Integral Equation for Scattering by a Dielectric

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Abstract—**The determination of the scattered and transmitted transient electromagnetic waves produced by a uniform dielectric body is reduced to the solution of a singular integral equation of the first kind for one tangential vector field defined on the surface. All derivations are carried out within the heuristic approach to Green functions and delta functions. The electric and magnetic fields are expressed in terms of the sources, initial values, and the boundary values by means of the Green function for the scalar wave equation. The appropriate integral equation is derived, and the integrals for the scattered and transmitted fields are given. The simpler problem of scattering of scalar waves is developed first. Formulas for the scattering of monochromatic fields are also given in the scalar and electromagnetic cases when transmitted fields do not vanish.**

I. INTRODUCTION

WHEN AN ELECTROMAGNETIC wave is scattered by a perfect conductor, the scattered field can be determined by integration when the surface current density induced on the surface of the conductor is known. This surface current density can be determined from the incident field by solving either the magnetic field integral equation (MFIE) or electric field integral equation (EFIE), which are consequences of the boundary conditions at the conducting surface. If the perfect conductor is replaced by a dielectric, there is a nonvanishing transmitted field and the boundary conditions are more complicated. The procedure followed for the perfect conductor can be generalized by defining electric and magnetic current densities on the surface [1], [2], thus doubling the number of unknown functions.

A similar problem occurs in the scattering of electromagnetic waves by gratings, where a formalism was developed to generalize the theory from perfect conductors to dielectrics or gratings of finite conductivity without increasing the number of unknown functions [3]. This approach has been used successfully for numerical calculations in the scattering of monochromatic plane waves by a metallic grating. The problem is essentially reduced to the solution of a one-dimensional integral equation for a scalar function over a finite interval.

We generalize this procedure to the scattering of an arbitrary electromagnetic wave by a given uniform dielectric body. In this case, we reduce the problem to a two-dimensional vector integral equation for a single tangential vector field over the surface of the three-dimensional body.

We have previously developed a mathematically rigorous treatment of this problem within the context of the theory of distributions [4]; in this paper we show how the equations can be obtained from the heuristic approach more familiar to engineers and physicists, where delta functions and Green functions are used in the appropriate versions of Green's theorem as if they were ordinary functions.

We first develop the formalism for transient fields. The resulting integral equations can be solved by a stepping-in-time procedure, which takes advantage of the physical principle of causality. In Section II we consider the problem for a scalar field, because it is simpler than the electromagnetic field and it is often used in approximate theories. An incident wave is scattered by an arbitrary body of uniform composition, and we reduce the determination of the scattered and transmitted waves to the solution of one singular integral equation of the first kind for a function defined on the surface of the body.

We then extend this procedure to the scattering of electromagnetic waves by a dielectric body, and in Section III we define a tangential vector field on the surface that obeys a singular vector integral equation of the first kind.

In Section IV we present the minor modifications that are required to apply the equations to monochromatic fields. For these fields, a simple generalization allows us to include dispersive media with a finite conductivity.

This approach should facilitate numerical calculations for three-dimensional scattering problems.

II. SCALAR WAVE SCATTERING

We first develop the new method to reduce the determination of scattered and transmitted scalar waves to the solution of a single integral equation on the surface separating two homogeneous media. Space is divided into two regions V_1 and V_2, separated by a surface S, and the media are characterized by the speed of propagation of a scalar wave. The incident wave is specified at the initial time $t = 0$ in the region V_1.

The field ψ satisfies the equations

$$\Box_i \psi(\vec{x}, t) = 0, \quad \vec{x} \in V_i, \quad i = 1, 2, \tag{1}$$

$$\Box = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{2}$$

and the boundary conditions

$$\psi_+(\vec{x}, t) = \psi_-(\vec{x}, t), \qquad \vec{x} \in S, \tag{3}$$

$$\left[\frac{\partial \psi(\vec{x}, t)}{\partial n}\right]_+ = \alpha \left[\frac{\partial \psi(\vec{x}, t)}{\partial n}\right]_-, \qquad \vec{x} \in S, \tag{4}$$

where the subindex on a d'Alembertian operator refers to the speed of propagation, α has a constant value, the normal $\hat{n}$ points into V_2, and the subscripts plus and minus indicate the limits as $\vec{x}$ approaches S from V_2 and V_1, respectively.

The elementary solution $G(\vec{x}, t)$ satisfies the scalar wave equation with a Dirac delta function source,

$$\Box G(\vec{x}, t) = \delta^{(3)}(\vec{x})\delta(t), \tag{5}$$

and vanishes for $t < 0$. We know that the solution is

$$G(\vec{x}, t) = \delta(t - r/v)/(4\pi r), \tag{6}$$

where $r = |\vec{x}|$. Then the free-space retarded Green function is

$$G_R^{(0)}(\vec{x}, t; \vec{x}', t') = G(\vec{x} - \vec{x}', t - t'). \tag{7}$$

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We obtain a form of Kirchoff's integral representation of a function ψ defined in the volume V and bounded by a surface S with sources $\rho(\vec{x}, t)$ and given initial and boundary values by substituting ψ and $G_R^{(0)}$ in the appropriate form of Green's theorem. We get

$$\psi(\vec{x}, t) = \int_0^t dt' \int_V dV' \rho(\vec{x}', t') G_R^{(0)}(\vec{x}, t; \vec{x}', t') - \frac{1}{v^2} \int_V dV' \left[\psi(\vec{x}', 0) \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} - \dot{\psi}(\vec{x}', 0) G_R^{(0)}(\vec{x}, t; \vec{x}', 0) \right] - \int_0^t dt' \oint_S dS' \left[\psi(\vec{x}', t') \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', t')}{\partial n'} - \frac{\partial \psi(\vec{x}', t')}{\partial n'} G_R^{(0)}(\vec{x}, t; \vec{x}', t') \right]. \tag{8}$$

Since either ψ or $\partial\psi/\partial n$ is given on S, this equation does not give a solution for ψ until we determine the function that is not given. If we let $\vec{x}$ approach the surface from the outside, the left side of (8) vanishes, and we obtain an integral equation for the unknown function on the surface. Once this function is determined, (8) gives ψ inside V for positive t; ψ vanishes for values of $\vec{x}$ outside V or for negative times.

If, on the other hand, two arbitrary functions are given for ψ and $\partial\psi/\partial n$ on S, $\psi(\vec{x}, t)$ as given by (8) is discontinuous across S, but does not have to vanish outside V. The given functions are equal to the jumps of ψ and $\partial\psi/\partial n$ across S.

The field in V_1 is separated into the incident and scattered parts,

$$\psi = \psi^{\text{in}} + \psi^{\text{sc}}, \quad \vec{x} \in V_1, \tag{9}$$

and the incident field is determined from the initial conditions as if S and V_2 did not exist. We find from (8) that

$$\psi^{\text{in}}(\vec{x}, t) = -\frac{1}{v_1^2} \int_{V_1} dV' \left[\psi(\vec{x}', 0) \frac{\partial G_{R1}^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} - \dot{\psi}(\vec{x}', 0) G_{R1}^{(0)}(\vec{x}, t; \vec{x}', 0) \right]. \tag{10}$$

If either ψ or $\partial\psi/\partial n$ vanishes on S, the transmitted field vanishes and the scattered field can be computed from a single function on S that obeys an integral equation. If the transmitted field does not vanish, it would seem that we have to determine two unknown functions on S. Following [3], we show how to define a single function on S that obeys an integral equation and provides both the scattered and transmitted fields by integration.

We first study a generic field χ that obeys the same wave equation in V_1 and V_2 and has given jumps for χ and $\partial\chi/\partial n$ on the surface S. We then define two auxiliary fields ψ_1 and ψ_2, express them in terms of a single function η on S, and proceed to derive an integral equation for η. Once this equation is solved, ψ can be found from η by integrations.

We use (8) to find a field $\chi(\vec{x}, t)$ that obeys the homogeneous wave equation in all space, satisfies homogeneous initial conditions and has given jumps $\phi = \Delta\chi$ and $\eta = \Delta(\partial\chi/\partial n)$ for positive times on S. We obtain

$$\chi(\vec{x}, t) = \int_0^t dt' \oint_S dS' \left[\phi(\vec{x}', t') \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', t')}{\partial n'} - \eta(\vec{x}', t') G_R^{(0)}(\vec{x}, t; \vec{x}', t') \right] \tag{11}$$

(the boundary values in (8) are the negative of the jumps). We use (6) and

$$\nabla G(\vec{x}, t) = -\delta\left(t - \frac{r}{v}\right) \frac{\vec{x}}{4\pi r^3} - \delta'\left(t - \frac{r}{v}\right) \frac{\vec{x}}{4\pi v r^2} \tag{12}$$

to reduce (11) to

$$\chi(\vec{x}, t) = \oint_S dS' \left[\left(\frac{\phi_{\text{ret}}}{R^3} + \frac{\dot{\phi}_{\text{ret}}}{vR^2} \right) \frac{\hat{n}' \cdot \vec{R}}{4\pi} - \frac{\eta_{\text{ret}}}{4\pi R} \right], \tag{13}$$

where we use the notation

$$f_{\text{ret}}(\vec{x}', t') = f(\vec{x}', \tau) \tag{14}$$

for function of the retarded time $\tau = t - R/v$, where $\vec{R} = \vec{x} - \vec{x}'$, $R = |\vec{R}|$. This expression defines χ when $\vec{x}$ is not on S; we extend this definition to S by choosing the principal value when an integral is singular. We also introduce the notation

$$\chi(\vec{x}, t) = G\{\eta\} + N\{\phi\}, \tag{15}$$

where G and N are functionals defined by

$$G\{\eta\} = -\oint_S dS' \frac{\eta_{\text{ret}}}{4\pi R}, \tag{16}$$

$$N\{\phi\} = \oint_S d\vec{S}' \cdot \frac{\vec{R}}{4\pi v R^2} \dot{\phi}_{\text{ret}} + \mathrm{P} \oint_S d\vec{S}' \cdot \frac{\vec{R}}{4\pi R^3} \phi_{\text{ret}}. \tag{17}$$

The functionals defined in (16) and (17) produce functions of $\vec{x}$ and t when functions $\eta(\vec{x}, t)$ and $\phi(\vec{x}, t)$ are given for $\vec{x}$ on the surface S and for all t. Since $d\vec{S}$ contains a factor of the order of R, the two integrals that are not preceded by P are not singular. The time derivative in (17) can also be moved outside the integral if this facilitates numerical computations.

The values of singular integrals can be determined by separating the surface into a small patch S_1 about the limiting field point $\vec{x}_0$ and the remainder $\bar{S}$. We follow Van Bladel [6] and let $\vec{x}$ approach $\vec{x}_0$ along the normal $\hat{n}$; we set

$$\vec{x} = \vec{x}_0 + h\hat{n} \tag{18}$$

and take the limit $h \to 0$. Then we take the limit as the size of the patch, characterized by a parameter a, tends to zero. On the patch S_1, the slowly varying functions η and ϕ can be approximated by their constant values at $\vec{x}_0$, which are multiplied by integrals that can be evaluated in this double limit; we have

$$\lim_{a \to 0} \lim_{h \to 0\pm} \int_{S_1} dS' \frac{1}{R} = 0, \tag{19}$$

$$\lim_{a\to 0}\lim_{h\to 0\pm}\int_{S_1} d\vec{S}'\cdot\frac{\vec{R}}{R^3}=\pm 2\pi, \tag{20}$$

$$\lim_{a\to 0}\lim_{h\to 0\pm}\int_{S_1} d\vec{S}'\cdot\frac{\vec{R}}{R^2}=0, \tag{21}$$

to show that

$$\chi_\pm(\vec{x},t)=\pm\tfrac{1}{2}\,\phi(\vec{x},t)+\chi(\vec{x},t),\quad \vec{x}\in S. \tag{22}$$

The first term in the limit implied in $\chi_\pm$ comes from the contribution of the patch S_1 and the second term corresponds to the integral over $\bar{S}$. We compute $\nabla\chi$ in the special case when $\phi = 0$ and find

$$\nabla\chi(\vec{x},t)\,|_{\phi=0} = \mathrm{P}\oint_S dS'\left(\frac{\vec{R}}{4\pi v R^2}\,\dot{\eta}_{\text{ret}}+\frac{\vec{R}}{4\pi R^3}\,\eta_{\text{ret}}\right), \tag{23}$$

whence

$$\left(\frac{\partial\chi}{\partial n}\right)_\pm=\pm\tfrac{1}{2}\,\eta+\frac{\partial\chi}{\partial n},\qquad \vec{x}\in S, \tag{24}$$

for $\phi = 0$, where we used

$$\lim_{a\to 0}\lim_{h\to 0\pm}\int_{S_1} dS'\,\frac{\vec{R}}{R^3}=\pm 2\pi\hat{n}, \tag{25}$$

$$\lim_{a\to 0}\lim_{h\to 0\pm}\int_{S_1} dS'\,\frac{\vec{R}}{R^2}=0. \tag{26}$$

If ϕ does not vanish, the expression for $\nabla\chi$ contains integrals that diverge when the field point is on S [4]. We define the functional N' by setting

$$N'\{\eta\}=\hat{n}\cdot\left(\oint_S dS'\,\frac{\vec{R}}{4\pi v R^2}\,\dot{\eta}_{\text{ret}}+\mathrm{P}\oint_S dS'\,\frac{\vec{R}}{4\pi R^3}\,\eta_{\text{ret}}\right), \tag{27}$$

so that

$$\left(\frac{\partial\chi}{\partial n}\right)_{\phi=0}=N'\{\eta\}. \tag{28}$$

We now define two auxiliary fields ψ_1 and ψ_2 by

$$\psi_1=\psi^{\text{sc}},\quad \vec{x}\in V_1, \tag{29}$$

$$\Box_1\psi_1=0,\quad \vec{x}\in V_2, \tag{30}$$

$$\psi_{1+}=\psi_{1-}=\psi^{\text{sc}},\qquad \vec{x}\in S, \tag{31}$$

$$\Delta\left(\frac{\partial\psi_1}{\partial n}\right)=\eta,\qquad \vec{x}\in S, \tag{32}$$

$$\psi_2=0,\qquad \vec{x}\in V_1, \tag{33}$$

$$\psi_2=\psi=\psi^{\text{tr}},\qquad \vec{x}\in V_2. \tag{34}$$

Both fields vanish initially, and each obeys the same wave equation in regions V_1 and V_2. We proceed to show that both ψ_1 and ψ_2 can be obtained from η by integrations, and find the integral equation η obeys.

Since ψ_1 is continuous across S, (15) reduces to

$$\psi_1=G_1\{\eta\} \tag{35}$$

in a space filled with a medium where waves propagate with speed v_1. The jumps in ψ_2 and $\partial\psi_2/\partial n$ are determined by the boundary conditions (3) and (4); they are

$$\Delta\psi_2=\psi_{2+}=\psi_+=\psi_-=\psi^{\text{in}}+\psi_{1-}, \tag{36}$$

$$\Delta\left(\frac{\partial\psi_2}{\partial n}\right)=\left(\frac{\partial\psi_2}{\partial n}\right)_+=\left(\frac{\partial\psi}{\partial n}\right)_+=\alpha\left(\frac{\partial\psi}{\partial n}\right)_-=\alpha\frac{\partial\psi^{\text{in}}}{\partial n}+\alpha\left(\frac{\partial\psi_1}{\partial n}\right)_-. \tag{37}$$

To express these jumps in terms of η we use (35) for ψ_1 and (24) and (28) applied to $\partial\psi_1/\partial n$ to obtain

$$\Delta\psi_2=\psi^{\text{in}}+G_1\{\eta\}, \tag{38}$$

$$\Delta\left(\frac{\partial\psi_2}{\partial n}\right)=\alpha\frac{\partial\psi^{\text{in}}}{\partial n}-\tfrac{1}{2}\,\alpha\eta+\alpha N_1'\{\eta\}, \tag{39}$$

which substituted into (15) for a medium where waves propagate with speed v_2 give

$$\psi_2=\alpha G_2\left\{\frac{\partial\psi^{\text{in}}}{\partial n}-\tfrac{1}{2}\,\eta+N_1'\{\eta\}\right\}+N_2\{\psi^{\text{in}}+G_1\{\eta\}\}. \tag{40}$$

This function does not vanish in V_1 for arbitrary η; we impose this requirement by setting

$$\psi_{2-}=0, \tag{41}$$

which by (22) and (35) gives the singular integral equation of the first kind

$$[\tfrac{1}{2}G_1+N_2G_1+\alpha(\tfrac{1}{2}G_2-G_2N_1')]\{\eta\}+\tfrac{1}{2}\,\psi^{\text{in}}-N_2\{\psi^{\text{in}}\}-\alpha G_2\{\partial\psi^{\text{in}}/\partial n\}=0, \tag{42}$$

where we use the notation

$$N_2G_1\{\eta\}=N_2\{G_1\{\eta\}\}. \tag{43}$$

The functional N_2G_1 is well defined, since given η on S we can compute $G_1\{\eta\}$ in all space, and, in particular, on S. This function then serves as the argument of the functional N_2.

Once η is determined, ψ can be found by integrations as shown by (35) and (40), and we write

$$\psi(\vec{x},t)=\psi^{\text{in}}(\vec{x},t)+G_1\{\eta\},\qquad \vec{x}\in V_1, \tag{44}$$

$$\psi(\vec{x},t)=N_2\{\psi^{\text{in}}\}+\alpha G_2\{\partial\psi^{\text{in}}/\partial n\}+[N_2G_1+\alpha(-\tfrac{1}{2}G_1-G_2N_1')]\{\eta\},\qquad \vec{x}\in V_2. \tag{45}$$

It is now fairly straightforward to show [4] that ψ satisfies all required conditions.

III. SCATTERING BY DIELECTRICS

We now follow a similar procedure to determine the scattered and transmitted electromagnetic fields that result from the interaction of an incident field in a region V_1 of permittivity ϵ_1 and permeability μ_1 with a dielectric body of volume V_2 and constants ϵ_2 and μ_2.

We first express the electromagnetic field in terms of its sources, initial values, and boundary values. The field obeys Maxwell's equations in a region V filled with a homogeneous dielectric of constants ϵ and μ,

$$\nabla \cdot \vec{E} = \rho/\epsilon, \tag{46}$$

$$\nabla \cdot \vec{B} = 0, \tag{47}$$

$$\nabla \times \vec{E} = -\partial\vec{B}/\partial t, \tag{48}$$

$$\nabla \times \vec{B} = \mu\vec{j} + \epsilon\mu\partial\vec{E}/\partial t, \tag{49}$$

where ρ is the charge density and $\vec{j}$ the current density. These sources satisfy the charge conservation equation

$$\nabla \cdot \vec{j} + \dot{\rho} = 0, \tag{50}$$

so that ρ has to be given only at the initial time. From (48) and (49) we derive the relationships

$$\frac{\partial(\hat{n} \cdot \vec{B})}{\partial t} = \nabla_s \cdot (\hat{n} \times \vec{E}), \tag{51}$$

$$\frac{\partial(\hat{n} \cdot \vec{E})}{\partial t} = -\frac{1}{\epsilon\mu} \nabla_s \cdot (\hat{n} \times \vec{B}) - \frac{1}{\epsilon} \hat{n} \cdot \vec{j}, \tag{52}$$

which allow us to calculate the normal components of $\vec{B}$ and $\vec{E}$ in terms of the tangential components of $\vec{E}$ and $\vec{B}$, and the initial values of the normal components. The expression $\nabla_s \cdot \vec{u}$ stands for the surface divergence of a vector field $\vec{u}$ defined on a surface.

We can use the Green function for the scalar wave equation with a speed of propagation $v = (\epsilon\mu)^{-1/2}$ to obtain [5], [7]

$$\begin{aligned}
&\vec{E}(\vec{x}, t) \\
&= \int_0^t dt' \int_V dV' \left[\mu\vec{j}(\vec{x}', t') \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', t')}{\partial t'} \right. \\
&\quad \left. + (1/\epsilon)\rho(\vec{x}', t')\nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \right] \\
&\quad - \int_V dV' \left[\frac{1}{v^2} \vec{E}(\vec{x}', 0) \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} \right. \\
&\quad - \vec{B}(\vec{x}', 0) \times \nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', 0)] \\
&\quad - \int_0^t dt' \oint_S dS' \left[\{\hat{n}' \times \vec{E}(\vec{x}', t')\} \times \nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \right. \\
&\quad + \hat{n}' \cdot \vec{E}(\vec{x}', t')\nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \\
&\quad \left. + \hat{n}' \times \vec{B}(\vec{x}', t') \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', t')}{\partial t'} \right],
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\vec{B}(\vec{x}, t) \\
&= \mu \int_0^t dt' \int_V dV' \vec{j}(\vec{x}', t') \times \nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \\
&\quad - \frac{1}{v^2} \int_V dV' \left[\vec{E}(\vec{x}', 0) \times \nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', 0) \right. \\
&\quad \left. + \vec{B}(\vec{x}', 0) \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} \right] \\
&\quad - \int_0^t dt' \oint_S dS' \left[\{\hat{n}' \times \vec{B}(\vec{x}', t')\} \times \nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \right. \\
&\quad + \hat{n}' \cdot \vec{B}(\vec{x}', t')\nabla' G_R^{(0)}(\vec{x}, t; \vec{x}', t') \\
&\quad \left. - \frac{1}{v^2} \hat{n}' \times \vec{E}(\vec{x}', t') \frac{\partial G_R^{(0)}(\vec{x}, t; \vec{x}', t')}{\partial t'} \right].
\end{aligned} \tag{54}$$

To determine the fields in a region V, it is sufficient to give the sources $\vec{j}$ for all times and ρ only at the initial time, the initial values of $\vec{E}$ and $\vec{B}$ subject to the constraints (46) and (47), and the tangential component of either $\vec{E}$ or $\vec{B}$ on the surface S. The other tangential component can be determined from (53) or (54) used as an integral equation, plus the relations (51) and (52) for the normal components. The initial values of the normal components are determined from the initial values of the fields.

When both tangential fields are given arbitrarily, the fields $\vec{E}$ and $\vec{B}$ determined by (53) and (54) do not vanish outside V, and the given tangential fields are not the values of the tangential components of $\vec{E}$ and $\vec{B}$ on S but their jumps across the surface.

In the scattering problem, the total fields $\vec{E}$ and $\vec{B}$ obey the homogeneous Maxwell equations in the regions V_1 and V_2, that is, we set $\rho = 0$ in (46) and rewrite (49) as

$$\nabla \times \vec{B} - (1/v_i^2)\partial\vec{E}/\partial t = 0 \quad \vec{x} \in V_i, i = 1, 2. \tag{55}$$

The boundary conditions that have to be satisfied are

$$\hat{n} \times \vec{E}_+ = \hat{n} \times \vec{E}_-, \qquad \vec{x} \in S, \tag{56}$$

$$\hat{n} \times \vec{B}_+ = \alpha\hat{n} \times \vec{B}_-, \qquad \vec{x} \in S, \tag{57}$$

where $\alpha = \mu_2/\mu_1$.

The fields in the region V_1 are separated into incident and scattered fields,

$$\vec{E}(\vec{x}, t) = \vec{E}^{\text{in}}(\vec{x}, t) + \vec{E}^{\text{sc}}(\vec{x}, t), \tag{58}$$

$$\vec{B}(\vec{x}, t) = \vec{B}^{\text{in}}(\vec{x}, t) + \vec{B}^{\text{sc}}(\vec{x}, t), \tag{59}$$

where the incident fields by definition obey the homogeneous Maxwell equations and propagate in a homogeneous space, thus satisfying no boundary conditions on S. We assume that the incident fields are restricted initially to the region V_1 so that the scattered fields vanish at $t = 0$ and the incident fields satisfy the initial conditions. Equations (53) and (54) reduce to

$$\begin{aligned}
&\vec{E}^{\text{in}}(\vec{x}, t) \\
&= -\int_{V_1} dV' \left[\frac{1}{v_1^2} \vec{E}(\vec{x}', 0) \frac{\partial G_{R_1}^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} \right. \\
&\quad \left. - \vec{B}(\vec{x}', 0) \times \nabla' G_{R_1}^{(0)}(\vec{x}, t; \vec{x}', 0) \right],
\end{aligned} \tag{60}$$

$$\vec{B}^{\text{in}}(\vec{x}, t) = -\frac{1}{v_1^2} \int_{V_1} dV' \left[\vec{E}(\vec{x}', 0) \times \nabla' G_{R_1}^{(0)}(\vec{x}, t; \vec{x}', 0) + \vec{B}(\vec{x}', 0) \frac{\partial G_{R_1}^{(0)}(\vec{x}, t; \vec{x}', 0)}{\partial t'} \right]. \tag{61}$$

We now consider generic fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ that obey the homogeneous Maxwell equations with the same constants in all space, vanish at $t = 0$, and satisfy jump conditions $\Delta\vec{\mathcal{E}} = \vec{\phi}$ and $\Delta\vec{\mathcal{B}} = \vec{\eta}$ on S. The fields are given by the appropriate terms in (53) and (54),

$$\vec{\mathcal{E}} = \int_0^t dt' \oint_S dS' [(\hat{n}' \times \vec{\phi}) \times \nabla' G_R^{(0)} + \hat{n}' \cdot \vec{\phi} \nabla' G_R^{(0)} + \hat{n}' \times \vec{\eta} \partial G_R^{(0)}/\partial t'], \tag{62}$$

$$\vec{\mathcal{B}} = \int_0^t dt' \oint_S dS' [(\hat{n}' \times \vec{\eta}) \times \nabla' G_R^{(0)} + \hat{n}' \cdot \vec{\eta} \nabla' G_R^{(0)} - v^{-2} \hat{n}' \times \vec{\phi} \partial G_R^{(0)}/\partial t']. \tag{63}$$

Equations (51) and (52) imply that the normal components of the jumps are related to the tangential components by

$$\frac{\partial(\hat{n} \cdot \vec{\eta})}{\partial t} = \nabla_s \cdot (\hat{n} \times \vec{\phi}), \tag{64}$$

$$\frac{\partial(\hat{n} \cdot \vec{\phi})}{\partial t} = -v^2 \nabla_s \cdot (\hat{n} \times \vec{\eta}). \tag{65}$$

Since $\vec{\phi}$ and $\vec{\eta}$ vanish at $t = 0$, these equations determine $\hat{n} \cdot \vec{\eta}$ and $\hat{n} \cdot \vec{\phi}$ in terms of $\hat{n} \times \vec{\phi}$ and $\hat{n} \times \vec{\eta}$, respectively.

We substitute $G_R^{(0)}$ from (6), (7), and (12) into (62) and (63) and integrate over time to obtain expressions of the form

$$\vec{\mathcal{E}} = \vec{L}\{\hat{n} \times \vec{\phi}\} + \vec{M}\{\hat{n} \times \vec{\eta}\}, \tag{66}$$

$$\vec{\mathcal{B}} = \vec{L}\{\hat{n} \times \vec{\eta}\} + \vec{M}'\{\hat{n} \times \vec{\phi}\}, \tag{67}$$

where the functionals $\vec{L}$, $\vec{M}$, and $\vec{M}'$ are defined by

$$\vec{L}\{\hat{n} \times \vec{\phi}\} = \mathrm{P} \oint_S dS' \left[\hat{n}' \times \left(\frac{1}{v} \left(\frac{\partial \vec{\phi}}{\partial t'} \right)_{\text{ret}} + \frac{\vec{\phi}_{\text{ret}}}{R} \right) \right] \times \frac{\vec{R}}{4\pi R^2}, \tag{68}$$

$$\vec{M}\{\hat{n} \times \vec{\eta}\} = P \oint_S dS' \left[\hat{n}' \cdot \left(\frac{1}{v} \left(\frac{\partial \vec{\phi}}{\partial t'} \right)_{\text{ret}} + \frac{\vec{\phi}_{\text{ret}}}{R} \right) \frac{\vec{R}}{4\pi R^2} - \hat{n}' \times \left(\frac{\partial \vec{\eta}}{\partial t'} \right)_{\text{ret}} \frac{1}{4\pi R} \right], \tag{69}$$

$$M'\{\hat{n} \times \vec{\phi}\} = \mathrm{P} \oint_S dS' \left[\hat{n}' \cdot \left(\frac{1}{v} \left(\frac{\partial \vec{\eta}}{\partial t'} \right)_{\text{ret}} + \frac{\vec{\eta}_{\text{ret}}}{R} \right) \frac{\vec{R}}{4\pi R^2} + \hat{n}' \times \left(\frac{\partial \vec{\phi}}{\partial t} \right)_{\text{ret}} \frac{1}{4\pi v^2 R} \right], \tag{70}$$

where $n \cdot \vec{\eta}$ and $\hat{n} \cdot \vec{\phi}$ are functionals of $\hat{n} \times \vec{\phi}$ and $\hat{n} \times \vec{\eta}$ via (64) and (65). The fields depend on v both explicitly and through retardation effects. Also in this case the time derivatives can be moved outside the integrals which do not become singular on the surface S.

We let again the field point approach S from either side, and the resulting limiting values of the fields are

$$\vec{\mathcal{E}}_\pm = \pm\tfrac{1}{2}\,\vec{\phi} + \vec{\mathcal{E}}, \tag{71}$$

$$\vec{\mathcal{B}}_\pm = \pm\tfrac{1}{2}\,\vec{\eta} + \vec{\mathcal{B}}. \tag{72}$$

We proceed to define two sets of auxiliary fields. The fields $\vec{E}_1$ and $\vec{B}_1$ propagate in a medium of constants ϵ_1 and μ_1, they are equal to the scattered field in V_1, and the tangential component of $\vec{E}_1$ is continuous across S. We set $\hat{n} \times \vec{\phi} = 0$ in (66) and (67), and we obtain

$$\vec{E}_1 = \vec{M}_1\{\hat{n} \times \vec{\eta}\} \tag{73}$$

$$\vec{B}_1 = \vec{L}_1\{\hat{n} \times \vec{\eta}\}. \tag{74}$$

The tangential vector field $\hat{n} \times \vec{\eta}$ is the unknown field for which we have to find an integral equation. The field $\hat{n} \times \vec{\eta}$ is analogous to the electric and magnetic surface current densities of the equivalence principle [2], but it does not have a particular physical meaning and is just a convenient intermediate variable in the computation of the electromagnetic fields. The fields $\vec{E}_2$ and $\vec{B}_2$ are equal to the transmitted fields in V_2, and they vanish in V_1. The discontinuities of the tangential components of $\vec{E}_2$ and $\vec{B}_2$ across S are determined by the boundary conditions (56) and (57), and they are

$$\hat{n} \times \Delta\vec{E}_2 = \hat{n} \times \vec{E}_{2+} = \hat{n} \times \vec{E}_+ = \hat{n} \times \vec{E}_- = \hat{n} \times \vec{E}^{\text{in}} + \hat{n} \times \vec{E}_{1-}, \tag{75}$$

$$\hat{n} \times \Delta\vec{B}_2 = \hat{n} \times \vec{B}_{2+} = \hat{n} \times \vec{B}_+ = \alpha\hat{n} \times \vec{B}_- = \alpha\hat{n} \times \vec{B}^{\text{in}} + \alpha\hat{n} \times \vec{B}_{1-}. \tag{76}$$

We substitute $\vec{E}_{1-}$ and $\vec{B}_{1-}$ from (71) to (74), and we find

$$\hat{n} \times \Delta\vec{E}_2 = \hat{n} \times \vec{E}^{\text{in}} + \hat{n} \times \vec{M}_1\{\hat{n} \times \vec{\eta}\}, \tag{77}$$

$$\hat{n} \times \Delta\vec{B}_2 = \alpha\hat{n} \times \vec{B}^{\text{in}} - \tfrac{1}{2}\,\alpha\hat{n} \times \vec{\eta} + \alpha\hat{n} \times \vec{L}_1\{\hat{n} \times \vec{\eta}\}; \tag{78}$$

these equations express the jumps of the tangential components of $\vec{E}_2$ and $\vec{B}_2$ in terms of $\hat{n} \times \vec{\eta}$. We use these jumps in the equations for $\vec{E}_2$ and $\vec{B}_2$ obtained from (66) and (67), namely

$$\vec{E}_2 = \vec{L}_2\{\hat{n} \times \Delta\vec{E}_2\} + \vec{M}_2\{\hat{n} \times \Delta\vec{B}_2\}, \tag{79}$$

$$\vec{B}_2 = \vec{L}_2\{\hat{n} \times \Delta\vec{B}_2\} + \vec{M}_2'\{\hat{n} \times \Delta\vec{E}_2\}. \tag{80}$$

We now have to find a field $\hat{n} \times \vec{\eta}$ that will make $\vec{E}_2$ and $\vec{B}_2$ vanish in V_1. We impose the condition

$$\hat{n} \times \vec{E}_{2-} = 0, \tag{81}$$

which leads to the integral equation

$$-\tfrac{1}{2}\hat{n} \times \Delta\vec{E}_2 + \hat{n} \times \vec{L}_2\{\hat{n} \times \Delta\vec{E}_2\} + \hat{n} \times \vec{M}_2\{\hat{n} \times \Delta\vec{B}_2\} = 0. \tag{82}$$

We make the dependence on $\hat{n} \times \vec{\eta}$ explicit and obtain

$$\hat{n} \times [\tfrac{1}{2}\,\vec{M}_1 - \vec{L}_2\vec{M}_1 + \alpha(\tfrac{1}{2}\vec{M}_2 - \vec{M}_2\vec{L}_1)]\{\hat{n} \times \vec{\eta}\} + \hat{n} \times (\tfrac{1}{2}\,\vec{E}^{\text{in}} - \vec{L}_2\{\hat{n} \times \vec{E}^{\text{in}}\} - \alpha\vec{M}_2\{\hat{n} \times \vec{B}^{\text{in}}\}) = 0, \tag{83}$$

where the composite operators are defined as in

$$\hat{n} \times \vec{L}_2 \vec{M}_1 \{\hat{n} \times \vec{\eta}\} = \hat{n} \times \vec{L}_2 \{\hat{n} \times \vec{M}_1 \{\hat{n} \times \vec{\eta}\}\}. \tag{84}$$

We note that given $\hat{n} \times \vec{\eta}$ on S, $\vec{M}_1\{\hat{n} \times \vec{\eta}\}$ is determined everywhere, and, in particular, on S, which in turn provides the argument of $\vec{L}_2$.

Once $\hat{n} \times \vec{\eta}$ is determined, the fields $\vec{E}$ and $\vec{B}$ are given by

$$\vec{E} = \vec{E}^{\text{in}} + \vec{M}_1\{\hat{n} \times \vec{\eta}\}, \qquad \vec{x} \in V_1, \tag{85}$$

$$\vec{B} = \vec{B}^{\text{in}} + \vec{L}_1\{\hat{n} \times \vec{\eta}\}, \qquad \vec{x} \in V_1, \tag{86}$$

$$\vec{E} = \vec{L}_2\{\hat{n} \times \vec{E}^{\text{in}}\} + \alpha \vec{M}_2\{\hat{n} \times \vec{B}^{\text{in}}\} + (\vec{L}_2 \vec{M}_1 - \tfrac{1}{2}\alpha \vec{M}_2 + \alpha \vec{M}_2 \vec{L}_1)\{\hat{n} \times \vec{\eta}\}, \qquad \vec{x} \in V_2, \tag{87}$$

$$\vec{B} = \vec{M}'_2\{n \times \vec{E}^{\text{in}}\} + \alpha \vec{L}_2\{\hat{n} \times \vec{B}^{\text{in}}\} + (\vec{M}'_2 \vec{M}_1 - \tfrac{1}{2}\alpha \vec{L}_2 + \alpha \vec{L}_2 \vec{L}_1)\{\hat{n} \times \vec{\eta}\}, \qquad \vec{x} \in V_2. \tag{88}$$

Again we can prove [4] that these fields satisfy all the imposed conditions. We first have to show that (81) implies that both $\vec{E}_2$ and $\vec{B}_2$ vanish in V_1, and then verify that the boundary conditions are actually satisfied.

IV. MONOCHROMATIC WAVES

The formalism developed in the previous sections can be applied almost unchanged to monochromatic fields, since we are dealing mostly with boundary conditions on the surface S.

In this section we give new meanings to many of the symbols used previously; we do not add a subindex ω as is sometimes done.

The scalar field is represented by a complex function $\psi(\vec{x})$, and the monochromatic field is then Re $[\psi(\vec{x})e^{-i\omega t}]$. The wave equation becomes the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\vec{x}) = \alpha(\vec{x}), \tag{89}$$

where the wavenumber is defined by $k^2 = \omega^2/v^2$.

Causality for transient waves is replaced by the outgoing wave condition, expressed by

$$\lim_{r \to \infty} r\left(\frac{\partial \psi}{\partial r} - ik\psi\right) = 0, \tag{90}$$

where r is the distance to a fixed origin. The elementary solution of the Helmholtz equation that satisfies this condition is

$$G(\vec{x}) = \frac{e^{ikr}}{4\pi r}, \tag{91}$$

and examination of the phase $kr - \omega t$ shows that constant phase spheres move away from the origin.

The determination of the scattered and transmitted scalar fields is again reduced to a single integral equation for an unknown function η on S. The field obeys the Helmholtz equation with speed v_1 in V_1 and with speed v_2 in V_2, boundary conditions of the form (3) and (4), and is related to a given incident field ψ^{in} by (9). The scattered field satisfies the outgoing wave condition, and so does the transmitted field when the region V_2 extends to infinity.

Fields ψ_1 and ψ_2 can be defined by (29) and (34) with only minor modifications, and the discontinuity η of $\partial\psi_1/\partial_n$ across S satisfies (42) with new definitions of the operators G, N and N', namely

$$G\{\eta\} = -\oint_S d\vec{S}'\,\eta(\vec{x}')\frac{e^{ikR}}{4\pi R}, \tag{92}$$

$$N\{\phi\} = \mathrm{P}\oint_S d\vec{S}' \cdot \phi(\vec{x}')\frac{e^{ikR}(1 - ikR)\vec{R}}{4\pi R^3}, \tag{93}$$

$$N'\{\eta\} = \hat{n} \cdot \mathrm{P}\oint_S dS'\,\eta(\vec{x}')\frac{e^{ikR}(1 - ikR)\vec{R}}{4\pi R^3}. \tag{94}$$

Then the field ψ can be obtained from η through (44) and (45).

Similarly, for monochromatic electromagnetic fields, Maxwell's equations of motion become

$$\nabla \times \vec{E} - i\omega\vec{B} = 0, \tag{95}$$

$$\nabla \times \vec{B} + i\omega v^{-2}\vec{E} = \mu\vec{j}, \tag{96}$$

which reduce to the vector Helmholtz equation

$$-\nabla \times (\nabla \times \vec{E}) + \omega^2 v^{-2}\vec{E} = -i\omega\mu\vec{j}. \tag{97}$$

In most media, μ is very close to the permeability of free space μ_0. The permittivity ϵ can often be approximated by a constant, although real media are dispersive, that is, ϵ is a complex function of ω. For a conducting medium we can approximately set

$$\vec{j} = \sigma(\omega)\vec{E}, \tag{98}$$

so that (96) becomes

$$i\omega\nabla \times \vec{B} - k^2\vec{E} = 0, \tag{99}$$

where k is the complex propagation constant defined by

$$k^2 = \omega^2\mu[\epsilon(\omega) + i\sigma(\omega)/\omega]. \tag{100}$$

The normal components of the fields on S are given in terms of the tangential components by (92) and (93), which become

$$\hat{n} \cdot \vec{\eta} = (i/\omega)\nabla_s \cdot (\hat{n} \times \vec{\phi}), \tag{101}$$

$$\hat{n} \cdot \vec{\phi} = -(i\omega^2/k^2)\nabla_s \cdot (\hat{n} \times \vec{\eta}). \tag{102}$$

These equations allow the explicit elimination of the normal components from the operators $\vec{L}$, $\vec{M}$, and $\vec{M}'$ given by (68)-(70). For monochromatic fields we have

$$\vec{L}\{\hat{n} \times \vec{\phi}\} = k\mathrm{P}\oint_S dS'\left(-i + \frac{1}{kR}\right)\frac{e^{ikR}(\hat{n}' \times \vec{\phi}) \times \vec{R}}{4\pi R^2}, \tag{103}$$

$$\vec{M}\{\hat{n} \times \vec{\eta}\} = \mathrm{P}\oint_S dS'\left[\frac{i\omega}{k^2}\left(ik - \frac{1}{R}\right)\frac{\nabla'_s \cdot (\hat{n}' \times \vec{\eta})\vec{R}}{4\pi R^2} + i\omega\frac{\hat{n}' \times \vec{\eta}}{4\pi R}\right]e^{ikR}, \tag{104}$$

$$\vec{M}'\{\hat{n} \times \vec{\phi}\} = \mathrm{P}\oint_S dS'\left[-\frac{1}{\omega}\left(ik - \frac{1}{R}\right)\frac{\nabla'_s \cdot (\hat{n}' \times \vec{\phi})\vec{R}}{4\pi R^2} - \frac{ik^2}{\omega}\frac{\hat{n}' \times \vec{\phi}}{4\pi R}\right]e^{ikR}. \tag{105}$$

We define the auxiliary fields $\vec{E}_1$, $\vec{B}_1$, $\vec{E}_2$ and $\vec{B}_2$ as in Section III, and these fields can be expressed in terms of the discontinuity $\hat{n} \times \vec{\eta}$ of the tangential component of $\vec{B}_1$ across S by (73), (74), (79), and (80) with the new definitions of the operators. The unknown tangential field $\hat{n} \times \vec{\eta}$ obeys the integral equation (83) and, once this equation is solved, the fields $\vec{E}$ and $\vec{B}$ can be found by integrations from (85)-(88).

V. CONCLUSION

In this paper we have shown how the method of solution of the problem of an electromagnetic wave scattered by a perfect conductor is extended to the case of a uniform dielectric without increasing the number of integral equations. We have also presented in detail the analogous problem for scalar waves, both as a preliminary and for its importance in other fields.

We have dealt with the mathematical difficulties in a nonrigorous way, but the results can all be justified [4] if we use the theory of distributions.

This method can be applied in several different ways to scalar and electromagnetic fields, and it can be generalized to other wave equations. For instance, we can obtain a somewhat different integral equation by setting $\hat{n} \times \vec{B}_{2-} = 0$, or we can assume that $\hat{n} \times \vec{B}_1$ is continuous across S and determine $\hat{n} \times \vec{\phi}$, or we can start from the vector wave equation for the electric field and use a dyadic Green function [4].

It is not likely that an extension of this method to the problem of scattering by inhomogeneous bodies can be formulated, because then we have to consider integral equations over the volume, not just the surface, of the scatterer. Nevertheless, similar considerations may reduce the number of unknowns and equations to a minimum.

The integral equation found by this method in electromagnetic scattering by gratings has been used successfully in the numerical solution of direct and inverse problems, and we intend to do likewise for small solids and arbitrary surfaces.

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