 4.1.2.]). Then, we have the following explicit description of $X(G)$.

Theorem 1.2. *([2, Theor. 3.2.])* $X(G) = \{\overline{x} \in X(F_k) \mid p_{ij}(\overline{x}) = 0,\ \forall i, j\}$, *where* F_k *is the free group in* k *generators.*

Example 1.3. **1)** $X(F_1) = \mathbb{C}, \quad X(F_2) = \mathbb{C}^3$.

2) Let $G = \langle x, y \mid xyx^{-1}y^{-1} \rangle$. It can be seen using the formulas given in (1) (see [7, Ex. 2] for instance) that $X(G) = \{(X, Y, Z) \in \mathbb{C}^3 \mid X^2 + Y^2 + Z^2 - XYZ - 4 = 0\}$. Observe that G is the fundamental group of the two-dimensional torus. In what follows we will denote $D(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$, this polynomial will play a very important role in our paper since it satisfies $D(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr}[A, B] - 2$ for all $A, B \in SL(2, \mathbb{C})$.

Remark 1.4. Since the character variety of $X(F_1)$ is the whole field $\mathbb{C}$, the map $\operatorname{tr} : SL(2, \mathbb{C}) \longrightarrow \mathbb{C}$ given by the trace of a matrix is surjective. Therefore if two polynomials in one variable coincide on $\operatorname{tr}(SL(2, \mathbb{C}))$ then they are equal as polynomials.

In the same way, since $X(F_2) = \mathbb{C}^3$, the map $t : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow \mathbb{C}^3$ given by $t(A,B) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$ is surjective and thus if one wants to see that two polynomials in three variables are equal, it is enough to check that these polynomials coincide on $t(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))$.

Note that, since $X(F_k) \neq \mathbb{C}^{\frac{k(k^2+5)}{6}}$ for all $k \geq 3$, the previous considerations do not generalize for an arbitrary k. In the sequel we will make use of these observations without an explicit reference.

We will end this section with a couple of technical results.

Lemma 1.5. *Let G and H be groups and let $\varphi : R(G) \longrightarrow R(H)$ be a polynomial map. Then, there exists a unique polynomial map $\psi : X(G) \longrightarrow X(H)$ such that $t \circ \varphi = \psi \circ t$.*

$$\begin{array}{ccc} R(G) & \xrightarrow{\ \varphi\ } & R(H) \\ {\scriptstyle t}\downarrow & \circlearrowleft & \downarrow{\scriptstyle t} \\ X(G) & \overset{\psi}{\dashrightarrow} & X(H) \end{array}$$

Corollary 1.6. *Let G and H be groups and let $f : H \longrightarrow G$ be a group homomorphism. Then, there exists a unique polynomial map $\psi : X(G) \longrightarrow X(H)$ such that $\psi(t(\rho)) = t(\rho \circ f)$ for every $\rho \in R(H)$.*

2 Some families of polynomials

In the forthcoming sections, we will need some particular families of polynomials whose definition and properties are given below.

Given $c_0(T), c_1(T) \in \mathbb{C}[T]$ two polynomials, we define $\mathcal{F}_k^{(c_0,c_1)}(T)$ as follows:

$$\mathcal{F}_k^{(c_0,c_1)} : \begin{cases} \mathcal{F}_0^{(c_0,c_1)}(T) = c_0, \\ \mathcal{F}_1^{(c_0,c_1)}(T) = c_1, \\ \mathcal{F}_k^{(c_0,c_1)}(T) = T\mathcal{F}_{k-1}^{(c_0,c_1)}(T) - \mathcal{F}_{k-2}^{(c_0,c_1)}(T). \end{cases}$$

Then we denote by f_k and h_k the polynomials $\mathcal{F}_k^{(2,T)}$ and $\mathcal{F}_k^{(0,1)}$ respectively. Note that these families are closely related to the Chebyshev polynomials (see [13]), in fact it can be shown that $f_k(2X) = 2T_k(X)$ and $h_k(2X) = U_{k-1}(X)$ for all k, where T_k (resp. U_k) is the Chebyshev polynomial of the first (resp. second) kind.

Using (1) it is possible to prove that f_k is the only polynomial in one variable which satisfies $f_k(\operatorname{tr} A) = \operatorname{tr} A^k$ for all $A \in SL(2,\mathbb{C})$. Analogously, given $a, b \in \mathbb{Z}$, it can be noted that there exists only one polynomial in three variables verifying $H(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr} A^a B^{-b}$ for all $A, B \in SL(2,\mathbb{C})$. We shall denote this polynomial by $F_{a,b}(X,Y,Z) \in \mathbb{C}[X,Y,Z]$. Now let us consider,

$$s_k(T) : \begin{cases} s_0(T) = 0, \\ s_1(T) = s_2(T) = 1, \\ s_3(T) = T + 1, \\ s_k(T) = Ts_{k-2}(T) - s_{k-4}(T). \end{cases} \qquad \sigma_k(T) : \begin{cases} \sigma_0(T) = 0, \\ \sigma_1(T) = \sigma_2(T) = 1, \\ \sigma_3(T) = T - 1, \\ \sigma_k(T) = T\sigma_{k-2}(T) - \sigma_{k-4}(T). \end{cases}$$

Although we have defined these families of polynomials for $k \in \mathbb{N}$, they can clearly be extended to arbitrary $k \in \mathbb{Z}$. Finally, let $\kappa : \mathbb{C}^3 \to \mathbb{C}^3$ be the involution given by $\kappa(X, Y, Z) = (-X, -Y, Z)$.

Proposition 2.1. *Let a, b, i, j, k be integers, then:*

1) $F_{a,b}(X, Y, Z) = F_{b,a}(Y, X, Z), \quad F_{-a,-b}(X, Y, Z) = F_{a,b}(X, Y, Z)$.

2) $F_{k,0}(X, Y, Z) = f_k(X), \quad F_{0,k}(X, Y, Z) = f_k(Y)$.

3) $s_{-k}(T) = -s_k(T), \quad \sigma_{-k}(T) = (-1)^{k-1}\sigma_k(T), \quad h_{-k}(T) = -h_k(T), \quad f_{-k}(T) = f_k(T)$.

4) $\kappa(f_k(X)) = (-1)^k f_k(X), \quad \kappa(s_k(X)) = (-1)^{\left[\frac{k-1}{2}\right]}\sigma_k(X)$ *and* $\kappa(F_{a,b}(X, Y, Z)) = (-1)^{a-b}F_{a,b}(X, Y, Z)$.

5) *If m is a positive integer, then*

$$s_m(T) = \begin{cases} 1 + \sum_{i=1}^{\frac{m-1}{2}} f_i(T) & \text{if } m \text{ is odd}, \\ \sum_{i=1}^{\frac{m}{4}} f_{2i-1}(T) & \text{if } m \equiv 0, \\ 1 + \sum_{i=1}^{\frac{m-2}{4}} f_{2i}(T) & \text{if } m \equiv 2, \end{cases} \qquad (-1)^{\frac{m-1}{2}}\sigma_m(T) = 1 + \sum_{i=1}^{\frac{m-1}{2}} (-1)^i f_i(T) \quad \text{if } m \text{ is odd},$$

$$h_m(T) = \begin{cases} 1 + \sum_{i=1}^{\frac{m-1}{2}} f_{2i}(T) & \text{if } m \text{ is odd}, \\ \sum_{i=1}^{\frac{m}{2}} f_{2i-1}(T) & \text{if } m \text{ is even}, \end{cases}$$

where the congruences are taken modulo 4.

6) $f_i(T) \cdot f_j(T) = f_{i+j}(T) + f_{i-j}(T)$.

7) $h_{2k+1}(T) = s_{2k+1}(T)\sigma_{2k+1}(T), \quad h_{2k}(T) = s_{2k}(T) f_k(T)$.

Proof. 1) - 5) Follow by definition and/or inductive arguments. For 6) take $T = \operatorname{tr} A$ with $A \in SL(2, \mathbb{C})$. Then $f_{i+j}(T) + f_{i-j}(T) = \operatorname{tr} A^i A^j + \operatorname{tr} A^i A^{-j} = \operatorname{tr} A^i \cdot \operatorname{tr} A^j = f_i(T) \cdot f_j(T)$. 7) It is an easy consequence of 5) and 6) when k is positive. The negative case follows immediately from the positive one together with 3) and the first part of 4). □

3 The main result

In what follows m and n will be assumed to be nonzero integers. Let $G_{m,n}$ be the group with presentation $G_{m,n} = \langle x, y \mid x^m = y^n \rangle$. Note that if $\gcd(m, n) = 1$ this group is isomorphic to the fundamental group of the exterior of the (m, n)-torus knot. We are interested in computing $X(G_{m,n})$, its character variety in $SL(2, \mathbb{C})$. Let $w = x^m y^{-n}$, then from Theorem 1.2, the ideal J corresponding to $X(G_{m,n})$ can be generated by $t_w - 2$, $t_{wx} - t_x$ and $t_{wy} - t_y$ in the ring of polynomials $\mathbb{C}[t_x, t_y, t_{xy}]$. In other words $J = \langle F_{m,n}(X, Y, Z) - 2, F_{m+1,n}(X, Y, Z) - X, F_{m,n-1}(X, Y, Z) - Y \rangle$, where $X = t_x$, $Y = t_y$ and $Z = t_{xy}$. Since $x^i y^{-k} = x^j y^{-l} \in G_{m,n}$ whenever $m = i - j$ and $n = k - l$, all the polynomials $t_{x^i y^{-k}} - t_{x^j y^{-l}} = F_{i,k}(X, Y, Z) - F_{j,l}(X, Y, Z)$ must belong to J. Therefore

$J = \langle F_{i,k}(X,Y,Z) - F_{j,l}(X,Y,Z) \mid m = i-j, n = k-l \rangle \subset \mathbb{C}[X,Y,Z]$. It is also possible to verify that $F_{i,k} - F_{j,l} \in J$ when $m = i+j$ and $n = k+l$, and hence we can write, if necessary, $m = i+j$ and $n = k+l$ instead of $m = i-j$ and $n = k-l$ in the above expression of J. Now, associated with (m,n), let us define some ideals in the ring $\mathbb{C}[X,Y,Z]$ which are closely related to J.

$$I_1 = \langle s_m(X), s_n(Y) \rangle, \qquad I_2 = \begin{cases} \langle \sigma_m(X), \sigma_n(Y) \rangle & \text{if } m,n \text{ are odd,} \\ \langle \sigma_m(X), f_{\frac{n}{2}}(Y) \rangle & \text{if } m \text{ is odd and } n \text{ is even,} \\ \langle f_{\frac{m}{2}}(X), \sigma_n(Y) \rangle & \text{if } m \text{ is even and } n \text{ is odd,} \\ \langle f_{\frac{m}{2}}(X), f_{\frac{n}{2}}(Y) \rangle & \text{if } m,n \text{ are even.} \end{cases}$$

$$I_3 = J + D.$$

The aim of this section is to present the main result of this paper. Specifically, we will prove the following theorem:

Theorem 3.1. $V(J) = V(I_1 \cap I_2 \cap I_3) = V(I_1) \cup V(I_2) \cup V(I_3)$.

The proof of this theorem is rather technical and will be split into two parts. In the first part we will prove that J is contained in $I_1 \cap I_2 \cap I_3$ and in the second one we will prove the inclusion $I_1 I_2 I_3 \subseteq J$. This suffices since we will have $I_1 I_2 I_3 \subseteq J \subseteq I_1 \cap I_2 \cap I_3$, consequently $\sqrt{I_1 I_2 I_3} \subseteq \sqrt{J} \subseteq \sqrt{I_1 \cap I_2 \cap I_3}$ and, as we are working over an algebraically closed field, it follows that $\sqrt{I_1 I_2 I_3} = \sqrt{I_1 \cap I_2 \cap I_3}$ which completes the proof.

3.1 The first inclusion

In order to prove that J is contained in $I_1 \cap I_2 \cap I_3$, we need some preliminary lemmas which give us some relations between $F_{a,b}$, f_k, s_k and σ_k.

Lemma 3.2. *For all integers a, b, i, j, k, l, the following expressions hold:*

1) $F_{a,k}(X,Y,Z) f_b(X) = F_{a+b,k}(X,Y,Z) + F_{a-b,k}(X,Y,Z)$.

2) $F_{j,a}(X,Y,Z) f_b(Y) = F_{j,a+b}(X,Y,Z) + F_{j,a-b}(X,Y,Z)$.

3) *In particular, if $m = i-j$ is even then $F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z) \in \langle f_{\frac{m}{2}}(X) \rangle$, and if $n = k-l$ is even then $F_{j,k}(X,Y,Z) + F_{j,l}(X,Y,Z) \in \langle f_{\frac{n}{2}}(Y) \rangle$.*

Proof. 1) Let $A, B \in SL(2,\mathbb{C})$ and put $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. Then

$$\begin{aligned} F_{a+b,k}(X,Y,Z) + F_{a-b,k}(X,Y,Z) &= \operatorname{tr} A^{a+b}B^{-k} + \operatorname{tr} A^{a-b}B^{-k} = \\ = \operatorname{tr} A^b A^a B^{-k} + \operatorname{tr} A^{-b} A^a B^{-k} = \operatorname{tr} A^b \cdot \operatorname{tr} A^a B^{-k} &= f_b(x) F_{a,k}(X,Y,Z). \end{aligned}$$

2) Follows from 1), since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$.

3) Take $i = a+b$ and $j = a-b$ in 1). One obtains

$$F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z) = F_{\frac{i+j}{2},k}(X,Y,Z) f_{\frac{m}{2}}(X) \in \langle f_{\frac{m}{2}}(X) \rangle.$$

In a similar way we can prove the second part of 3). In this case, take $k = a+b$ and $l = a-b$ in 2). ☐

Lemma 3.3. *Let $\{p_k\}_{k\in\mathbb{Z}}$ be a family of polynomials satisfying the recursive equation $p_k = Tp_{k-1} - p_{k-2}$ for all $k \in \mathbb{Z}$ and let $\lambda, \mu : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be two maps verifying the following conditions:*

$$\begin{cases}\lambda(i,j) = \lambda(i-1,j+1)\\ \lambda(j+1,j) = j+1\\ \lambda(j+2,j) = j+2\end{cases} \qquad \begin{cases}\mu(i,j) = \mu(i-1,j+1)\\ \mu(j+1,j) = j\\ \mu(j+2,j) = j\end{cases} \qquad \forall i,j \in \mathbb{Z}. \tag{2}$$

Then $p_i - p_j = s_{i-j}(T)\left(p_{\lambda(i,j)} - p_{\mu(i,j)}\right)$ for all $i, j \in \mathbb{Z}$.

Proof. First, suppose that $i - j \geq 0$ and let us prove the assertion by induction on $a = i - j \geq 0$. The claim is clear for $a = 0, 1, 2$, since $s_0(T) = 0$, $s_1(T) = s_2(T) = 1$ and λ, μ satisfy the conditions in (2) by hypothesis. For $a = 3$ we note that $\lambda(j+3, j) = \lambda(j+2, j) = j+2$ and $\mu(j+3, j) = \mu(j+2, j+1) = j+1$, and hence

$$\begin{aligned} s_3(T)\left(p_{\lambda(j+3,j)} - p_{\beta(j+3,j)}\right) &= (T+1)(p_{j+2} - p_{j+1}) = \\ (Tp_{j+2} - p_{j+1}) - (Tp_{j+1} - p_{j+2}) &= p_{j+3} - p_j. \end{aligned}$$

Let $a = i - j \geq 4$ and assume that the formula is true for all integers less than a and greater than or equal to 0. Then

$$\begin{aligned} s_{i-j}(T)\left(p_{\lambda(i,j)} - p_{\mu(i,j)}\right) &= \left(Ts_{i-j-2}(T) - s_{i-j-4}(T)\right)\left(p_{\lambda(i,j)} - p_{\mu(i,j)}\right) = \\ = Ts_{(i-1)-(j+1)}(T)&\left(p_{\lambda(i-1,j+1)} - p_{\mu(i-1,j+1)}\right) - \\ &- s_{(i-2)-(j+2)}(T)\left(p_{\lambda(i-2,j+2)} - p_{\mu(i-2,j+2)}\right) \overset{\text{(IH)}}{=} \\ \overset{\text{(IH)}}{=} T(p_{i-1} - p_{j+1}) - (p_{i-2} - p_{j+2}) &= (Tp_{i-1} - p_{i-2}) - (Tp_{j+1} - p_{j+2}) = p_i - p_j. \end{aligned}$$

Now suppose that $a = i - j < 0$. Since $s_{-k}(T) = -s_k(T)$, to finish the proof, it is enough to see that $\lambda(i,j) = \lambda(j,i)$ and $\mu(i,j) = \mu(j,i)$. Assume, for instance, that $i > j$. There exists a positive integer k such that $i = j + k$. Hence

$$\lambda(i,j) = \lambda(j+k,j) = \lambda(j+k-1,j+1) = \cdots = \lambda(j,j+k) = \lambda(j,i).$$

Analogously, $\mu(i,j) = \mu(j,i)$ and this finishes the proof. □

Remark 3.4. In fact, λ and μ are uniquely determined by conditions (2).

Corollary 3.5. *Using the notation in Section 2, we have:*

1) $F_{i,k}(X,Y,Z) - F_{j,k}(X,Y,Z) = s_{i-j}(X)\left(F_{\left[\frac{i+j+2}{2}\right],k}(X,Y,Z) - F_{\left[\frac{i+j-1}{2}\right],k}(X,Y,Z)\right).$

2) $F_{i,k}(X,Y,Z) - F_{i,l}(X,Y,Z) = s_{k-l}(Y)\left(F_{i,\left[\frac{k+l+2}{2}\right]}(X,Y,Z) - F_{i,\left[\frac{k+l-1}{2}\right]}(X,Y,Z)\right).$

In particular, if $m = i - j$ then $F_{i,k}(X,Y,Z) - F_{j,k}(X,Y,Z) \in \langle s_m(X)\rangle$, and if $n = k - l$ then $F_{i,k}(X,Y,Z) - F_{i,l}(X,Y,Z) \in \langle s_n(Y)\rangle$. Moreover, if $m = i - j$ and $n = k - l$ are odd then $F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z)$ belongs to the ideal generated by $\sigma_m(X)$ while $F_{j,k} + F_{j,l}$ belongs to the ideal generated by $\sigma_n(Y)$.

Proof. 1) Note that $\lambda(i,j) = [\frac{i+j+2}{2}]$ and $\mu(i,j) = [\frac{i+j-1}{2}]$ satisfy the conditions (2) in Lemma 3.3, and we can take $p_i(X) = F_{i,k}(X,Y,Z)$ which verifies the corresponding recursive equation. Since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$, 2) follows from 1).

Now, assume that m and n are odd and let us apply to expression 1) the involution $\kappa : \mathbb{C}^3 \to \mathbb{C}^3$ given by $\kappa(X,Y,Z) = (-X,-Y,Z)$. Then the claim follows by Proposition 2.1 (4) and (1). □

Theorem 3.6. *$J \subseteq I_1 \cap I_2 \cap I_3$.*

Proof. Let i,j,k,l be integers such that $m = i - j$ and $n = k - l$. Recall that I_3 is by definition the ideal $J + D$. Therefore J is contained in I_3. We divide the rest of the proof in two parts.

1) $\underline{J \subset I_1}$. The difference $F_{i,k}(X,Y,Z) - F_{j,l}(X,Y,Z)$ can be written in the form

$$F_{i,k} - F_{j,l} = (F_{i,k} - F_{j,k}) + (F_{j,k} - F_{j,l}),$$

which belongs to $\langle s_m(X), s_n(Y)\rangle = I_1$ from Corollary 3.5.

2) $\underline{J \subset I_2}$. Essentially, three cases can occur.

- m and n are odd. In this case, the involution κ can be used. Since $J \subseteq I_1$, $\kappa(J) \subseteq \kappa(I_1)$. Now observe that from the fourth part of Proposition 2.1, $\kappa(J) = J$ and $\kappa(I_1) = I_2$.
- m is odd and n is even. Here, $F_{i,k} - F_{j,l}$ can be written in the form

$$F_{i,k} - F_{j,l} = (F_{i,k} + F_{j,k}) - (F_{j,k} + F_{j,l})$$

which belongs to the ideal $\langle \sigma_m(X), f_{\frac{n}{2}}(Y)\rangle$ from Lemma 3.2 and Corollary 3.5.
- m and n are even. We write the difference $F_{i,k} - F_{j,l}$ as before. Now, from Lemma 3.2, it belongs to the ideal $\langle f_{\frac{m}{2}}(X), f_{\frac{n}{2}}(Y)\rangle$.

□

3.2 The second inclusion

We shall now show that $I_1I_2I_3 \subseteq J$. In order to do this, it is enough to study if I_1I_2D is contained in J, since $I_3 = J + D$. If necessary, we will write $J_{m,n}$ instead of just J. Since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$, if a polynomial $H(X,Y,Z)$ belongs to $J_{m,n}$ then $H(Y,X,Z)$ belongs to $J_{n,m}$. This easy remark allows us to simplify the proofs.

Since $V(J) = X(G_{m,n}) = \{(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) \mid A, B \in SL(2,\mathbb{C}),\ A^m = B^n\}$, we can assume $A^m = B^n$ when we work modulo $J_{m,n}$

Lemma 3.7. *$h_m(X)D \in J_{m,n}$ for all $m,n \in \mathbb{Z}$. As a consequence, $h_n(Y)D \in J_{m,n}$ for all $m,n \in \mathbb{Z}$.*

Proof. Let A, B be two matrices in $SL(2,\mathbb{C})$ and put $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. It can easily be proved by induction on m that $h_m(X)D = \operatorname{tr} A^m BA^{-1}B^{-1} - f_{m-1}(X)$, since both members of the equality satisfy the same recursive equation. Now, recall that $A^m = B^n$ modulo $J_{m,n}$. Therefore $\operatorname{tr} A^m BA^{-1}B^{-1} \equiv \operatorname{tr} B^n A^{-1}$ and thus $h_m(X)D \equiv F_{1,n} - F_{m-1,0} \in J_{m,n}$. □

Lemma 3.8. *Take $(X,Y,Z) \in \mathbb{C}^3$ and let $A, B \in SL(2,\mathbb{C})$ be two matrices such that $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. Then*

$$s_m(X)D = \begin{cases} \operatorname{tr} A^{\frac{m+1}{2}} BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}} BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} & \text{if } m \text{ is odd,} \\ \operatorname{tr} A^{\frac{m}{2}} BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} & \text{if } m \text{ is even.} \end{cases}$$

Proof. As before, the claim follows since both members of the expression verify the same recursive equation and it is obvious for $m = -1, 0, 1, 2$. □

Lemma 3.9. *$s_m(X) f_{\frac{n}{2}}(Y) D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with n even. As a consequence, $f_{\frac{m}{2}}(X) s_n(Y) D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with m even.*

Proof. We will make use of the previous lemma, so it is clear that we must work out separately the cases m odd and m even. We will only develop the odd case here. Let us recall that $f_{\frac{n}{2}}(Y) = \operatorname{tr} B^{\frac{n}{2}}$ and apply Lemma 3.8 together with formulas (1).

$$\begin{aligned}
&f_{\frac{n}{2}}(Y) s_m(X) D = \\
&\quad = \operatorname{tr} B^{\frac{n}{2}} \left(\operatorname{tr} A^{\frac{m+1}{2}} BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}} BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} \right) = \\
&\quad = \operatorname{tr} B^{\frac{n-2}{2}} A^{\frac{m+1}{2}} BA^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} AB^{-1} + \operatorname{tr} B^{\frac{n-2}{2}} A^{\frac{m-1}{2}} BA^{-1} + \\
&\qquad + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} AB^{-1} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-1}{2}} - \\
&\qquad - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} = \\
&\quad = \operatorname{tr} B^{\frac{n}{2}} \operatorname{tr} A^{\frac{m-1}{2}} B - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} AB^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} AB^{-1} + \\
&\qquad + \operatorname{tr} B^{\frac{n}{2}} \operatorname{tr} A^{\frac{m-3}{2}} B - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}} AB^{-1} + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} AB^{-1} - \\
&\qquad - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-1}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} = \\
&\quad = \operatorname{tr} B^{\frac{n-4}{2}} A^{\frac{-m+1}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} \operatorname{tr} AB^{-1} + \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n}{2}} + \\
&\qquad + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} \operatorname{tr} AB^{-1} - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n+4}{2}} + \operatorname{tr} B^{\frac{n-4}{2}} A^{\frac{-m+3}{2}} - \\
&\qquad - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}} \operatorname{tr} AB^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n}{2}} + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} \operatorname{tr} AB^{-1} - \\
&\qquad - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+4}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} =
\end{aligned} \tag{3}$$

$$
\begin{aligned}
&= \left(\operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n}{2}}\right) + \\
&\quad + \left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+4}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n+4}{2}}\right) + \\
&\quad + \operatorname{tr} AB^{-1} \left[\left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}}\right) + \right. \\
&\quad \left. + \left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}}\right)\right] =
\end{aligned}
$$

$$
\begin{aligned}
&= \left(F_{\frac{m+3}{2},\frac{n}{2}} - F_{\frac{m-3}{2},\frac{n}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n}{2}} - F_{\frac{m-1}{2},\frac{n}{2}}\right) + \\
&\quad + \left(F_{\frac{m-1}{2},\frac{n-4}{2}} - F_{\frac{m+1}{2},\frac{n+4}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n-4}{2}} - F_{\frac{m+3}{2},\frac{n+4}{2}}\right) + \\
&\quad + F_{1,1}\left[\left(F_{\frac{m+1}{2},\frac{n+2}{2}} - F_{\frac{m-1}{2},\frac{n-2}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n+2}{2}} - F_{\frac{m+1}{2},\frac{n-2}{2}}\right)\right].
\end{aligned}
$$

Thus $s_m(X)f_{\frac{n}{2}}(Y)D \in J_{m,n}$ as claimed. □

Lemma 3.10. *$s_m(X)\sigma_n(Y)D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with n odd. As a consequence, $\sigma_m(X)s_n(Y)D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with m odd.*

Proof. We will carry out the case when m is odd and n is positive (the case when n is negative follows from the positive one and from Proposition 2.1 (3) and (1)). First of all note that, n being odd, $(-1)^{\frac{n-1}{2}}\sigma_n(Y) = 1 + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j$. Now applying again Lemma 3.8 we can show, after some straightforward computations as in the previous lemma, that:

$$
\begin{aligned}
s_m(X)\sigma_n(Y)D = &\left(F_{\frac{m+3}{2},\frac{n-1}{2}} - F_{\frac{m-3}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right) + \\
&+ \left(F_{\frac{m-1}{2},\frac{n+3}{2}} - F_{\frac{m+1}{2},\frac{n-3}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n+3}{2}} - F_{\frac{m+3}{2},\frac{n-3}{2}}\right) + \\
&+ F_{1,-1}\left[\left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n-1}{2}} - F_{\frac{m+1}{2},\frac{n+1}{2}}\right)\right].
\end{aligned} \tag{4}
$$

and thus $s_m(X)\sigma_n(Y)D \in J_{m,n}$ as claimed. □

The last two lemmas can also be proved by double induction on m and n, since both members of the corresponding expression satisfy the same recursive equation and hence the proofs are reduced to the base cases. However, we did not proceed in that direction because finding those identities would require to work out all the computations as in the proof of Lemma 3.9. We refer to the appendix for complete details on how to find and show those complicated formulas.

Theorem 3.11. *$I_1I_2I_3 \subseteq J$.*

Proof. The above two lemmas imply that $I_1I_2D \subseteq J$. Now since $I_3 = J + D$, we have that $I_1I_2I_3 = I_1I_2(J + D) = I_1I_2J + I_1I_2D \subseteq J$. □

As we already remarked, Theorems 3.6 and 3.11 obviously imply Theorem 3.1. Based on computational evidence, the authors conjecture that in fact $J = I_1 \cap I_2 \cap I_3$, but unfortunately no proof has been found.

4 Irreducible components of $X(G_{m,n})$

From Theorem 3.1, the character variety $X(G_{m,n})$ can be decomposed into the algebraic sets $V(I_1)$, $V(I_2)$ and $V(I_3)$. Therefore, in order to obtain an explicit description of $X(G_{m,n})$, we need to factorize the polynomials $s_k(T)$, $\sigma_k(T)$ and $f_k(T)$, and to find a nicer expression for $I_3 = J + D$.

4.1 Finding the factorization of $s_k(T)$, $\sigma_k(T)$ and $f_k(T)$

In this section k will be a nonzero integer. Let us recall the recursive definition of the cyclotomic polynomials $\{c_\ell(T)\}_{\ell\in\mathbb{N}}$ given by

$$\prod_{\ell|k} c_\ell(T) = T^k - 1, \tag{5}$$

where $c_\ell(T)$ is the minimal polynomial of the primitive ℓ-th roots of unity. In the same way we will denote by $r_\ell(T)$ the minimal polynomial of the ℓ-th primitive roots of -1. This definition gives an expression similar to (5):

$$\prod_{\substack{\ell|k \\ \frac{k}{\ell}\ \text{odd}}} r_\ell(T) = T^k + 1. \tag{6}$$

In fact it can be seen that if ℓ is odd $r_\ell(T) = c_\ell(-T)$ and if ℓ is even $r_\ell(T) = c_{2\ell}(T)$.

Note that for every $3 \le \ell \in \mathbb{N}$, there exists a polynomial (irreducible over $\mathbb{Z}$) $q_\ell(T)$ such that:

$$c_\ell(T) = T^{\frac{\varphi(\ell)}{2}} q_\ell\left(T + \frac{1}{T}\right),$$

where φ is Euler's phi function and $\varphi(\ell) = \deg\, c_\ell(T)$. For the sake of completeness, we will denote $q_1(T) = T - 2$ and $q_2(T) = T + 2$.

Lemma 4.1. *If $\ell \ge 3$, $q_\ell(T)$ has $\frac{\varphi(\ell)}{2}$ distinct real roots and its set of zeroes is $Z[q_\ell] = \{2Re\ z \mid z \text{ is a primitive } \ell\text{-th root of unity}\}$ for all $\ell \in \mathbb{N}$. Moreover if $\ell_1 \ne \ell_2$, then $Z[q_{\ell_1}] \cap Z[q_{\ell_2}] = \emptyset$.*

We can now obtain a factorization of the polynomial f_k which will allow us to find its roots.

Proposition 4.2. $f_k(T) = \displaystyle\prod_{\substack{\ell|k \\ \frac{k}{\ell}\ odd}} q_{4\ell}(T).$

Proof.

$$f_k\left(T + \frac{1}{T}\right) = T^k + \frac{1}{T^k} = \frac{T^{2k}+1}{T^k} = \frac{1}{T^k} \prod_{\substack{\ell|2k \\ \frac{2k}{\ell}\ odd}} r_\ell(T) = \frac{1}{T^k} \prod_{\substack{\ell|2k \\ \frac{2k}{\ell}\ odd}} T^{\varphi(\ell)} q_{2\ell}\left(T + \frac{1}{T}\right) =$$

$$= \prod_{\substack{\ell|2k \\ \frac{2k}{\ell}\ odd}} q_{2\ell}\left(T + \frac{1}{T}\right) = \prod_{\substack{\ell|k \\ \frac{k}{\ell}\ odd}} q_{4\ell}\left(T + \frac{1}{T}\right),$$

where the last equality follows readily from the fact that $\{2\ell \mid \ell|2k,\ \frac{2k}{\ell} \text{ odd}\} = \{4\ell \mid \ell|k,\ \frac{k}{\ell} \text{ odd}\}$ and from the identity $\displaystyle\sum_{\ell|2k,\ \frac{2k}{\ell} \text{ odd}} \varphi(\ell) = k.$ □

Corollary 4.3. *The polynomial $f_k(T)$ has k distinct real roots and its set of zeroes is $Z[f_k] = \{2Re\ z \mid z$ is a primitive 4ℓ-th root of unity, $\frac{k}{\ell}$ odd$\} = \{2Re\ z \mid z^{2k} = -1\}$.*

Proof. The first equality is an immediate consequence of the previous proposition and Lemma 4.1. The second one can be verified by direct calculations. □

Now, we turn to the polynomials s_k and σ_k. First we present a technical lemma that can easily be proved by induction.

Lemma 4.4.

$$s_k\left(T + \frac{1}{T}\right) = \begin{cases} \dfrac{T^k - 1}{T^{\frac{k-1}{2}}(T-1)} & \text{if } k \text{ is odd.} \\ \dfrac{T^k - 1}{T^{\frac{k-2}{2}}(T^2-1)} & \text{if } k \text{ is even.} \end{cases}$$

Using this result together with (5) and Lemma 4.1, and recalling Proposition 2.1, we have the following result.

Proposition 4.5. $s_k(T) = \displaystyle\prod_{1,2\neq\ell|k} q_\ell(T), \quad \sigma_k(T) = (-1)^{\left[\frac{k-1}{2}\right]} \prod_{1,2\neq\ell|k} q_\ell(-T).$
Hence, $s_k(T)$ and $\sigma_k(T)$ have $\left[\frac{k-1}{2}\right]$ distinct real roots. In particular their sets of zeroes are $Z[s_k] = \{2Re\ z \mid z \neq 1, 2; \quad z^k = 1\} = -Z[\sigma_k]$.

Remark 4.6. In so far k was implicitly assumed to be positive. Observe that if we shift the sign of k the sets of zeroes described above remain invariant. Namely $Z[f_k] = Z[f_{-k}]$, $Z[s_k] = Z[s_{-k}]$ and $Z[\sigma_k] = Z[\sigma_{-k}]$. Moreover, if k is negative, then $f_k(T)$ has $|k|$ distinct real roots while $s_k(T)$ and $\sigma_k(T)$ have $\left[\frac{|k|-1}{2}\right]$.

As a consequence of the previous discussion we will be able to count the number of irreducible components of $V(I_1)$ and $V(I_2)$ which, due to the form of the ideals I_1 and I_2, are disjoint straight lines.

4.2 Another description for V(I_3)

Now, we are interested in computing the number of irreducible components of the affine algebraic variety associated with I_3. For this aim, our definition of $I_3 = J + D$ doesn't seem to be very useful. That is why we need another expression for $V(I_3)$. From now on we assume that $d = \gcd(m, n)$ and we will write $m = m'd$ and $n = n'd$.

Lemma 4.7. $V(I_3) = \{(u + u^{-1}, v + v^{-1}, uv + (uv)^{-1}) \mid u, v \in \mathbb{C}^*,\ u^m = v^n\}$.

Proof. Let $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$ be the polynomial map defined in Section 1 and let us recall that by definition $V(J) = X(G_{m,n}) = t(R(G_{m,n}))$. Since $D(t(\rho)) = \operatorname{tr}\rho[x,y] - 2$ when $\rho \in R(G_{m,n})$ and $V(I_3) = V(J + D) = V(J) \cap V(D)$, it is clear that $V(I_3) = \{t(\rho) \mid \rho \in R(G_{m,n}),\ \operatorname{tr}\rho[x,y] = 2\}$. Now the claim follows from Proposition 1.1(2) and the identity $t(Red) = t(Diag)$, where Red is the set of all reducible representations in $R(G_{m,n})$ and $Diag$ the set of all diagonal ones, (see the proof of Corollary 1.4.5 in [1]). $\square$

Using the previous lemma, since $u^m - v^n = \prod_{i=1}^{d}(u^{m'} - \zeta_i v^{n'})$ where $\{\zeta_1, \ldots, \zeta_d\}$ is the set of all d-th roots of unity, the variety associated with I_3 can be decomposed as the union $\bigcup_{i=1}^{d} W_{\zeta_i}$ with $W_{\zeta_i} = \{(u + u^{-1}, v + v^{-1}, uv + (uv)^{-1}) \mid u, v \in \mathbb{C}^*,\ u^{m'} = \zeta_i v^{n'}\}$. Once we remove the redundant components, the above union provides a decomposition of $V(I_3)$ into irreducible components. This fact is clarified in the lemma below.

Lemma 4.8. *The following properties hold:*

1) W_{ζ_i} *is an irreducible algebraic set for all* i*.*

2) $W_{\zeta_i} = W_{\zeta_j}$ *if and only if* $\zeta_i = \zeta_j^{\pm 1}$*. Moreover, if* $W_{\zeta_i} \neq W_{\zeta_j}$ *then they are disjoint.*

Proof. 1) Let us first show that the set W_{ζ_i} is algebraic. Take $Diag \subset R(G_{m,n})$ the algebraic set of all diagonal representations and $Diag^{\zeta_i}$ the algebraic subset

$$Diag^{\zeta_i} = \left\{ \left[\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \right] \in SL(2,\mathbb{C})^2 \mid u^{m'} = \zeta_i v^{n'} \right\}.$$

Since the map $\mathbb{C}^* \longrightarrow \mathbb{C}$ given by $a \mapsto a + a^{-1}$ is proper with the usual topology over $\mathbb{C}$, then so is $t|_{Diag}$ and, consequently, also $t|_{Diag^{\zeta_i}}$ and it follows (see [15] and [3]) that $t_{|Diag^{\zeta_i}}$ is closed with the Zariski topology. Now observe that $W_{\zeta_i} = t(Diag^{\zeta_i})$ and $Diag^{\zeta_i}$ is Zariski-closed, (see also the proof of Corollary 1.4.5 in [1]). Finally, the irreducibility of W_{ζ_i} follows from the irreducibility of $Diag^{\zeta_i}$ which is birationally equivalent to the plane curve $\mathcal{C}_{\zeta_i} : \{u^{m'} = \zeta_i v^{n'}\}$.

2) Note that the equation $a + a^{-1} = b + b^{-1}$ with $a, b \in \mathbb{C}^*$ has the only solutions $a = b^{\pm 1}$. Now, the assertion is a clear consequence of this fact. $\square$

4.3 Counting the irreducible components of $X(G_{m,n})$

Finally we present the main result of this section, which allows us to explicitly count the number of irreducible components of $X(G_{m,n})$.

Theorem 4.9. *The number of irreducible components of* $X(G_{m,n})$ *is:*

$$\begin{cases} \dfrac{(|m|-1)(|n|-1)}{2} + \dfrac{d+1}{2} & \textit{if } d \textit{ is odd,} \\[2ex] \dfrac{(|m|-1)(|n|-1)+1}{2} + \dfrac{d+2}{2} & \textit{if } d \textit{ is even,} \end{cases}$$

where the first summand corresponds to the number of straight lines in $V(I_1) \cup V(I_2)$ *and the second one corresponds to the number of irreducible components of* $V(I_3)$*.*

Note that if $d = 1$ the genus of the torus knot $K_{m,n}$ is precisely the number of straight lines in its character variety.

5 Irreducible components of $R(G_{m,n})$

Given the set $R(G_{m,n})$ we define the following subsets:

$$Irr = \{\rho \in R(G_{m,n}) \mid \rho \text{ is irreducible}\}$$

$$Red = \{\rho \in R(G_{m,n}) \mid \rho \text{ is reducible}\}$$

$$Ab = \{\rho \in R(G_{m,n}) \mid \rho(G_{m,n}) \text{ is abelian}\}$$

$$M_R = \{\rho \in Red \mid \rho(G_{m,n}) \text{ is metabelian}\}$$

Clearly we have the partitions $R(G_{m,n}) = Irr \cup Red = Irr \cup Ab \cup M_R$. We know that Ab is an algebraic set, while Irr and M_R are not. We can obtain a decomposition of $R(G_{m,n})$ into closed subsets just by taking closures, but the unions will no longer be disjoint. Namely, we have $R(G_{m,n}) = \overline{Irr} \cup Ab \cup \overline{M_R}$. We will study these three subsets separately in order to count the number of irreducible components of $R(G_{m,n})$.

Note that we have $t(\overline{Irr}) = V(I_1) \cup V(I_2)$, $t(Ab) = V(I_3) = t(Red)$ and $t(\overline{M_R}) = (V(I_1) \cup V(I_2)) \cap V(I_3) = t(\overline{Irr}) \cap t(Ab)$. In particular this implies (t being continuous) that the previous decompositions are not redundant.

5.1 The irreducible components of $\overline{Irr}$

The following result can be shown by elementary facts of general topology.

Lemma 5.1. *Let $\varphi : X \longrightarrow Y$ be a continuous surjective map between two topological spaces. Then the number of irreducible components of Y is less than or equal to the number of irreducible components of X. Moreover, if this numbers are different then there exist X_1 and X_2 two different components of X such that $\overline{\varphi(X_1)} \subseteq \overline{\varphi(X_2)}$.*

Note that if X_1 and X_2 are two different components of $R(G_{m,n})$ which contain irreducible representations, then $t(X_1) \cap t(X_2) = \emptyset$ and thus $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow t(\overline{Irr})$ preserves the number of irreducible components.

Remark 5.2. Let G be an arbitrary finitely presented group. From Propositions 1.5.2 and 1.1.1 in [1], $t(X_1) \cap t(X_2) \subseteq t(Red)$ and hence $t(X_1) \nsubseteq t(X_2)$. Therefore $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow t(\overline{Irr})$ always preserves the number of irreducible.

As we proved in the previous section, $t(\overline{Irr})$ has either $\frac{(|m|-1)(|n|-1)}{2}$ irreducible components if d is odd or $\frac{(|m|-1)(|n|-1)+1}{2}$ if d is even. Thus we have found the number of irreducible components of $\overline{Irr}$. Note that that due to [1, Cor. 1.5.3.] all these irreducible components are of dimension 4.

Since it is known that $\dim R(G_{m,n}) = 4$ and we will see that $\dim Ab = \dim \overline{M_R} = 3$, we have; in particular, the following result:

Theorem 5.3. *The number or irreducible 4-dimensional components of* $R(G_{m,n})$ *is:*

a) $\dfrac{(|m|-1)(|n|-1)}{2}$ *if* d *is odd.*

b) $\dfrac{(|m|-1)(|n|-1)+1}{2}$ *if* d *is even.*

This result can be found in [10, Theorem A], where a more direct approach is used. Note that we obtained it as a consequence of our study of $X(G_{m,n})$.

5.2 The irreducible components of Ab

This section is devoted to count the number of irreducible components of Ab. First we note that $Ab = R(G^{\mathrm{ab}}_{m,n})$ where $G^{\mathrm{ab}} = G/G'$ denotes the abelianization of G. In our case $G^{\mathrm{ab}}_{m,n} = \langle x, y \mid x^m = y^n,\ [x,y] = 1\rangle$. In the following lemma we give another presentation of $G^{\mathrm{ab}}_{m,n}$ which will be easier to work with.

Lemma 5.4. $G^{ab}_{m,n} \cong H_d = \langle a, b \mid a^d = 1 = [a,b]\rangle$*, where* $d = \gcd(m,n)$*.*

Proof. Put $m' = \frac{m}{d}$ and $n' = \frac{n}{d}$ and consider Bezout's identity $\alpha m - \beta n = d$. The claimed isomorphism is then given by $\phi : G^{\mathrm{ab}} \longrightarrow H_d$ with $\phi(x) = b^{n'}a^{\alpha}$, $\phi(y) = b^{m'}a^{\beta}$ and $\psi : H_d \longrightarrow G^{\mathrm{ab}}$ with $\psi(a) = x^{m'}y^{-n'}$, $\psi(b) = x^{-\beta}y^{\alpha}$. □

Thus, $Ab \cong R(H_d)$ and we will count the irreducible components of the latter. To do so we introduce some notation. If $\{\zeta_1, \zeta_2, \dots, \zeta_d\}$ are the d-th roots of unity we put $A_{\zeta_i} = \left(\begin{smallmatrix} \zeta_i & 0 \\ 0 & \zeta_i^{-1}\end{smallmatrix}\right)$ and we define the set $V_{\zeta_i} = \{P^{-1}A_{\zeta_i}P \mid P \in SL(2,\mathbb{C})\}$. Note that $V_1 = \{I_2\}$, $V_{-1} = \{-I_2\}$ and if $\zeta_i \neq \pm 1$ then V_{ζ_i} is an irreducible affine algebraic variety of dimension 2 (see [9, Cor. 1.5.]).

Lemma 5.5. *If* $X \in SL(2,\mathbb{C})$ *is such that* $X^d = I_2$*, then* $X \in V_{\zeta_i}$ *for some* ζ_i *d-th root of unity.*

Proof. Since $\mathbb{C}$ is algebraically closed there exists $P \in SL(2,\mathbb{C})$ such that $PXP^{-1} = \left(\begin{smallmatrix} a & \alpha \\ 0 & a^{-1}\end{smallmatrix}\right) = Y$. Clearly $I_2 = Y^d = \left(\begin{smallmatrix} a^d & \alpha h_d(a+a^{-1}) \\ 0 & a^{-d}\end{smallmatrix}\right)$ so $a^d = 1$ and $\alpha h_d(a+a^{-1}) = 0$ and two cases arise.

1) If $a = \pm 1$, since $h_d(\pm 2) \neq 0$ it must be $\alpha = 0$ and $X = Y = \pm I_2 \in V_{\pm 1}$.

2) If $a = \zeta_i \neq \pm 1$ then $a \neq a^{-1}$ and X is diagonalizable so there exists $P \in SL(2,\mathbb{C})$ such that $X = P^{-1}A_{\zeta_i}P \in V_{\zeta_i}$.

□

If we now define $M_{\zeta_i} = \{(A,B) \in SL(2,\mathbb{C}) \mid A \in V_{\zeta_i},\ [A,B] = I_2\}$, then the previous lemma shows that $R(H_d) = \bigcup_{i=1}^{d} M_{\zeta_i}$. Clearly $M_1 = \{I_2\} \times SL(2,\mathbb{C})$ and $M_{-1} = \{-I_2\} \times SL(2,\mathbb{C})$ are irreducible affine algebraic varieties of dimension 3. Now, we want to study the case M_{ζ_i} with $\zeta_i \neq \pm 1$.

Proposition 5.6. *M_{ζ_i} is an affine irreducible algebraic variety of dimension 3 $\forall i \in \{1,\ldots,d\}$.*

Proof. We can assume $\zeta_i \neq \pm 1$. In this case we define $\Psi : M_{\zeta_i} \longrightarrow V_{\zeta_i} \times \mathbb{C}^*$ as follows: given $(A,B) \in M_{\zeta_i}$ there exists $P \in SL(2,\mathbb{C})$ such that $PAP^{-1} = A_{\zeta_i}$, now since A and B commute, PBP^{-1} and A_{ζ_i} must also commute and it follows that $PBP^{-1} = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)$ must be diagonal. We define $\Psi(A,B) = (A,b)$.

Let us see that Ψ is well defined: if $PAP^{-1} = A_{\zeta_i} = QAQ^{-1}$, then QP^{-1} commute with A_{ζ_i} and it must be diagonal. Consequently $QBQ^{-1} = QP^{-1}\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)(QP^{-1})^{-1} = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1}\end{smallmatrix}\right)$ and $\Psi(A,B)$ does not depend on the choice of P.

Now, we claim that Ψ is bijective. Let us suppose that $\Psi(A_1,B_1) = (A_1,b_1) = (A_2,b_2) = \Psi(A_2,B_2)$, then $A_1 = A_2$ and $b_1 = b_2$. Since $A_1 = A_2$ and Ψ is well defined there must exist $P \in SL(2,\mathbb{C})$ such that $PB_1P^{-1} = \begin{pmatrix} b_1 & 0 \\ 0 & b_1^{-1}\end{pmatrix} = \begin{pmatrix} b_2 & 0 \\ 0 & b_2^{-1}\end{pmatrix} = PB_2P^{-1}$ so we have that $(A_1,B_1) = (A_2,B_2)$ and Ψ is injective. Since surjectivity of Ψ is obvious the claim follows.

To finish the proof it is enough to observe that Ψ induces a birational equivalence between M_{ζ_i} and $V_{\zeta_i} \times \mathbb{C}$. □

Now, in order to be able to count the number of irreducible components, we must remove the redundant components in the decomposition $R(H_d) = \bigcup_{i=1}^d M_{\zeta_i}$. This is done as follows.

Proposition 5.7. *$M_{\zeta_i} = M_{\zeta_j}$ if and only if $\zeta_i = \zeta_i^{\pm 1}$. Moreover, if $M_{\zeta_i} \neq M_{\zeta_j}$, then they are disjoint.*

Proof. $M_{\zeta_i} = M_{\zeta_j} \Leftrightarrow V_{\zeta_i} = V_{\zeta_j} \Leftrightarrow A_{\zeta_i} \in V_{\zeta_j} \overset{[9,\,1.4.]}{\Leftrightarrow} \operatorname{tr} A_{\zeta_i} = \operatorname{tr} A_{\zeta_j} \Leftrightarrow \zeta_i = \zeta_j^{\pm 1}$. □

As a consequence of this proposition we have that Ab has $\frac{d+1}{2}$ irreducible components if d is odd and $\frac{d+2}{2}$ if d is even.

5.3 The irreducible components of $\overline{M_R}$

Now we turn to the set M_R of reducible metabelian representations of $G_{m,n}$. In the following lemma we give some properties of such representations which will be useful in the sequel.

Lemma 5.8. *Let $\rho \in M_R$. Then $\rho(x)^m = \rho(y)^n = \pm I_2$ and $\operatorname{tr}\rho(x), \operatorname{tr}\rho(y) \neq 2$.*

Proof. We can assume that $\rho(x) = X = \left(\begin{smallmatrix} a & \alpha \\ 0 & a^{-1}\end{smallmatrix}\right)$ and $\rho(y) = Y = \left(\begin{smallmatrix} b & \beta \\ 0 & b^{-1}\end{smallmatrix}\right)$. Let us suppose that $\operatorname{tr} X = 2$. Then $a = 1$ and $b^n = 1$ and two cases arise:

1) If $b = \pm 1$ then $\rho(G)$ is abelian, which contradicts the hypothesis $p \in M_R$.

2) If $b \neq \pm 1$ then $I_2 = Y^n = X^m = \left(\begin{smallmatrix} 1 & m\alpha \\ 0 & 1\end{smallmatrix}\right)$ and $m\alpha = 0$, thus $\alpha = 0$ and $X = I_2$ which implies again that $\rho(G)$ is abelian.

Analogously it can be proved that $\operatorname{tr} X \neq -2$ and $\operatorname{tr} Y \neq \pm 2$.

Now, suppose that $X^m = Y^n \neq \pm I_2$, in which case $a^m, b^n \neq \pm 1$ and $h_m(a+a^{-1}) \neq 0 \neq h_n(b+b^{-1})$. From $A^m = B^n$ it follows that $\alpha h_m(a+a^{-1}) = \beta h_n(b+b^{-1})$, so $\alpha = 0$ if and only if $\beta = 0$ and, since $\alpha = \beta = 0$ implies that $\rho(G)$ is abelian we deduce that $\alpha\beta \neq 0$. Finally $a^m + a^{-m} = b^n + b^{-n} \Leftrightarrow (a - a^{-1})h_m(a+a^{-1}) = (b+b^{-1})h_n(b+b^{-1}) \Leftrightarrow \beta(a-a^{-1}) = \alpha(b-b^{-1}) \Leftrightarrow \rho(G)$ is abelian. This, again, is a contradiction and the lemma follows. $\square$

We will now introduce some notation.

$$\Theta = \{\zeta \mid \zeta^m = \pm 1,\ \zeta \neq \pm 1\} = \{\zeta \mid \zeta^m = 1,\ \zeta \neq \pm 1\} \cup \{\zeta \mid \zeta^m = -1,\ \zeta \neq -1\} = \Theta^+ \cup \Theta^-$$

$$\Upsilon = \{\eta \mid \eta^n = \pm 1,\ \eta \neq \pm 1\} = \{\eta \mid \eta^n = 1,\ \eta \neq \pm 1\} \cup \{\eta \mid \eta^n = -1,\ \eta \neq -1\} = \Upsilon^+ \cup \Upsilon^-$$

Now, given $\zeta \in \Theta$ and $\eta \in \Upsilon$ we put $A_\zeta = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $B_\eta = \begin{pmatrix} \eta & 1 \\ 0 & \eta^{-1} \end{pmatrix}$. Finally, let us define $V_{\zeta,\eta} = \{(P^{-1}A_\zeta P, P^{-1}B_\eta P) \mid P \in SL(2,\mathbb{C})\}$.

Lemma 5.9. *If $\rho \in M_R$, there exist $\zeta \in \Theta$ and $\eta \in \Upsilon$ such that $(\rho(x), \rho(y)) \in V_{\zeta,\eta}$.*

Proof. Put $X = \rho(x)$ and $Y = \rho(y)$. By Lemma 5.8 we have that there exists $P \in SL(2,\mathbb{C})$ such that $PXP^{-1} = \begin{pmatrix} \zeta & \alpha \\ 0 & \zeta^{-1} \end{pmatrix} = X'$ and $PYP^{-1} = \begin{pmatrix} \eta & \beta \\ 0 & \eta^{-1} \end{pmatrix} = Y'$ for some $\zeta \in \Theta$, $\eta \in \Upsilon$ and $\alpha, \beta \in \mathbb{C}$. By straightforward computations it is easy to see that there exists an upper triangular matrix $Q \in SL(2,\mathbb{C})$ such that $Q^{-1}X'Q = A_\zeta$ and $Q^{-1}Y'Q = B_\eta$. This completes the proof. $\square$

As a consequence of this lemma we obtain a decomposition

$$\overline{M_R} = \bigcup_{\substack{\zeta \in \Theta^+ \\ \eta \in \Upsilon^+}} \overline{V_{\zeta,\eta}} \cup \bigcup_{\substack{\zeta \in \Theta^- \\ \eta \in \Upsilon^-}} \overline{V_{\zeta,\eta}}.$$

Therefore, in order to find the dimension and the number of irreducible components of $\overline{M_R}$ we will study each $\overline{V_{\zeta,\eta}}$ separatedly.

Proposition 5.10. *Given $\zeta \in \Theta^\pm$ and $\eta \in \Upsilon^\pm$, the set $\overline{V_{\zeta,\eta}}$ is an affine irreducible algebraic variety of dimension 3.*

Proof. Let us define $\Phi : V_{\zeta,\eta} \longrightarrow PSL(2,\mathbb{C})$ as follows: given $(A,B) \in V_{\zeta,\eta}$, there exists $P \in SL(2,\mathbb{C})$ such that $PAP^{-1} = A_\zeta$ and $PBP^{-1} = B_\eta$; we define $\Phi(A,B) = [P]$.

Now we will see that Φ is well defined. If $PAP^{-1} = A_\zeta = QAQ^{-1}$, and $PBP^{-1} = B_\eta = QBQ^{-1}$ it follows that PQ^{-1} commutes with A_ζ, so it must be diagonal. Moreover as PQ^{-1} commutes with B_η and it is diagonal it must be $PQ^{-1} = \pm I_2$ so $P = \pm Q$ and $[P] = [Q]$ in $PSL(2,\mathbb{C})$.

Since Φ is trivially surjective, it is enough to prove the injectivity. If $(A_1,B_1),(A_2,B_2) \in V_{\zeta,\eta}$ are such that $\Phi(A_1,B_1) = \Phi(A_2,B_2) = [P]$, then either $PA_1P^{-1} = A_\zeta = PA_2P^{-1}$ or $PA_1P^{-1} = (-P)A_2(-P)^{-1}$. But in any case $A_1 = A_2$ and analogously $B_1 = B_2$.

Now, since Φ clearly induces a birational equivalence between $V_{\zeta,\eta}$ and $PSL(2,\mathbb{C}) \cong SO_3(\mathbb{C})$, the proof is complete. $\square$

Note that $V_{\zeta_i,\eta_j} = V_{\zeta_k,\eta_l}$ if and only if $\zeta_i = \zeta_k$ and $\eta_j = \eta_l$. Moreover if $V_{\zeta_i,\eta_j} \neq V_{\zeta_k,\eta_l}$ then they are disjoint. Finally, as in the previous sections, we have to remove the redundant components in our decomposition of $\overline{M_R}$.

Proposition 5.11. *$\overline{V_{\zeta_i,\eta_j}} = \overline{V_{\zeta_k,\eta_l}}$ if and only if $\zeta_i = \zeta_k$ and $\eta_j = \eta_l$.*

Proof. Let us suppose that $\overline{V_{\zeta_i,\eta_j}} = \overline{V_{\zeta_k,\eta_l}}$, then $(\zeta_i + \zeta_i^{-1}, \eta_j + \eta_j^{-1}, \zeta_i\eta_j + (\zeta_i\eta_j)^{-1}) = t\left(\overline{V_{\zeta_i,\eta_j}}\right) = t\left(\overline{V_{\zeta_k,\eta_l}}\right) = (\zeta_k + \zeta_k^{-1}, \eta_l + \eta_l^{-1}, \zeta_k\eta_l + (\zeta_k\eta_l)^{-1})$. This implies that either $\zeta_i = \zeta_k$ and $\eta_j = \eta_l$ or $\zeta_i = \zeta_k^{-1}$ and $\eta_j = \eta_l^{-1}$.

Now, if $\zeta_i = \zeta_k^{-1}$ and $\eta_j = \eta_l^{-1}$ we know that $V_{\zeta_i,\eta_j} \cap V_{\zeta_k,\eta_l} = \emptyset$. Consequently $V_{\zeta_i,\eta_j} \subseteq \overline{V_{\zeta_k,\eta_l}} - V_{\zeta_k,\eta_l}$. This is a contradiction since $\dim(\overline{V_{\zeta_k,\eta_l}} \setminus V_{\zeta_k,\eta_l}) < \dim V_{\zeta_k,\eta_l} = \dim V_{\zeta_i,\eta_j}$ (see [8, §8.3.] for instance).

The converse is obvious. ☐

As a consequence of this result, we obtain that the number of irreducible components of $\overline{M_R}$ is $2(|m|-1)(|n|-1)$ if d is odd and $2[(|m|-1)(|n|-1)+1]$ if d is even.

5.4 Counting the irreducible components of $R(G_{m,n})$

Finally we can combine the results obtained in the previous sections to explicitly compute the number of irreducible components of $R(G_{m,n})$. Namely we have the following.

Theorem 5.12. *The number of irreducible components of $R(G_{m,n})$ is*

$$\begin{cases} \dfrac{(|m|-1)(|n|-1)}{2} + 2(|m|-1)(|n|-1) + \dfrac{d+1}{2} & \text{if } d \text{ is odd,} \\[2ex] \dfrac{(|m|-1)(|n|-1)+1}{2} + 2[(|m|-1)(|n|-1)+1] + \dfrac{d+2}{2} & \text{if } d \text{ is even,} \end{cases}$$

where the first summand corresponds to the number of irreducible components in $\overline{Irr}$, the second one to the irreducible components of $\overline{M_R}$, and the last one to the irreducible components of Ab.

Note that, in light of Theorem 4.9, the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$ does not preserve the number of irreducible components. Nevertheless, if we consider the restrictions $t|_{\overline{Irr}} : \overline{Irr} \longrightarrow V(I_1) \cup V(I_2)$ and $t|_{Ab} : Ab \longrightarrow V(I_3)$, we have that both of them preserve the number of irreducible components. On the other hand $t|_{\overline{M_R}}$ maps every irreducible component of $\overline{M_R}$ to a single point which lies in $t(\overline{Irr}) \cap t(Ab)$.

This situation is essentially different when we consider the variety of characters in $PSL(2,\mathbb{C})$, where the projection $t : R(G_{m,n}) \longrightarrow X(G_{m,n})$ always induces a bijection between irreducible components (see [4, Rem. 3.17.]).

Also note that, due to [1, Cor. 1.5.3.], if Irr_0 is an irreducible component of $\overline{Irr}$, then $\dim Irr_0 = \dim t(Irr_0) + 3$. In our case $\dim Irr_0 = 4$ and $\dim t(Irr_0) = 1$. Although the aforementioned result in [1] cannot be applied, the same relation holds for any irreducible component of $\overline{M_R}$. In fact if M_{R_0} is an irreducible component of $\overline{M_R}$, then we have seen that $\dim M_{R_0} = 3$ while $\dim t(M_{R_0}) = 0$. Nevertheless, the result is not true in general since we have that $\dim Ab_0 = 3$ and $\dim t(Ab_0) = 1$ for every irreducible component of Ab.

6 Comments and applications

6.1 The abelian component

Since $V(I_3) = t(Ab)$ we will call it the abelian component of $X(G_{m,n})$. In Section 4.2 we gave a simple description of this component and its decomposition into irreducible components. Namely, we proved that $V(I_3) = \bigcup_{i=1}^{d} W_{\zeta_i}$ with $W_{\zeta_i} = \{(u + u^{-1}, v + v^{-1}, uv + (uv)^{-1}) \mid v, v \in \mathbb{C}^*,\ u^{m'} = \zeta_i v^{n'}\}$ and $W_{\zeta_i} = W_{\zeta_j}$ if and only if $\zeta_i = \zeta_j^{\pm 1}$.

Now, if $\mathcal{C}_{\zeta_i}$ is the plane curve given by $u^{m'} = \zeta_i v^{n'}$, it is clear that $\mathcal{C}_{\zeta_i}$ is isomorphic to $\mathcal{C}_1$ for all $i = 1, \ldots, d$. Since W_{ζ_i} is birationally equivalent to $\mathcal{C}_{\zeta_i}$ it follows that W_{ζ_i} is birationally equivalent to W_1 for all $i = 1, \ldots, d$. Therefore, if we want to study the curves W_{ζ_i} it is enough to study W_1 and, as we will see, it admits a parametrization.

Lemma 6.1. *Let r, s be coprime integers. If A, B are two commuting matrices in $SL(2, \mathbb{C})$ such that $A^r = B^s$, then there exists $C \in SL(2, \mathbb{C})$ such that $A = C^s$ and $B = C^r$.*

Proof. It is a straightforward consequence of the fact that the group $\langle a, b \mid a^r = b^s,\ [a, b] = 1\rangle$ is cyclic if $\gcd(r, s) = 1$. ☐

With this lemma we are ready to give a parametrization of W_1.

Proposition 6.2. $W_1 = \{(f_{n'}(t), f_{m'}(t), f_{n'+m'}(t)) \mid t \in \mathbb{C}\}$.

Proof. We know that $W_1 = \{(u + u^{-1}, v + v^{-1}, uv + (uv)^{-1}) \mid v, v \in \mathbb{C}^*,\ u^{m'} = v^{n'}\}$ with $\gcd(m', n') = 1$. Now if we put $A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$, we can apply the previous lemma to find $C \in SL(2, \mathbb{C})$ such that $A = C^{n'}$ and $B = C^{m'}$. Thus, $u + u^{-1} = \operatorname{tr} A = \operatorname{tr} C^{n'} = f_{n'}(\operatorname{tr} C)$; analogously $v + v^{-1} = f_{m'}(\operatorname{tr} C)$ and $uv + (uv)^{-1} = f_{m'+n'}(\operatorname{tr} C)$. Finally it is enough to put $t = \operatorname{tr} C$ and the result follows. ☐

Finally we want to point out that if we project W_1 over the plane $Z = 0$ we essentially obtain a Lissajous figure. This is due to the relation between $\{f_k\}_{k \in \mathbb{N}}$ and the Chebyshev polynomials of the first kind $\{T_k\}_{k \in \mathbb{N}}$ and the well-known parametrization of the Lissajous curves.

6.2 The mirror image of $K_{m,n}$ and its character variety

In this section we are interested in the relationship between the character variety of the torus knot $K_{m,n}$ and that of its mirror image (which is precisely $K_{m,-n}$). We know that $X(G_{m,n}) = V(J_{m,n}) = V(I_1^{m,n}) \cup V(I_2^{m,n}) \cup V(I_3^{m,n})$ and $X(G_{m,-n}) = V(J_{m,-n}) = V(I_1^{m,-n}) \cup V(I_2^{m,-n}) \cup V(I_3^{m,-n})$.

Let us define $\phi : R(G_{m,n}) \longrightarrow R(G_{m,-n})$ given by $\phi(A, B) = (A, B^{-1})$. Clearly it is an isomorphism and due to Lemma 1.5 it induces an isomorphism (an involution in fact) $\psi : X(G_{m,n}) \longrightarrow X(G_{m,-n})$ which can be seen to be given by $\psi(X, Y, Z) = (X, Y, XY - Z)$. By definition we have that $\psi(V(I_i^{m,n})) = V(I_i^{m,-n})$ for $i = 1, 2, 3$. Now it is easy to see that, due to Remark 4.6., ψ fixes (not pointwise) $V(I_1^{m,n})$ and $V(I_2^{m,n})$ so $X(G_{m,n})$ and $X(G_{m,-n})$ share the set of straight lines. Moreover we have the decompositions

$$V(I_3^{m,n}) = \bigcup_{i=1}^{d} W_{\zeta_i}^{m,n}, \quad V(I_3^{m,-n}) = \bigcup_{i=1}^{d} W_{\zeta_i}^{m,-n}$$

and it can be trivially seen that $\psi(W^{m,n}_{\zeta_i}) = W^{m,-n}_{\zeta_i}$.

We will end this section studying the intersection $\mathcal{I} = V(I_3^{m,n}) \cap V(I_3^{m,-n})$. Recall that $V(I_3^{m,n}) = \{(u+u^{-1}, v+v^{-1}, uv+(uv)^{-1}) \mid u^m = v^n\}$ and $V(I_3^{m,-n}) = \{(u+u^{-1}, v+v^{-1}, uv+(uv)^{-1}) \mid u^m = v^{-n}\}$. An elementary analysis of the situation allows us to see that $(u+u^{-1}, v+v^{-1}, uv+(uv)^{-1}) \in \mathcal{I}$ if and only if $u^m = v^n = \pm 1$. This implies that $\operatorname{card}(\mathcal{I}) = mn+1$.

6.3 Another description for the character variety of $K_{m,2}$

If m is an odd integer, then $K_{m,2}$ is a two-bridge knot and it is possible to find a presentation of $G_{m,2} := G(K_{m,2})$ generated by two meridians. Namely, we have an isomorphism

$$H_m = \langle x, y \mid \overbrace{xyxy\ldots yx}^{\text{length } m} = \underbrace{yxyx\ldots xy}_{\text{length } m} \rangle \cong \langle a, b \mid a^m = b^2 \rangle = G_{m,2}$$

given by $x \mapsto b^{-1}a^{\frac{m+1}{2}}$, $y \mapsto a^{-\frac{m-1}{2}}b$ which, due to Corollary 1.6., induces an isomorphism between $X(H_m)$ and $X(G_{m,2})$ given by $(X,Y,Z) \mapsto \Big(F_{\frac{m+1}{2},1}(X,Y,Z), X\Big)$ for all $(X,Y,Z) \in X(G_{m,2})$. Now, it can be proved using the techniques in [12] that

$$X(H_m) \cong \{(X,Y) \in \mathbb{C}^2 \mid (X^2 - Y - 2)\sigma_m(Y) = 0\}$$

and we have a description of $X(G_{m,2})$ as an algebraic complex curve in $\mathbb{C}^2$.

Since for $|m|, |n| > 2$ the knot $K_{m,n}$ has more than two bridges, the Wirtinger presentation of $G_{m,n}$ has more than two generators and the character variety is not a curve in $\mathbb{C}^2$. Nevertheless is would be interesting to find an explicit isomorphism between the Wirtinger presentation and the one given in Section 1 in order to find an easier description of $X(G_{m,n})$ and, in particular, of its abelian component.

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A Appendix

Some formulas which appear in this article like (3) and (4) are really easy to prove by double induction on the corresponding indices, since both members of the expression satisfy the same recursive equation and hence the proofs are reduced to the base cases. However, finding those identities would require to work out all the computations as in the proof of Lemma 3.9. In this appendix we explain how to find them and show those complicated formulas in detail without using induction. For the proofs given $(X,Y,Z) \in \mathbb{C}^3$, let us consider $A, B \in SL(2,\mathbb{C})$ two matrices such that $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$.

Lemma A.1.

$$s_m(X)D = \begin{cases} \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} & \textit{if } m \textit{ is odd,} \\ \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. We will make use of Proposition 2.1(5), so it is clear that we must work out separately three different cases. Also we will assume that m is a non-negative integer. The negative case follows immediately from the positive one together with the third part of Proposition 2.1 and the formulas given in (1).

-) m odd.

$$s_m(X)D(X,Y,Z) = \Big(1 + \sum_{i=1}^{\frac{m-1}{2}} f_i(X)\Big)D(X,Y,Z) = \Big(1 + \sum_{i=1}^{\frac{m-1}{2}} \operatorname{tr} A^i\Big)\Big(\operatorname{tr} ABA^{-1}B^{-1} - 2\Big) =$$

$$= D(X,Y,Z) + \sum_{i=1}^{\frac{m-1}{2}} \Big[\operatorname{tr} A^{i+1}BA^{-1}B^{-1} + \operatorname{tr}(A^{i-1})(BAB^{-1})\Big] - 2\sum_{i=1}^{\frac{m-1}{2}} \operatorname{tr} A^i =$$

$$= D + \sum_{i=1}^{\frac{m-1}{2}} \Big[\operatorname{tr} A^{i+1}BA^{-1}B^{-1} + (\operatorname{tr} A^{i-1} \operatorname{tr} A) - \operatorname{tr} A^{i-1}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m-1}{2}} \operatorname{tr} A^i =$$

$$= D + \sum_{i=1}^{\frac{m-1}{2}} \Big[\operatorname{tr} A^{i+1}BA^{-1}B^{-1} + (\operatorname{tr} A^{i} + \operatorname{tr} A^{i-2}) - \operatorname{tr} A^{i-1}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m-1}{2}} \operatorname{tr} A^i =$$

$$= D + \sum_{i=1}^{\frac{m-1}{2}} \Big(\operatorname{tr} A^{i-2} - \operatorname{tr} A^i\Big) + \sum_{i=1}^{\frac{m-1}{2}} \Big(\operatorname{tr} A^{i+1}BA^{-1}B^{-1} - \operatorname{tr} A^{i-1}BA^{-1}B^{-1}\Big) =$$

$$= \Big(\operatorname{tr} ABA^{-1}B^{-1} - 2\Big) + \Big(\operatorname{tr} A^{-1} + \operatorname{tr} A^0 - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}}\Big) +$$

$$+ \Big(\operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^0BA^{-1}B^{-1} - \operatorname{tr} ABA^{-1}B^{-1}\Big) =$$

$$= \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}}.$$

-) $m \equiv 0$ modulo 4.

$$s_m(X)D(X,Y,Z) \;=\; \Big(\sum_{i=1}^{\frac{m}{4}} f_{2i-1}(X)\Big)D(X,Y,Z) = \Big(\sum_{i=1}^{\frac{m}{4}} \operatorname{tr} A^{2i-1}\Big)\Big(\operatorname{tr} ABA^{-1}B^{-1} - 2\Big) =$$

$$= \sum_{i=1}^{\frac{m}{4}} \Big[\operatorname{tr} A^{2i}BA^{-1}B^{-1} + \operatorname{tr}(A^{2i-2})(BAB^{-1})\Big] - 2\sum_{i=1}^{\frac{m}{4}} \operatorname{tr} A^{2i-1} =$$

$$= \sum_{i=1}^{\frac{m}{4}} \Big[\operatorname{tr} A^{2i}BA^{-1}B^{-1} + (\operatorname{tr} A^{2i-2} \operatorname{tr} A) - \operatorname{tr} A^{2i-2}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m}{4}} \operatorname{tr} A^{2i-1} =$$

$$= \sum_{i=1}^{\frac{m}{4}} \Big[\operatorname{tr} A^{2i}BA^{-1}B^{-1} + (\operatorname{tr} A^{2i-1} + \operatorname{tr} A^{2i-3}) - \operatorname{tr} A^{2i-2}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m}{4}} \operatorname{tr} A^{2i-1} =$$

$$= \sum_{i=1}^{\frac{m}{4}} \Big(\operatorname{tr} A^{2i-3} - \operatorname{tr} A^{2i-1}\Big) + \sum_{i=1}^{\frac{m}{4}} \Big(\operatorname{tr} A^{2i}BA^{-1}B^{-1} - \operatorname{tr} A^{2i-2}BA^{-1}B^{-1}\Big) =$$

$$= \Big(\operatorname{tr} A^{-1} - \operatorname{tr} A^{\frac{m-2}{2}}\Big) + \Big(\operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{0}BA^{-1}B^{-1}\Big) =$$

$$= \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}}.$$

-) $m \equiv 2$ modulo 4.

$$s_m(X)D(X,Y,Z) \;=\; \Big(1 + \sum_{i=1}^{\frac{m-2}{4}} f_{2i}(X)\Big)\cdot D = \Big(1 + \sum_{i=1}^{\frac{m-2}{4}} \operatorname{tr} A^{2i}\Big)\Big(\operatorname{tr} ABA^{-1}B^{-1} - 2\Big) =$$

$$= D(X,Y,Z) + \sum_{i=1}^{\frac{m-2}{4}} \Big[\operatorname{tr} A^{2i+1}BA^{-1}B^{-1} + \operatorname{tr}(A^{2i-1})(BAB^{-1})\Big] - 2\sum_{i=1}^{\frac{m-2}{4}} \operatorname{tr} A^{2i} =$$

$$= D + \sum_{i=1}^{\frac{m-2}{4}} \Big[\operatorname{tr} A^{2i+1}BA^{-1}B^{-1} + (\operatorname{tr} A^{2i-1} \operatorname{tr} A) - \operatorname{tr} A^{2i-1}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m-2}{4}} \operatorname{tr} A^{2i} =$$

$$= D + \sum_{i=1}^{\frac{m-2}{4}} \Big[\operatorname{tr} A^{2i+1}BA^{-1}B^{-1} + (\operatorname{tr} A^{2i} + \operatorname{tr} A^{2i-2}) - \operatorname{tr} A^{2i-1}BA^{-1}B^{-1}\Big] - 2\sum_{i=1}^{\frac{m-2}{4}} \operatorname{tr} A^{2i} =$$

$$= D + \sum_{i=1}^{\frac{m-2}{4}} \Big(\operatorname{tr} A^{2i-2} - \operatorname{tr} A^{2i}\Big) + \sum_{i=1}^{\frac{m-2}{4}} \Big(\operatorname{tr} A^{2i+1}BA^{-1}B^{-1} - \operatorname{tr} A^{2i-1}BA^{-1}B^{-1}\Big) =$$

$$= \Big(\operatorname{tr} ABA^{-1}B^{-1} - 2\Big) + \Big(\operatorname{tr} A^{0} - \operatorname{tr} A^{\frac{m-2}{2}}\Big) + \Big(\operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} ABA^{-1}B^{-1}\Big) =$$

$$= \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}}.$$

□

Lemma A.2. *k an integer.*

$$\sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j \operatorname{tr} A^k B A^{-1} B^{-1} =$$

$$\stackrel{(1)}{=} \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j A^{k-1} + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^{j+2} A^{-k+1} - \operatorname{tr} A^{-k+1} B + \operatorname{tr} A^{-k} BAB -$$
$$- \ (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}} A^{-k} BA \ - \ (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}} A^{-k} BA =$$

$$\stackrel{(2)}{=} \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j A^{k-1} + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^{j-2} A^{-k+1} + \operatorname{tr} A^{-k+1} B^{-1} - \operatorname{tr} A^{-k} BAB^{-1} +$$
$$+ \ (-1)^{\frac{n-3}{2}} \operatorname{tr} A^{-k} B^{\frac{n-1}{2}} AB^{-1} \ + \ (-1)^{\frac{n-1}{2}} \operatorname{tr} A^{-k} B^{\frac{n+1}{2}} AB^{-1}.$$

Proof.

1) $$\sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j \operatorname{tr} BA^{-1}B^{-1}A^k \ = \ \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr}(B^{j+1})(A^{-1}B^{-1}A^k) + \operatorname{tr} B^{j-1} A^{-k} BA\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr} B^{j+1} \operatorname{tr} B^{-1} A^{k-1} - \operatorname{tr} B^{j+1} A^{-k} BA + \operatorname{tr} B^{j-1} A^{-k} BA\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr} B^j A^{k-1} + \operatorname{tr} B^{j+2} A^{-k+1} + \left(\operatorname{tr} B^{j-1} A^{-k} BA - \operatorname{tr} B^{j+1} A^{-k} BA\right)\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j A^{k-1} + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^{j+2} A^{-k+1} - \operatorname{tr} B^0 A^{-k} BA + \operatorname{tr} BA^{-k} BA +$$
$$- \ (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}} A^{-k} BA \ - \ (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}} A^{-k} BA.$$

2) $$\sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j \operatorname{tr} A^k BA^{-1}B^{-1} \ = \ \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr}(B^{j-1})(A^k BA^{-1}) + \operatorname{tr} B^{j+1} AB^{-1} A^{-k}\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr} B^{j-1} \operatorname{tr} A^{k-1} B - \operatorname{tr} B^{j-1} AB^{-1} A^{-k} + \operatorname{tr} B^{j+1} AB^{-1} A^{-k}\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \left[\operatorname{tr} B^j A^{k-1} + \operatorname{tr} B^{j-2} A^{-k+1} + \left(\operatorname{tr} B^{j+1} AB^{-1} A^{-k} - \operatorname{tr} B^{j-1} AB^{-1} A^{-k}\right)\right] =$$
$$= \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j A^{k-1} + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^{j-2} A^{-k+1} + \operatorname{tr} B^0 AB^{-1} A^{-k} - \operatorname{tr} BAB^{-1} A^{-k} +$$
$$+ \ (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}} AB^{-1} A^{-k} \ + \ (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}} AB^{-1} A^{-k}.$$

□

Lemma A.3. *m and n odd.*

$$s_m(X)\sigma_n(Y)D = \left(F_{\frac{m+3}{2},\frac{n-1}{2}} - F_{\frac{m-3}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right) +$$
$$+ \left(F_{\frac{m-1}{2},\frac{n+3}{2}} - F_{\frac{m+1}{2},\frac{n-3}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n+3}{2}} - F_{\frac{m+3}{2},\frac{n-3}{2}}\right) +$$
$$+ F_{1,-1}\left[\left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n-1}{2}} - F_{\frac{m+1}{2},\frac{n+1}{2}}\right)\right].$$

Proof. We will use Lemma A.2(1) in $(*)$ taking $k = (m+1)/2$ and $k = (m-1)/2$.

$$(-1)^{\frac{n-1}{2}}\sigma_n(Y)s_m(X)D = \left(1 + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j\right)\left(\operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} -\right.$$
$$\left. - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}}\right) \overset{(*)}{=}$$

$$\overset{(*)}{=} \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} +$$
$$+ \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-3}{2}} + \sum(-1)^j \operatorname{tr} B^{j+2}A^{\frac{-m+3}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}}B + \operatorname{tr} A^{\frac{-m+1}{2}}BAB -$$
$$- (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m+1}{2}}BA - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m+1}{2}}BA +$$
$$+ \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-1}{2}} + \sum(-1)^j \operatorname{tr} B^{j+2}A^{\frac{-m+1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}}B + \operatorname{tr} A^{\frac{-m-1}{2}}BAB -$$
$$- (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m-1}{2}}BA - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m-1}{2}}BA - \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-3}{2}} -$$
$$- \sum(-1)^j \operatorname{tr} B^j A^{\frac{-m+3}{2}} - \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-1}{2}} - \sum(-1)^j \operatorname{tr} B^j A^{\frac{-m+1}{2}} =$$

$$= \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}}B +$$
$$+ \operatorname{tr} A^{\frac{-m+1}{2}}BAB - (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m+1}{2}}BA - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m+1}{2}}BA -$$
$$- \operatorname{tr} A^{\frac{-m+1}{2}}B + \operatorname{tr} A^{\frac{-m-1}{2}}BAB - (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m-1}{2}}BA -$$
$$- (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m-1}{2}}BA + \sum(-1)^j\left[\operatorname{tr} B^{j+2}A^{\frac{-m+3}{2}} - \operatorname{tr} B^j A^{\frac{-m+3}{2}}\right] +$$
$$+ \sum(-1)^j\left[\operatorname{tr} B^{j+2}A^{\frac{-m+1}{2}} - \operatorname{tr} B^j A^{\frac{-m+1}{2}}\right] =$$

$$= \operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}}B +$$
$$+ \operatorname{tr} A^{\frac{-m+1}{2}}BAB - (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m+1}{2}}BA - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m+1}{2}}BA -$$
$$- \operatorname{tr} A^{\frac{-m+1}{2}}B + \operatorname{tr} A^{\frac{-m-1}{2}}BAB - (-1)^{\frac{n-3}{2}}\left[\operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m-1}{2}}BA - \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m-1}{2}}BA\right] +$$
$$+ \left[\operatorname{tr} BA^{\frac{-m+3}{2}} - \operatorname{tr} B^2A^{\frac{-m+3}{2}} + (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m+3}{2}} + (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+3}{2}}A^{\frac{-m+3}{2}}\right] +$$
$$+ \left[\operatorname{tr} BA^{\frac{-m+1}{2}} - \operatorname{tr} B^2A^{\frac{-m+1}{2}} + (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n+1}{2}}A^{\frac{-m+1}{2}} + (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+3}{2}}A^{\frac{-m+1}{2}}\right] =$$

$$= \operatorname{tr} A^{\frac{m+1}{2}} B A^{-1} B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}} B A^{-1} B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} + \operatorname{tr} A^{\frac{-m+1}{2}} BAB -$$
$$- (-1)^{\frac{n-3}{2}} \left[\operatorname{tr} B^{\frac{n-1}{2}} A^{\frac{-m+1}{2}} \operatorname{tr} BA - \operatorname{tr} B^{\frac{n-3}{2}} A^{\frac{-m-1}{2}}\right] - (-1)^{\frac{n-1}{2}} \left[\operatorname{tr} B^{\frac{n+1}{2}} A^{\frac{-m+1}{2}} \operatorname{tr} BA -\right.$$
$$\left. - \operatorname{tr} B^{\frac{n-1}{2}} A^{\frac{-m-1}{2}}\right] + \operatorname{tr} A^{\frac{-m-1}{2}} BAB - (-1)^{\frac{n-3}{2}} \left[\operatorname{tr} B^{\frac{n-1}{2}} A^{\frac{-m-1}{2}} \operatorname{tr} BA - \operatorname{tr} B^{\frac{n-3}{2}} A^{\frac{-m-3}{2}}\right] -$$
$$- (-1)^{\frac{n-1}{2}} \left[\operatorname{tr} B^{\frac{n+1}{2}} A^{\frac{-m-1}{2}} \operatorname{tr} BA - \operatorname{tr} B^{\frac{n-1}{2}} A^{\frac{-m-3}{2}}\right] - \operatorname{tr} B^2 A^{\frac{-m+3}{2}} +$$
$$+ (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n+1}{2}} A^{\frac{-m+3}{2}} + (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+3}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^2 A^{\frac{-m+1}{2}} +$$
$$+ (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n+1}{2}} A^{\frac{-m+1}{2}} + (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n+3}{2}} A^{\frac{-m+1}{2}} =$$

$$= (-1)^{\frac{n-1}{2}} \operatorname{tr} AB \left[\left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+1}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(\operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n+1}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+3}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-3}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n+3}{2}} - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n-3}{2}}\right)\right] +$$
$$+ \left(\operatorname{tr} A^{\frac{m+1}{2}} B A^{-1} B^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} BAB - \operatorname{tr} A^{\frac{m-1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^2\right) +$$
$$+ \left(\operatorname{tr} A^{\frac{m-1}{2}} B A^{-1} B^{-1} + \operatorname{tr} A^{\frac{-m+1}{2}} BAB - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}} B^2\right) \overset{(**)}{=}$$

$$= (-1)^{\frac{n-1}{2}} \operatorname{tr} AB \left[\left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+1}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(\operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n+1}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-1}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+3}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-3}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n+3}{2}} - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n-3}{2}}\right)\right] =$$

$$= (-1)^{\frac{n-1}{2}} F_{1,-1} \left[\left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n-1}{2}} - F_{\frac{m+1}{2},\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(F_{\frac{m+3}{2},\frac{n-1}{2}} - F_{\frac{m-3}{2},\frac{n+1}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}\right)\right] +$$
$$+ (-1)^{\frac{n-1}{2}} \left[\left(F_{\frac{m-1}{2},\frac{n+3}{2}} - F_{\frac{m+1}{2},\frac{n-3}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n+3}{2}} - F_{\frac{m+3}{2},\frac{n-3}{2}}\right)\right].$$

$$(**) \qquad \begin{aligned} \operatorname{tr} A^k B A^{-1} B^{-1} &= \operatorname{tr} B A^{-1} B^{-1} A^k = \operatorname{tr} B \operatorname{tr} A^{k-1} B^{-1} - \operatorname{tr} B A^{-k} BA = \\ &= \operatorname{tr} A^{k-1} + \operatorname{tr} A^{-k+1} B^2 - \operatorname{tr} A^{-k} BAB. \end{aligned}$$

We have used $(**)$ taking $k = (m+1)/2$ and $k = (m-1)/2$. □

Lemma A.4. *m even and n odd.*

$$s_m(X)\sigma_n(Y)D = \left(F_{\frac{m+2}{2},\frac{n+1}{2}} - F_{\frac{m-2}{2},\frac{n-1}{2}}\right) + \left(F_{\frac{m-2}{2},\frac{n-3}{2}} - F_{\frac{m+2}{2},\frac{n+3}{2}}\right) + \\ + F_{1,1}\left(F_{\frac{m}{2},\frac{n+1}{2}} - F_{\frac{m}{2},\frac{n-1}{2}}\right).$$

Proof. We will use Lemma A.2(2) in $(*)$ taking $k = m/2$.

$$(-1)^{\frac{n-1}{2}}\sigma_n(Y)s_m(X)D = \left(1 + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j \operatorname{tr} B^j\right)\left(\operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}}\right) =$$

$$= \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} + \sum_{j=1}^{\frac{n-1}{2}}(-1)^j\left[\operatorname{tr} B^j \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} B^j \operatorname{tr} A^{\frac{m-2}{2}}\right] \overset{(*)}{=}$$

$$\overset{(*)}{=} \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} + \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-2}{2}} + \sum(-1)^j \operatorname{tr} B^{j-2}A^{\frac{-m+2}{2}} + \\ + \operatorname{tr} A^{\frac{-m+2}{2}}B^{-1} - \operatorname{tr} A^{\frac{-m}{2}}BAB^{-1} + (-1)^{\frac{n-3}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-1}{2}}AB^{-1} + \\ + (-1)^{\frac{n-1}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+1}{2}}AB^{-1} - \sum(-1)^j \operatorname{tr} B^j A^{\frac{m-2}{2}} - \sum(-1)^j \operatorname{tr} B^j A^{\frac{-m+2}{2}} =$$

$$= \operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} + \operatorname{tr} A^{\frac{-m+2}{2}}B^{-1} - \operatorname{tr} A^{\frac{-m}{2}}BAB^{-1} + \\ + (-1)^{\frac{n-3}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-1}{2}}AB^{-1} + (-1)^{\frac{n-1}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+1}{2}}AB^{-1} - \operatorname{tr} B^{-1}A^{\frac{-m+2}{2}} + \\ + \operatorname{tr} B^0 A^{\frac{-m+2}{2}} - (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-3}{2}}A^{\frac{-m+2}{2}} - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m+2}{2}} =$$

$$= (-1)^{\frac{n-3}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-1}{2}} \operatorname{tr} AB^{-1} - (-1)^{\frac{n-3}{2}} \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+1}{2}} + \\ + (-1)^{\frac{n-1}{2}} \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+1}{2}} \operatorname{tr} AB^{-1} - (-1)^{\frac{n-1}{2}} \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+3}{2}} - \\ - (-1)^{\frac{n-3}{2}} \operatorname{tr} B^{\frac{n-3}{2}}A^{\frac{-m+2}{2}} - (-1)^{\frac{n-1}{2}} \operatorname{tr} B^{\frac{n-1}{2}}A^{\frac{-m+2}{2}} =$$

$$= (-1)^{\frac{n-1}{2}}\left[\left(\operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+1}{2}} - \operatorname{tr} A^{\frac{-m+2}{2}}B^{\frac{n-1}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+2}{2}}B^{\frac{n-3}{2}} - \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+3}{2}}\right) + \right. \\ \left. + \operatorname{tr} AB^{-1}\left(\operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+1}{2}} - \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-1}{2}}\right)\right] =$$

$$= (-1)^{\frac{n-1}{2}}\left[\left(F_{\frac{m+2}{2},\frac{n+1}{2}} - F_{\frac{m-2}{2},\frac{n-1}{2}}\right) + \left(F_{\frac{m-2}{2},\frac{n-3}{2}} - F_{\frac{m+2}{2},\frac{n+3}{2}}\right) + \right. \\ \left. + F_{1,1}\left(F_{\frac{m}{2},\frac{n+1}{2}} - F_{\frac{m}{2},\frac{n-1}{2}}\right)\right].$$

□

Lemma A.5. *m odd, n even.*

$$\begin{aligned}s_m(X)f_{\frac{n}{2}}(Y)D = &\left(F_{\frac{m+3}{2},\frac{n}{2}} - F_{\frac{m-3}{2},\frac{n}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n}{2}} - F_{\frac{m-1}{2},\frac{n}{2}}\right) + \\ &+ \left(F_{\frac{m-1}{2},\frac{n-4}{2}} - F_{\frac{m+1}{2},\frac{n+4}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n-4}{2}} - F_{\frac{m+3}{2},\frac{n+4}{2}}\right) + \\ &+ F_{1,1}\left[\left(F_{\frac{m+1}{2},\frac{n+2}{2}} - F_{\frac{m-1}{2},\frac{n-2}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n+2}{2}} - F_{\frac{m+1}{2},\frac{n-2}{2}}\right)\right].\end{aligned}$$

Proof.

$$f_{\frac{n}{2}}(Y)s_m(X)D =$$

$$= \operatorname{tr} B^{\frac{n}{2}}\left(\operatorname{tr} A^{\frac{m+1}{2}}BA^{-1}B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}}\right) =$$

$$\begin{aligned}= \ &\operatorname{tr} B^{\frac{n-2}{2}}A^{\frac{m+1}{2}}BA^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n-2}{2}}AB^{-1} + \operatorname{tr} B^{\frac{n-2}{2}}A^{\frac{m-1}{2}}BA^{-1} + \\ &+ \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n+2}{2}}AB^{-1} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-1}{2}} - \\ &- \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+1}{2}} =\end{aligned}$$

$$\begin{aligned}= \ &\operatorname{tr} B^{\frac{n}{2}}\operatorname{tr} A^{\frac{m-1}{2}}B - \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n-2}{2}}AB^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n+2}{2}}AB^{-1} + \\ &+ \operatorname{tr} B^{\frac{n}{2}}\operatorname{tr} A^{\frac{m-3}{2}}B - \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n-2}{2}}AB^{-1} + \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n+2}{2}}AB^{-1} - \\ &- \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-1}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+1}{2}} =\end{aligned}$$

$$\begin{aligned}= \ &\operatorname{tr} B^{\frac{n-4}{2}}A^{\frac{-m+1}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n-2}{2}}\operatorname{tr} AB^{-1} + \operatorname{tr} A^{\frac{-m-3}{2}}B^{\frac{n}{2}} + \\ &+ \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n+2}{2}}\operatorname{tr} AB^{-1} - \operatorname{tr} A^{\frac{-m-3}{2}}B^{\frac{n+4}{2}} + \operatorname{tr} B^{\frac{n-4}{2}}A^{\frac{-m+3}{2}} - \\ &- \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n-2}{2}}\operatorname{tr} AB^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n}{2}} + \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n+2}{2}}\operatorname{tr} AB^{-1} - \\ &- \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n+4}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+1}{2}} =\end{aligned}$$

$$\begin{aligned}= \ &\left(\operatorname{tr} A^{\frac{-m-3}{2}}B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}}B^{\frac{n}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n}{2}}\right) + \\ &+ \left(\operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n+4}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+3}{2}}B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-3}{2}}B^{\frac{n+4}{2}}\right) + \\ &+ \operatorname{tr} AB^{-1}\left[\left(\operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n-2}{2}}\right) + \right. \\ &\left. + \left(\operatorname{tr} A^{\frac{-m+1}{2}}B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}}B^{\frac{n-2}{2}}\right)\right] =\end{aligned}$$

$$\begin{aligned}= \ &\left(F_{\frac{m+3}{2},\frac{n}{2}} - F_{\frac{m-3}{2},\frac{n}{2}}\right) + \left(F_{\frac{m+1}{2},\frac{n}{2}} - F_{\frac{m-1}{2},\frac{n}{2}}\right) + \\ &+ \left(F_{\frac{m-1}{2},\frac{n-4}{2}} - F_{\frac{m+1}{2},\frac{n+4}{2}}\right) + \left(F_{\frac{m-3}{2},\frac{n-4}{2}} - F_{\frac{m+3}{2},\frac{n+4}{2}}\right) + \\ &+ F_{1,1}\left[\left(F_{\frac{m+1}{2},\frac{n+2}{2}} - F_{\frac{m-1}{2},\frac{n-2}{2}}\right) + \left(F_{\frac{m-1}{2},\frac{n+2}{2}} - F_{\frac{m+1}{2},\frac{n-2}{2}}\right)\right].\end{aligned}$$

□

Lemma A.6. *m, n even.*

$$s_m(X)f_{\frac{n}{2}}(Y)D = \left(F_{\frac{m+2}{2},\frac{n}{2}} - F_{\frac{m-2}{2},\frac{n}{2}}\right) + \left(F_{\frac{m-2}{2},\frac{n-4}{2}} - F_{\frac{m+2}{2},\frac{n+4}{2}}\right) + \\ + F_{1,1}\left(F_{\frac{m}{2},\frac{n+2}{2}} - F_{\frac{m}{2},\frac{n-2}{2}}\right).$$

Proof.

$$\begin{aligned}
f_{\frac{n}{2}}(Y)s_m(X)D &= \operatorname{tr} B^{\frac{n}{2}}\left(\operatorname{tr} A^{\frac{m}{2}}BA^{-1}B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}}\right) = \\
&= \operatorname{tr} B^{\frac{n-2}{2}}A^{\frac{m}{2}}BA^{-1} + \operatorname{tr} B^{\frac{n+2}{2}}AB^{-1}A^{\frac{-m}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-2}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+2}{2}} = \\
&= \operatorname{tr} B^{\frac{n-2}{2}} \operatorname{tr} A^{\frac{m-2}{2}}B - \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-2}{2}}AB^{-1} + \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+2}{2}}AB^{-1} - \\
&\quad - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-2}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+2}{2}} = \\
&= \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-2}{2}} + \operatorname{tr} B^{\frac{n-4}{2}}A^{\frac{-m+2}{2}} - \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-2}{2}} \operatorname{tr} AB^{-1} + \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n}{2}} + \\
&\quad + \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+2}{2}} \operatorname{tr} AB^{-1} - \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+4}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{m-2}{2}} - \operatorname{tr} B^{\frac{n}{2}}A^{\frac{-m+2}{2}} = \\
&= \left(\operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+2}{2}}B^{\frac{n}{2}}\right) + \left(\operatorname{tr} A^{\frac{-m+2}{2}}B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-2}{2}}B^{\frac{n+4}{2}}\right) + \\
&\quad + \operatorname{tr} AB^{-1}\left(\operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m}{2}}B^{\frac{n-2}{2}}\right) = \\
&= \left(F_{\frac{m+2}{2},\frac{n}{2}} - F_{\frac{m-2}{2},\frac{n}{2}}\right) + \left(F_{\frac{m-2}{2},\frac{n-4}{2}} - F_{\frac{m+2}{2},\frac{n+4}{2}}\right) + F_{1,1}\left(F_{\frac{m}{2},\frac{n+2}{2}} - F_{\frac{m}{2},\frac{n-2}{2}}\right).
\end{aligned}$$

□