An improved approximation algorithm for the ATSP with parameterized triangle inequality

Tongquan Zhang\textsuperscript{a,\ast}, Weidong Li\textsuperscript{b,\ast}, Jianping Li\textsuperscript{b,*}

\textsuperscript{a} Center for Nonlinear Complex Systems, School of Physical Science and Technique, Yunnan University, Kunming 650091, China
\textsuperscript{b} Department of Mathematics, School of Mathematics and Statistics, Yunnan University, Kunming 650091, China

\begin{abstract}
In this paper, we consider the asymmetric traveling salesman problem with the $\gamma$-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. That means the edge weights in the given complete graph $G = (V, E, \omega)$ satisfy $\omega(u, w) \leq \gamma \cdot (\omega(u, v) + \omega(v, w))$ for all distinct nodes $u, v, w \in V$. L.S. Chandran and L.S. Ram gave the first constant factor approximation algorithm with polynomial running time for this problem. They achieve performance ratio $\frac{2}{1-\gamma}$. M. Bläser, B. Manthey and J. Sgall obtain a $\frac{1+\gamma}{2-\gamma}$-approximation algorithm. We devise an approximation algorithm with performance ratio $\max\{1 + \frac{\gamma^2}{1-\gamma}, \frac{\gamma^2+1}{2-\gamma} + \frac{\gamma^3}{1-\gamma}\}$, which is better than both $\frac{1+\gamma}{2-\gamma}$ and $\frac{\gamma^2+1}{2-\gamma}$ for almost all $\gamma \in [\frac{1}{2}, 1)$. © 2008 Elsevier Inc. All rights reserved.
\end{abstract}

1. Introduction

The Asymmetric Traveling Salesman Problem (ATSP) is to find a minimum weight directed Hamiltonian cycle in a complete digraph $G = (V, E; \omega)$ with weight function $\omega : E \to \mathbb{R}^+$ associated with the edges. A Hamiltonian cycle will be referred to as a tour. We denote the weight of an optimal tour in ATSP by $\gamma$-ATSP tours.

Hamiltonian cycles are special cases of cycle covers; we have AP($G$) $\leq$ ATSP($G$). It is known that ATSP is NP-hard [2], whereas AP is polynomial time solvable. So the problem of finding the minimum weight cycle cover in $G$ plays an important role in the design of approximation algorithms for ATSP. An $f$-factor approximation algorithm for ATSP seeks to find a tour whose weight is at most $f \cdot$ ATSP. Designing a constant factor approximation algorithm for ATSP (assuming the triangle inequality) is one of the most important open problems in the field of approximation algorithms. The best approximation algorithms for this problem to date achieve only factors of $O(\log n)$ [3,4]. In fact, all of the currently known approaches for designing approximation algorithms for ATSP rely on (directly or in a more sophisticated way) using AP($G$) as a lower bound. In a seminal paper Karp [5] proved via the analysis of an $O(n^2)$-time algorithm that $\frac{\text{ATSP}(G)}{\text{AP}(G)} = 1 + o(1)$ with high
Lemma 2.1. Assume that the weight function satisfies the triangle inequality (i.e. the metric property)
\[ \omega(u, w) \leq \omega(u, v) + \omega(v, w) \quad \text{for all } u, v, w \in V. \] (1.1)

If the weight function of an instance of ATSP fulfills inequality (1.1), we name it \( \triangle \)-ATSP. The factors for it to date are \( O(\log n) \) [3,4].

Here, we consider the ATSP with a strengthening of the triangle inequality (1.1), which allows a constant factor approximation: Let \( \gamma \) be some constant with \( \frac{1}{2} \leq \gamma < 1 \). An instance of the problem \( \triangle(\gamma) \)-ATSP is a complete loopless directed graph \( G \) with node set \( V \) and a weight function \( \omega \) assigned to each edge of \( G \). The weight function fulfills the \( \gamma \)-parameterized triangle inequality, i.e.
\[ \omega(u, w) \leq \gamma \cdot (\omega(u, v) + \omega(v, w)) \quad \text{for all } u, v, w \in V. \] (1.2)

Chandran and Ram [1] studied the asymmetric traveling salesman problem with the \( \gamma \)-parameterized triangle inequality for \( \gamma \in [\frac{1}{2}, 1) \). They designed a constant factor approximation algorithm with performance ratio (asymptotically) \( \frac{\gamma}{1-\gamma} \). Bläser gave an approximation algorithm with performance ratio \( \frac{2}{\gamma-\gamma^2} \) [8]. Bläser, Manthey and Sgall obtain an approximation algorithm in [9].

As our main result, we present an approximation algorithm with performance ratio \( \max(1+\frac{\gamma^3}{1-\gamma^2}, \frac{\gamma+\gamma^2+1}{2}, \frac{\gamma^3}{1-\gamma^2}) \). Here, \( k = \min\{|C| \mid i = 1, 2, \ldots, K\} \), \( C_1, C_2, \ldots, C_K \) is an optimal solution of the minimum cycle cover problem in \( G \), which are better than \( \frac{1+y}{1-\gamma^2} \) and \( \frac{1}{\gamma} \) for almost all \( \gamma \in [\frac{1}{2}, 1) \). For \( \gamma \) close to 1,
\[
\lim_{\gamma \to 1} \frac{\gamma + \gamma^2 + 1}{2}, \frac{\gamma}{1-\gamma^2} = \frac{1}{2}, \quad \text{and} \quad \lim_{\gamma \to 1} \frac{\gamma + 2 \gamma^2 + 1}{\frac{2}{\gamma-\gamma^2}} = \frac{2}{3},
\]
the improvement is by a factor close to 2 and 1.5 respectively. Our algorithm (based on algorithms of AP or minimum cycle cover problem) can be computed in \( O(n^3) \) [10].

2. The approximation algorithm

We are given a complete directed graph \( G = (V, E; \omega) \) with weight function \( \omega : E \to R \) satisfying the \( \gamma \)-parameterized triangle inequality, i.e. \( \omega(u, w) \leq \gamma \cdot (\omega(u, v) + \omega(v, w)) \) for all distinct nodes \( u, v, w \in V \) with \( \gamma \in [\frac{1}{2}, 1) \). We define \( \omega(S) = \sum_{e \in E} \omega(e) \) for \( S \subseteq E \). Also assume that the weight of the optimal tour in \( G \) is \( \text{ATSP}(G) \). Let \( |V| = n \).

Lemma 2.1. Let \( C = (u_1, u_2, \ldots, u_k) \) be a directed cycle in \( G \) with \( 2 \leq k \leq n \) and \( \gamma \in [\frac{1}{2}, 1) \). Let \( u_{k+1} = u_1 \), for any \( w, w \in \{V - \{u_1, u_2, \ldots, u_k\}\} \).

Then, each edge \( (u_j, w) \) \((w, u_j)\) for \( j = 1, 2, \ldots, k \) satisfies
\[ \omega(u_j, w) \omega(w, u_j) \leq \frac{\gamma}{1 - \gamma^k} \cdot \omega(C). \]

Proof. If the edge is \( (u_j, w) \), then \( u_j, u_{j+1}, w \) satisfy the \( \gamma \)-parameterized triangle inequality, i.e.
\[ \omega(u_j, w) \leq \gamma \left( \omega(u_j, u_{j+1}) + \omega(u_{j+1}, w) \right). \] (2.1)

Again,
\[ \omega(u_{j+1}, w) \leq \gamma \left( \omega(u_{j+1}, u_{j+2}) + \omega(u_{j+2}, w) \right). \] (2.2)

We have
\[ \omega(u_j, w) \leq \gamma \omega(u_j, u_{j+1}) + \omega(u_{j+1}, w) \leq \gamma \omega(u_j, u_{j+1}) + \gamma^2 \omega(u_{j+1}, u_{j+2}) + \gamma^2 \omega(u_{j+2}, w) \]
by inequalities (2.1) and (2.2). After repeating the foregoing steps \( k \) times we have
\[ \omega(u_j, w) \leq \gamma \omega(u_j, u_{j+1}) + \gamma^2 \omega(u_{j+1}, u_{j+2}) + \cdots + \gamma^k \omega(u_{j-1}, u_j) + \gamma^k \omega(u_j, w). \] (2.3)

At the same time \( \gamma \in [\frac{1}{2}, 1) \), then \( \gamma > \gamma^2 > \cdots > \gamma^k \). Therefore inequality (2.3) can be transformed to the following inequality by amplifying its right side
\[ \omega(u_j, w) \leq \gamma \omega(u_j, u_{j+1}) + \gamma \omega(u_{j+1}, u_{j+2}) + \cdots + \gamma \omega(u_{j-1}, u_j) + \gamma^k \omega(u_j, w). \] (2.4)
At last, we have \(\omega(u, w) \leq \frac{\gamma}{1 - \gamma^2} \cdot \omega(C)\) from the transformation of inequality (2.4).

If the edge is \((w, u_j)\), then \(u_j, u_j, w\) satisfy the \(\gamma\)-parameterized triangle inequality, i.e.
\[
\omega(w, u_j) \leq \gamma (\omega(u_{j-1}, u_j) + \omega(w, u_{j-1})).
\] (2.5)

Again,
\[
\omega(w, u_{j-1}) \leq \gamma (\omega(u_{j-1}, u_{j-1}) + \omega(w, u_{j-2})).
\] (2.6)

We have
\[
\omega(w, u_j) \leq \gamma (\omega(u_{j-1}, u_j) + \omega(w, u_{j-1})) \leq \gamma \omega(u_{j-1}, u_j) + \gamma^2 \omega(u_{j-2}, u_{j-1}) + \gamma^2 \omega(w, u_{j-2})
\]
by inequalities (2.5) and (2.6). After repeating the foregoing steps \(k\) times we have
\[
\omega(w, u_j) \leq \gamma \omega(u_{j-1}, u_j) + \gamma^2 \omega(u_{j-2}, u_{j-1}) + \cdots + \gamma^k \omega(u, u_j+1) + \gamma^k \omega(w, u_j).
\] (2.7)

At the same time \(\gamma \in [\frac{1}{2}, 1)\); then \(\gamma > \gamma^2 > \cdots > \gamma^k\). Therefore inequality (2.7) can be transformed to the following inequality by amplifying its right side
\[
\omega(w, u_j) \leq \gamma \omega(u_{j-1}, u_j) + \gamma^2 \omega(u_{j-2}, u_{j-1}) + \cdots + \gamma^k \omega(u, u_j+1) + \gamma^k \omega(w, u_j).
\] (2.8)

Therefore we have \(\omega(w, u_j) \leq \frac{\gamma^k}{1 - \gamma^k} \cdot \omega(C)\) from the transformation of inequality (2.8).

Thus, the proof is over. \(\Box\)

Lemma 2.2. Let \(C_1 = (u_1, u_2, \ldots, u_{|C_1|})\) and \(C_2 = (v_1, v_2, \ldots, v_{|C_2|})\) be two directed and disjoint cycles in \(G\) with \(2 \leq |C_1|, |C_2| \leq n\) and \(\gamma \in [\frac{1}{2}, 1)\). Let \(u_{|C_1|+1} = u_1\) and \(v_{|C_2|+1} = v_1\). The following inequality is always satisfied for any \((i, j) \in (1 \leq i \leq |C_1|, 1 \leq j \leq |C_2|)\).
\[
\omega(u_i, v_{j+1}) + \omega(v_j, u_{i+1}) \leq (\gamma + \gamma^2) \left[\omega(u_i, u_{i+1}) + \omega(v_j, v_{j+1})\right] + \frac{\gamma^3 \omega(C_1)}{1 - \gamma^{|C_1|}} + \frac{\gamma^3 \omega(C_2)}{1 - \gamma^{|C_2|}}.
\]

Proof. From Lemma 2.1 we know that
\[
\omega(u_{i+1}, v_j) \leq \frac{\gamma \omega(C_1)}{1 - \gamma^{|C_1|}}
\] (2.9)
and
\[
\omega(v_{j+1}, u_i) \leq \frac{\gamma \omega(C_2)}{1 - \gamma^{|C_2|}}.
\] (2.10)

The graph fulfills the \(\gamma\)-parameterized triangle inequality, so we have
\[
\omega(u_i, v_{j+1}) \leq \gamma \omega(u_i, u_{i+1}) + \gamma^2 \omega(u_{i+1}, v_j) + \gamma^2 \omega(v_j, v_{j+1})
\] (2.11)
and
\[
\omega(v_j, u_{i+1}) \leq \gamma \omega(v_j, v_{j+1}) + \gamma^2 \omega(v_{j+1}, u_i) + \gamma^2 \omega(u_i, u_{i+1}).
\] (2.12)

Inequalities (2.11) and (2.22) can be transformed by (2.9) and (2.10) to the following inequalities
\[
\omega(u_i, v_{j+1}) \leq \gamma \omega(u_i, u_{i+1}) + \frac{\gamma^3 \omega(C_1)}{1 - \gamma^{|C_1|}} + \gamma^2 \omega(v_j, v_{j+1})
\] (2.13)
and
\[
\omega(v_j, u_{i+1}) \leq \gamma \omega(v_j, v_{j+1}) + \frac{\gamma^3 \omega(C_2)}{1 - \gamma^{|C_2|}} + \gamma^2 \omega(u_i, u_{i+1}).
\] (2.14)

Then we can get
\[
\omega(u_i, v_{j+1}) + \omega(v_j, u_{i+1}) \leq (\gamma + \gamma^2) \left[\omega(u_i, u_{i+1}) + \omega(v_j, v_{j+1})\right] + \frac{\gamma^3 \omega(C_1)}{1 - \gamma^{|C_1|}} + \frac{\gamma^3 \omega(C_2)}{1 - \gamma^{|C_2|}}
\]
by adding (2.13) to (2.14). \(\Box\)

Corollary 2.1. Let \(C_1 = (u_1, u_2, \ldots, u_{|C_1|})\) and \(C_2 = (v_1, v_2, \ldots, v_{|C_2|})\) be two directed and disjoint cycles in \(G\) with \(2 \leq |C_1|, |C_2| \leq n\) and \(\gamma \in [\frac{1}{2}, 1)\). Let \(u_{|C_1|+1} = u_1\) and \(v_{|C_2|+1} = v_1\). Inequality
\[
\omega(u_i, v_{j+1}) + \omega(v_j, u_{i+1}) \leq (\gamma + \gamma^2) \left[\omega(u_i, u_{i+1}) + \omega(v_j, v_{j+1})\right] + \frac{2 \gamma^3 \omega(C_r)}{1 - \gamma^{|C_r|}}
\] (2.15)
is always satisfied for any \((i, j) \in (1 \leq i \leq |C_1|, 1 \leq j \leq |C_2|)\) and \(r \in \{1, 2\} \).
Now we present and examine the algorithm CombineCycles, which is based on the minimum cost cycle covers.

Algorithm 2.1 (CombineCycles).
Input: An instance of $\triangle(\gamma)$-ATSP graph $G = (V, E; \omega)$.
Output: A directed Hamiltonian cycle $H$ of graph $G = (V, E; \omega)$.

1: Compute a minimum weight cycle cover $C$ of $G$. If $C$ has a single cycle, let $H = C$ and stop.
2: Let $C_1, C_2, \ldots, C_K$ be the cycles of $C$. Denote the vertices in $C_i$ by $v_{i1}, v_{i2}, \ldots, v_{i|C_i|}$, $v_{i(|C_i|+1)} = v_{i1}$, in the order from the cycle beginning so that the edge $(v_{i1}, v_{i2})$ has the minimal weight of all the edges of $C_i$ if $\gamma \in [\frac{\sqrt{5} - 1}{2}, 1)$ or the edge $(v_{i1}, v_{i2})$ has the maximal weight of all the edges of $C_i$ if $\gamma \in [\frac{\sqrt{5} + 1}{2}, \sqrt{5} - 1)$. Set $H_0 = C_1, C_2, \ldots, C_K$.
3: Get $H_1$ by deleting the edge $(v_{i1}, v_{i2})$ of $C_i$ of $H_0$ for $i = 1, 2, \ldots, K$.
4: Connect $(v_{i1}, v_{(i+1)2})$ of $G$ in $H_1$ for $i = 1, 2, \ldots, K$ (if $i = K$, then $i + 1 = 1$); then we obtain a directed Hamiltonian cycle $H$ and output it.

Then we can estimate the approximation algorithm performance of our algorithm CombineCycles.

Theorem 2.1. The approximation ratio of the algorithm CombineCycles is bounded by $R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma^3 + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}$.

The running time of the algorithm is cubic in the number of nodes.

Proof. Hamiltonian cycles are special cases of cycle covers; we have $\text{AP}(G) \leq \text{ATSP}(G)$. Denote the output solution of the algorithm CombineCycles and its weight by $\text{OUT}(G)$. From the algorithm we have

$$\text{OUT}(G) = \text{AP}(G) - \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \omega(v_{i1}, v_{(i+1)2}) \quad (K + 1 = 1),$$

and from Eq. (2.13) or (2.14) of Lemma 2.2 we have

$$\omega(v_{i1}, v_{(i+1)2}) \leq \gamma \omega(v_{i1}, v_{i2}) + \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}} + \gamma^2 \omega(v_{(i+1)1}, v_{(i+1)2}).$$

Then

$$\sum_{i=1}^{K} \omega(v_{i1}, v_{(i+1)2}) \leq (\gamma + \gamma^2) \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}}.$$

Thus

$$\frac{\text{OUT}(G)}{\text{ATSP}(G)} = \frac{\text{AP}(G) - \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \omega(v_{i1}, v_{(i+1)2})}{\text{ATSP}(G)} \leq \frac{\text{AP}(G) + (\gamma + \gamma^2 - 1) \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}}}{\text{ATSP}(G)}.$$

i.e.

$$\frac{\text{OUT}(G)}{\text{ATSP}(G)} \leq \frac{\text{AP}(G) + (\gamma + \gamma^2 - 1) \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}}}{\text{ATSP}(G)}.$$

For any $i \in \{1, 2, \ldots, K\}$, $|C_i| \geq 2$, $\frac{1}{1 - \gamma^{|C_i|}} \leq \frac{1}{1 - \gamma^2}$.

If $(v_{i1}, v_{i2})$ has the minimal weight of all the edges of $C_i$, then $\omega(v_{i1}, v_{i2}) \leq \frac{\omega(C_i)}{2}$, $\gamma + \gamma^2 - 1 > 0$, and we can obtain

$$\frac{\text{OUT}(G)}{\text{ATSP}(G)} \leq \frac{\text{AP}(G) + (\gamma + \gamma^2 - 1) \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}}}{\text{ATSP}(G)} \leq (1 + \frac{\gamma + \gamma^2 - 1}{2} + \frac{\gamma^3}{1 - \gamma^2}) \frac{\text{AP}(G)}{\text{ATSP}(G)} \leq \gamma + \gamma^2 + 1 + \frac{\gamma^3}{2} + \frac{\gamma^3}{1 - \gamma^2}.$$

Furthermore,

$$\frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2} > \frac{1}{1 - \gamma^2} + \frac{\gamma^3}{1 - \gamma^2}.$$

And if $(v_{i1}, v_{i2})$ has the maximal weight of all the edges of $C_i$, then $\gamma + \gamma^2 - 1 < 0$,

$$\frac{\text{OUT}(G)}{\text{ATSP}(G)} \leq \frac{\text{AP}(G) + (\gamma + \gamma^2 - 1) \sum_{i=1}^{K} \omega(v_{i1}, v_{i2}) + \sum_{i=1}^{K} \frac{\gamma^3 \omega(C_i)}{1 - \gamma^{|C_i|}}}{\text{ATSP}(G)} \leq (1 + \frac{\gamma^2}{1 - \gamma^2}) \frac{\text{AP}(G)}{\text{ATSP}(G)} = \frac{1 + \gamma^3}{1 - \gamma^2}.$$

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Furthermore, 
\[1 + \frac{\gamma^3}{1 - \gamma^2} > \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}.\]

Thus, \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is proved.

Let \(S(n)\) denote the worst case running time of the algorithm on instances with \(n\) nodes. The complexity of step 1 is dominated by the time \(O(n^3)\) used to construct the minimum cycle cover \(C\); step 2 needs \(O(n^2 \lg n)\) units of time to be completed; steps 3 and 4 could be finished in \(O(n + n)\). At last, we get that the computing complexity of the algorithm 

\[S(n) = O(n^3) + O(n^2 \lg n) + O(n + n) = O(n^3).\]

\[\square\]

**Theorem 2.1.** The approximation ratio \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{\gamma^2}{1 - \gamma^2}\) for all \(\gamma \in [0.5550, 1)\).

**Proof.** The proof of this theorem is completed by solving the inequality \(\max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\} \leq \frac{\gamma^2}{1 - \gamma^2}\) for \(\gamma \in [0.5, 1)\). \(\square\)

**Theorem 2.2.** The approximation ratio \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{1 + \gamma}{2 - \gamma - \gamma^2}\) for all \(\gamma \in [0.5846, 0.7283]\).

**Proof.** We can finish the proof of this theorem by solving the inequality \(\max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\} \leq \frac{1 + \gamma}{2 - \gamma - \gamma^2}\) for \(\gamma \in [0.5, 1)\). \(\square\)

**Remark 2.1.** The number \(\min\{|C_i| \mid i = 1, 2, \ldots, K\}\) the bigger the superiority of our algorithm the better.

For example, if \(\min\{|C_i| \mid i = 1, 2, \ldots, K\} = 3\), then \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{\gamma^2}{1 - \gamma^2}\) for all \(\gamma \in [0.5437, 1)\), \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{1 + \gamma}{2 - \gamma - \gamma^2}\) for all \(\gamma \in [0.5437, 1)\). And if \(\min\{|C_i| \mid i = 1, 2, \ldots, K\} = 4\), then \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{1 + \gamma}{2 - \gamma - \gamma^2}\) for all \(\gamma \in [0.5395, 1)\), and \(R = \max\{1 + \frac{\gamma^3}{1 - \gamma^2}, \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2}\}\) is better than \(\frac{1 + \gamma}{2 - \gamma - \gamma^2}\) for all \(\gamma \in [0.5326, 1)\).

Lastly, we can get the conclusion by solving the limiting problem which reflects the changing of the formula when \(\gamma\) is close to 1.

**Conclusion 2.1.** For \(\gamma\) close to 1,
\[
\lim_{\gamma \to 1} \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2} = \frac{1}{2}, \quad \text{and}
\lim_{\gamma \to 1} \frac{\gamma + \gamma^2 + 1}{2} + \frac{\gamma^3}{1 - \gamma^2} = \frac{2}{3}.
\]

the improvement is by a factor close to 2 and 1.5 respectively.

**References**