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Common Normals of Two Tori

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Abstract. The present paper deals with the computation of the common normals of two tori. We use line geometry in order to describe these common normals as the solution of a system of algebraic equations. Further the non trivial configurations of two tori with infinitely many common normals are investigated.

Key Words: torus, normal, common normal, line geometry, line complex, congruence of surface normals, ruled surface.

MSC 2000: 51N20, 51M35, 51N35

1. Introduction

The computation of relatively extremal distances between two surfaces S_1 and S_2 in three-dimensional Euclidean space is equivalent to the problem of finding common normals of S_1 and S_2, respectively. It is important in different areas such as robotics (e.g. in collision detection), computer graphics, computer animation, and virtual reality, see [4, 12, 16].

The computation of distances between polyhedral objects is well understood. There are a lot of efficient algorithms for that. It is much more difficult to find the minimum or maximum distance between two curved objects. In general one has to solve a system of polynomial equations, which can be time consuming. In case the surfaces do not allow a convenient analytical representation the surfaces have to be replaced by sufficiently well approximating polyhedra.

In [12] line geometry was used to simplify the computation of distances between ellipsoids. The techniques presented in this paper can be applied to other algebraic surfaces as well. Thus the computation of distances (or equivalently of common normals) of further simple geometric objects as used in geometric modeling can be done this way. A different approach is presented in [4]. Because of the absence of convincing examples and the rather high degrees of involved algebraic equations this approach looks not very promising.

Another simple object which sometimes appears in geometric modeling is a torus. A torus is not a quadratic surface, it is of degree four, but it is rational and so it can be written as rational tensor product surface. There are algorithms to find the minimum and maximum distance between such surfaces. One of these algorithms uses recursive subdivision, see [3].

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In the following we show a way to find the common normals of two tori. On these normals the minimum and maximum distance can be found. We use some ideas from line geometry in order to simplify the computations and clarify the number of solutions. For this purpose we make the reader familiar with some basic concepts of line geometry. After that we discuss the general case, i.e., the two tori are in general position relatively to each other.

Then we focus on those cases where two tori have infinitely many common normals. Some of the configurations are trivial, others not.

2. Fundamentals of line geometry

In the following we use the projective closure $\mathbb{P}^3$ and the complex extension of Euclidean three-space as well, if necessary. We describe points P by their Cartesian coordinates $p = (p_x, p_y, p_z)$. Oriented lines L in Euclidean three-space can easily be described by *normalized Plücker coordinates* $L = (l, \bar{l}) = (L_1, L_2, L_3; L_4, L_5, L_6) \in \mathbb{R}^6$, where l is a unit vector parallel to L. Thus L becomes oriented. The vector $\bar{l}$ is called *momentum vector* of L. It is computed by $\bar{l} = p \times l$, if P is any point on L. The momentum vector is independent on the choice of P on L and thus the Plücker coordinates of L are also independent of P. Obviously

$$\langle l, \bar{l} \rangle = L_1 L_4 + L_2 L_5 + L_3 L_6 = 0 \tag{1}$$

holds for the Plücker coordinates of a line. Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. Conversely any six tupel of real (indeed complex) numbers L_i satisfying (1) are the coordinates of a uniquely determined (oriented) line.

When the direction vector l is scaled by a non vanishing factor the momentum vector is scaled by the same factor, and the Plücker coordinates describe the same line. The Plücker coordinates of a line are homogeneous, i.e., we give up the orientation and drop the norming condition of l. So the Plücker coordinates of L can be interpreted as homogeneous coordinates of points in a projective five-space $\mathbb{P}^5$.

The mapping $\gamma: \; L \mapsto (L_1, L_2, L_3; L_4, L_5, L_6)$ is called *Klein mapping*, see [6, 13]. It maps lines in (projective) three-space to points in $\mathbb{P}^5$. The Klein mapping is not onto. Only points whose coordinates satisfy (1) appear as γ-images of lines. Eq. (1) defines a quadratic hypersurface M_2^4 called *Klein quadric* or *Plücker quadric*. The subscript and superscript denote the algebraic degree and the dimension, respectively. The Klein quadric is a point model for the set of lines in (projective) three-space, see [6, 13].

The mapping γ maps manifolds of lines to submanifolds of M_2^4. For instance the normals of a regular C^1-surface are mapped to a two-dimensional submanifold of M_2^4. Line manifolds of dimension one, two, and three are called *ruled surfaces*, *congruences of lines*, and *complexes of lines*, respectively. At least one example of each type will appear in the present paper. The set of lines whose homogeneous Plücker coordinates satisfy a homogeneous algebraic equation of degree d is called *algebraic line complex of degree d*.

3. The set of normals of a torus

A torus T can be generated by rotating a circle c about an axis A contained in c's plane. If A passes through the center of c we obtain a sphere. Each position of c is called a *meridian circle* of T. The path s of the center of c is called *spine curve*. An example is illustrated in Fig. 1.

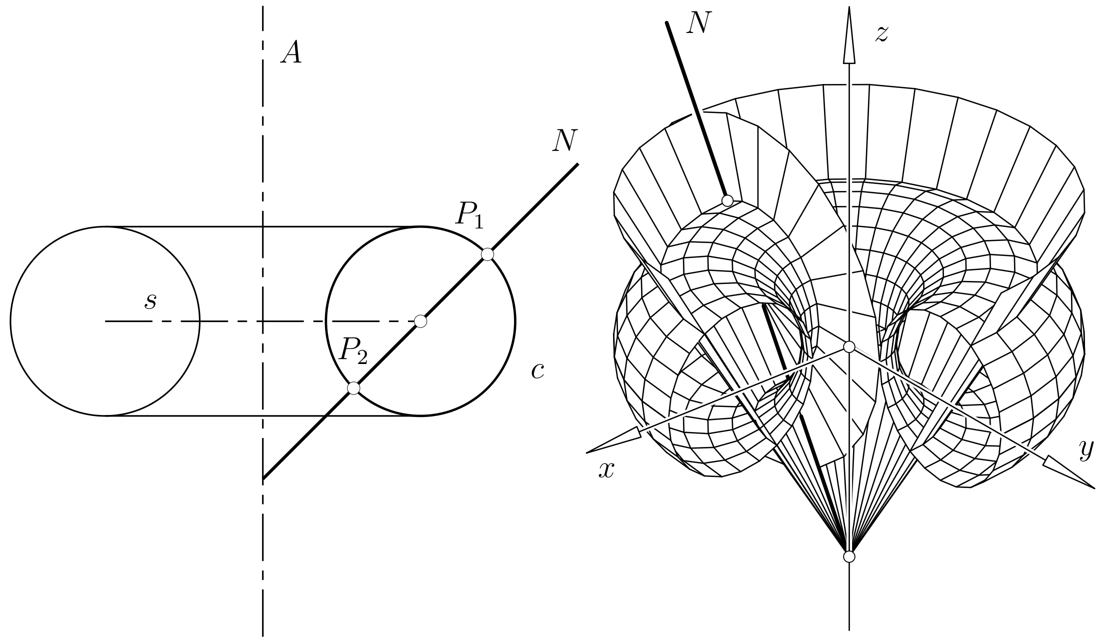


Figure 1: Torus as a surface of revolution

Sometimes it makes sense to distinguish between three types of tori: They correspond to the three possible set of intersection points of c and A. It will turn out that we are able to treat these types simultaneously, so we do not distinguish.

In order to represent the normals of T by Plücker coordinates we choose a Cartesian coordinate system and let A be the z-axis. Further we assume that $(R, 0, 0)$ with positive R is the center of c and $r > 0$ is its radius. Since c is contained in the xz-plane T admits the parametrization

$$T = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)\,, \tag{2}$$

where $(u, v) \in [0, 2\pi]^2$. Eliminating u and v from (2) we obtain

$$(x^2 + y^2 + z^2 - r^2 - R^2)^2 - 4R^2(r^2 - z^2) = 0$$

as an irreducible equation of T. Obviously T is an algebraic surface of degree four.

In order to compute the normals of T given by (2) we compute $n = \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}$. Normalizing n and computing the momentum vector, we find

$$N = (\cos u \cos v,\ \cos u \sin v,\ \sin u;\ R\sin u \sin v,\ R\sin u \cos v,\ 0) \tag{3}$$

as a parametrization of T's *congruence* $\mathcal{N}$ *of surface normals*.

We recall that a normal N of T is a normal of a meridian circle c. Thus N is the normal of T in at least two diametral points P_1 and P_2 on c, see Fig. 1. If either P_1 or P_2 lies on the equator of T, then N is normal in four points of T. Therefore any normal of T is at least a *double normal* of T.

Since N is a normal of c it contains the center of c and meets the axis A of T, see Fig. 1. So the normals of T along a meridian form a pencil of lines and the normals of T along a path circle of a point on c form a cone of revolution. The axis of the cone obviously is A. So the congruence $\mathcal{N}$ can be decomposed into a one-parameter family of cones of revolution sharing the axes and into a one-parameter family of pencils of lines all of whose vertices lie on a circle s in planes through A (orthogonal to s). Fig. 1 shows a torus with a cone and a pencil of normals.

The lines of $\mathcal{N}$ belong to a *singular linear line complex* $\mathcal{A}$, i.e., the set of lines intersecting a fixed straight line called the *axis* of the complex (cf. [6]). Here the axis A of s is the axis of $\mathcal{A}$. The Plücker coordinates of lines in $\mathcal{A}$ satisfy

$$N_6 = 0 \tag{4}$$

as can be seen from (3).

Now we eliminate the parameters u and v from (3). Besides of (1), (4) and the inhomogeneous norming condition $N_1^2 + N_2^2 + N_3^2 = 1$ we find

$$N_4^2 + N_5^2 = R^2 N_3^2. \tag{5}$$

This is the equation of the *quadratic complex* $\mathcal{Q}$ *of lines* which intersect the spine curve s.[1] Eqs. (4) and (5) describe the congruence $\mathcal{N}$ of normals of T.

Later we are interested in the normals of two tori, so we have to ask for the algebraic equations describing these congruences for tori in a more general position.

It is well known (see e.g. [6, 13]) that Euclidean motions induce automorphic collineations of M_2^4. We assume $\beta:\ \mathbb{R}^3 \to \mathbb{R}^3$ with $p \mapsto p' = Bp + b$, where $B^{-1} = B^T$ and $b = \text{const.} \in \mathbb{R}^3$, is a Euclidean motion. The mapping β transforms a torus into another one and it induces a transformation of the linear line complex (4) and the quadratic complex (5) too.

In order to write down the equations of the induced mapping we extend $\mathbb{R}^3$ projectively and thus β reads

$$\beta:\ \begin{bmatrix} 1 \\ p' \end{bmatrix} = \begin{bmatrix} 1 & 0^T \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ p \end{bmatrix}. \tag{6}$$

The induced automorphic collineation of M_2^4 is now described by the 6×6 matrix

$$\mathcal{B} = \begin{bmatrix} B & 0 \\ b \wedge B & B \end{bmatrix}, \tag{7}$$

where $b \wedge B$ is the 3×3-matrix of the linear mapping $x \mapsto b \times (Bx)$. Plücker coordinates $N = (N_1, N_2, N_3; N_4, N_5, N_6)$ and the equations of linear line complexes C transform according to

$$N' = \mathcal{B}N \quad \text{and} \quad C' = \mathcal{M}^{-1}\mathcal{B}^{-1}\mathcal{M}C, \tag{8}$$

where $\mathcal{M}$ is the matrix of the quadratic form (1). With $\mathcal{Q} = \text{diag}\,(0, 0, -R^2; 1, 1, 0)$ we can write down the equation of the transformed quadratic complex as

$$N^T \mathcal{B}^{-T} \mathcal{Q} \mathcal{B}^{-1} N = 0. \tag{9}$$

[1] $\mathcal{Q}$ has the *characteristic* [(22)2], which means that s is its singular surface. More on the classification of quadratic complexes can be found in [15].

4. Common normals of two tori

Let T_1 and T_2 be two tori with axes A_1, A_2 and pairs of radii (R_1, r_1) and (R_2, r_2), respectively. They may be centered at $C_i \in A_i$, $i \in \{1, 2\}$. The normals of T_i intersect both the axes A_i and the spine curves s_i, $i = 1, 2$. Thus the common normals of T_1 and T_2 belong to two singular linear line complexes $\mathcal{A}_i$ with axis A_i and two quadratic complexes $\mathcal{Q}_i$ with s_i for their singular curves.

In order to find common normals of two tori one has to write down the equations of the involved complexes. The underlying Cartesian coordinate system can always be choosen such that the linear and quadratic complex containing the congruence of surface normals $\mathcal{N}_1$ of T_1 have the equations (4) and (5). The equations of the complexes corresponding to $\mathcal{N}_2$ can always be obtained by transforming the ones determining T_1 and according to (7).

Taking into account that the coordinates of a common normal N of T_i have to satisfy (1) the solution of a system of two linear homogeneous and three quadratic homogeneous equations in the unknowns N_1, N_2, N_3, N_4, N_5, and N_6 has to be solved. Now we have the following:

Theorem 1 *Two tori T_1 and T_2 in general position have eight common normals. The number of common normals is independent on the meridian radii.*

Proof: We use Bézout's theorem. Since the meridian radius does not appear in any of the equations (1), (4), and (5) the solutions of the above mentioned system of algebraic equations is independent on the meridian radii. □

Note that each common normal N of T_1 and T_2 is a double normal of both T_1 and T_2. Thus N is four times a common normal of T_1 and T_2.

Example: Let $A_1 = (0, 0, 1; 0, 0, 0)$ and $A_2 = (-24, 0, 7; 28, 0, 96)$ be the axes of T_1 and T_2, respectively. The centers of T_1 and T_2 shall be given by $(0, 0, -1)$ and $(-24/25, 4, 7/25)$ and let further $R_1 = 6$ and $R_2 = 7$ be radii of the respective spine curves. The thus determined tori have (independent on r_i) eight real common normals with Plücker coordinates

$$\begin{array}{rrrrrr}
(\ 1.1539, & 2.5805, & 0.1302; & 1.8672, & -0.8350, & 0),\\
(-1.2300, & 16.9597, & 1.8424; & 5.9345, & 0.4304, & 0),\\
(-3.8744, & -3.1911, & 7.1665; & 24.1458, & -29.3160, & 0),\\
(-6.0906, & -6.3847, & -2.1113; & -15.5507, & 14.8343, & 0),\\
(-0.7265, & -8.4488, & 3.8430; & 14.5246, & -1.2489, & 0),\\
(-0.8284, & 3.8595, & -2.5857; & -11.3091, & -2.4273, & 0),\\
(\ 2.5020, & -6.6736, & 5.9281; & 26.6312, & 9.9841, & 0),\\
(\ 2.2539, & -1.4237, & -5.0933; & -17.7435, & -28.0909, & 0).
\end{array}$$

Fig. 3 shows T_1 (yellow), T_2 (orange), and the eight common normals (blue). In order to illustrate that normals intersect both axes and spine curves, we give Fig. 2. In this example the maximum number of eight real solutions is achieved.
Computations are done with Maple 9.5. The images displayed in Figs. 2, 3, 5, 6, and 7 are created with POV-Ray. ◇

As a consequence of Theorem 1 we can state:

Corollary 2 *Let s_1 and s_2 be circles in Euclidean three-space $\mathbb{R}^3$. There exists a two-parameter family of pairs (T_1, T_2) of tori with spine curves s_1, s_2 such that each pair of the two-parameter family has the same eight common normals.*

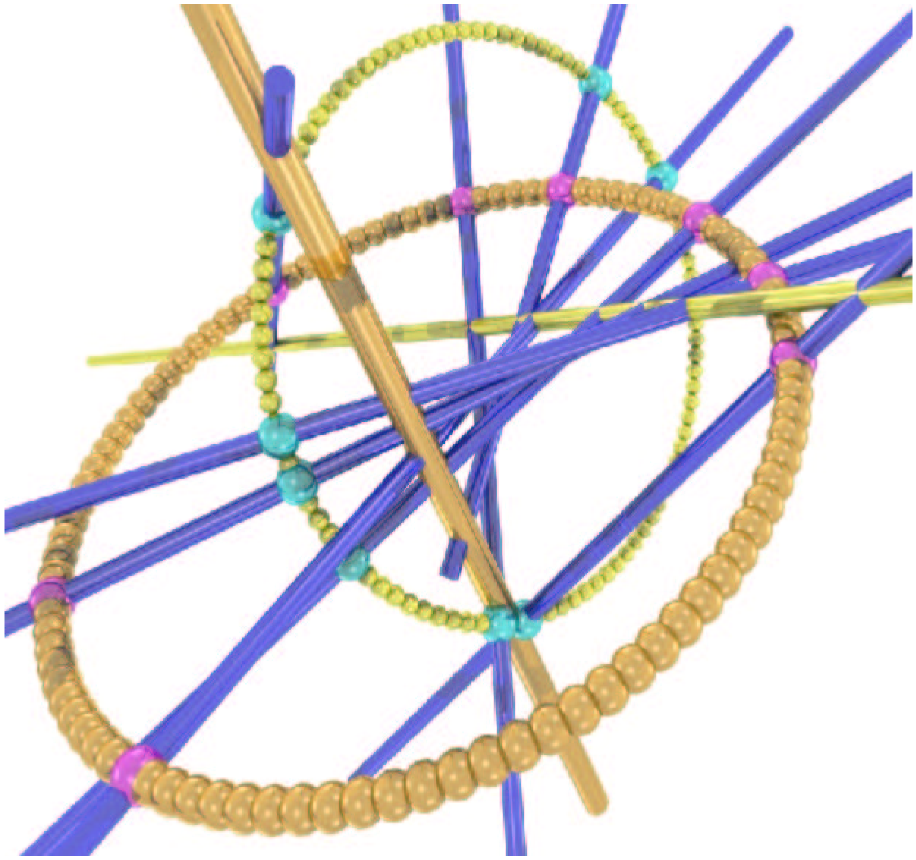


Figure 2: Eight common normals of two tori (blue): Each of them intersects both spine curves and both axes (yellow: A_1 and s_1, orange A_2, s_2)

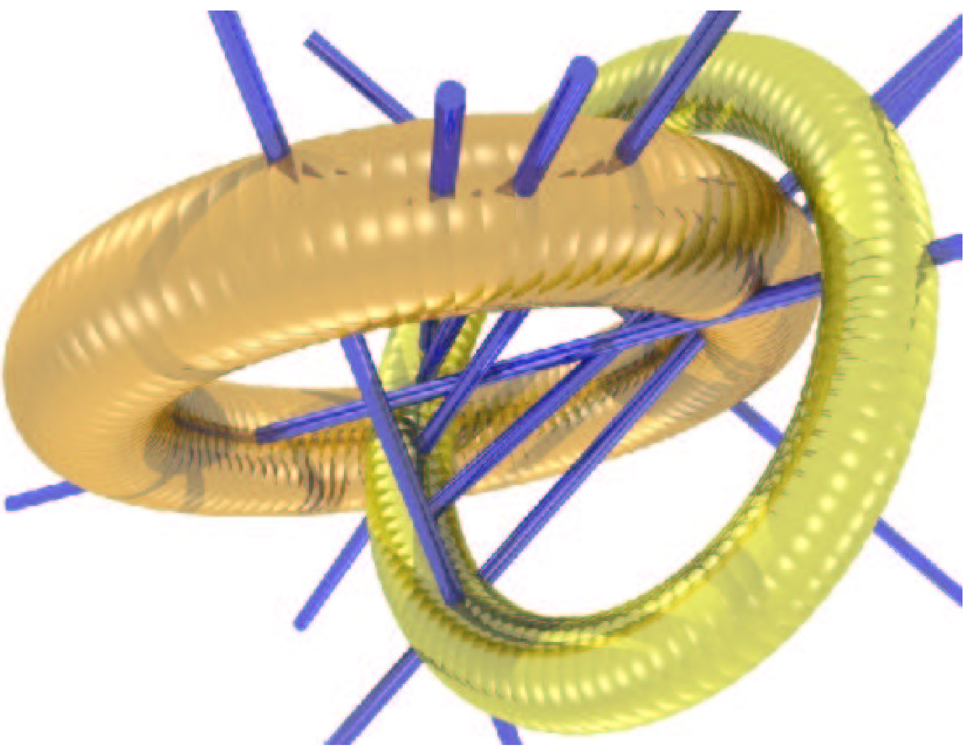


Figure 3: Tori with eight common normals

The radii of the respective meridian curves can serve as the coordinates in this two-parameter family of tori.

Before starting the discussion of particular cases we take a different point of view. We do already know that a normal N of T_1 meets the axis A_1 and the spine curve s_1. If N is a normal of T_2 at the same time it also intersects A_2 and s_2. Thus we are looking for lines N intersecting all A_1, A_2, s_1 and s_2. It is well known (see e.g. [5]) that the lines intersecting two lines A_1, A_2 and a circle s_1 are the set of rulings on an algebraic ruled surface Φ_1 of degree four and Sturm type 7.[2] One example is displayed in Fig. 6, and Fig. 5 shows a discrete version.

Therefore the computation or construction of common normals can be done by intersecting Φ_1 with s_2. Since Φ_1 is of degree four its intersection curve ϕ_1 with the plane π_1 carrying s_1 is (in general) an algebraic plane curve of degree four.

Applying Bézout's theorem we can expect up to eight points lying on both s_2 and ϕ_1, where multiplicities and the complex extension of π_1 have to be taken into account. The common normals of T_1 and T_2 are passing through the previously mentioned eight points.

Analogously we could take the algebraic ruled surface Φ_2 consisting of all lines intersecting A_1, A_2 and s_1.

The computation of common normals is obviously equivalent to the computation of common normals of the spine curves. In [16] the extreme distances between two spatial circles are computed. Constructive and algebraic methods are used in order to find the minimum and maximum distance between points and circle, line and circle, plane and circle, and finally, circle and circle. The authors even tried a line geometric approach. Unfortunately, they did not see that the computation of common normals of two tori is done automatically.

5. Special configurations

In this section we ask for configurations of two tori where common normals and extremal distances can be given explicitly. We also look for configurations of T_i such that they have infinitely many common normals. We use the preparations and considerations from Section 4 and compute the conditions for two quartic ruled surfaces Φ_i to coincide.

5.1. Skew axes

First we assume skew axes A_1, A_2 with the enclosed angle $2\varphi < \pi$ and distance $2d > 0$. Then a Cartesian coordinate system can always be chosen such that the z-axis is the common perpendicle of A_i and their Plücker coordinates are

$$A_1 = (c_\varphi, s_\varphi, 0; -ds_\varphi, dc_\varphi, 0) \quad \text{and} \quad A_2 = (c_\varphi, -s_\varphi, 0; -ds_\varphi, -dc_\varphi, 0), \tag{10}$$

where $c_\varphi := \cos\varphi$ and $s_\varphi := \sin\varphi$, see Fig. 4. Note that A_1 and A_2 are oriented.

With C_i we denote the center of T_i. The distance of C_i to the z-axis shall be denoted by

[2] Algebraic ruled surfaces of degree four and Sturm type 7 are characterized by having two skew straight lines as their double curve, here these lines are A_1 and A_2.

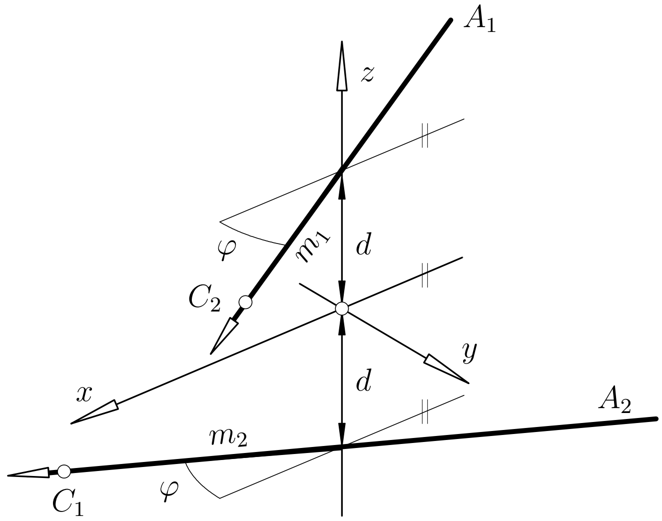


Figure 4: Choice of a coordinate system attached to two skew axes

m_i. The equation of the quartic ruled surface Φ_1 with directrices A_1, A_2 and s_1 is given by

$$\begin{aligned}\Phi_1:\ & 2(d^2-(m_1^2+R_1^2)s_\varphi^2c_\varphi^2)(-s_\varphi c_\varphi xy(z^2+d^2)+d(s_\varphi^2x^2+c_\varphi^2y^2)z)\\ &+s_\varphi^2c_\varphi^2(-m_1^2(z^2-d^2)^2+2d^2x^2y^2)-d^2(x^4s_\varphi^4+y^4c_\varphi^4)\\ &+(R_1^2c_\varphi^4-m_1^2s_\varphi^2c_\varphi^2-d^2)(s_\varphi^2x^2z^2+d^2c_\varphi^2y^2)+(R_1^2s_\varphi^4-m_1^2s_\varphi^2c_\varphi^2-d^2)(c_\varphi^2y^2z^2+d^2s_\varphi^2x^2)\\ &+2ds_\varphi c_\varphi(2d^2+2m_1^2s_\varphi^2c_\varphi^2-R_1^2(s_\varphi^4+c_\varphi^4))xyz\\ &+2dm_1s_\varphi c_\varphi(s_\varphi x+c_\varphi y)(z+d)((z-d)^2+(s_\varphi x-c_\varphi y)^2)=0\end{aligned}\tag{11}$$

The equation of Φ_2 is obtained by replacing d, φ, R_1, m_1 in (11) by $-d$, $-\varphi$, R_2, m_2, respectively. Again we observe that the existence of common normals is independent on the meridian radii r_i since they do not show up in the equations of Φ_i.

Now we are able to discuss the configurations of T_1 and T_2, where they have infinitely many common normals. This situation appears exactly if and only if the ruled surfaces Φ_1 and Φ_2 coincide. Then the set of lines intersecting A_1, A_2 and s_1 is the same as the set of lines intersecting A_1, A_2 and s_2.

Analogously we could look for conditions to the coefficients of Eq. (11) of Φ_1 such that s_2 is included, i.e., Φ_1 and s_2 have infinitely many common intersection points.

For that end we have to compare the coefficients of the monomials $x^ry^sz^t$ in the equations (11) of Φ_1 and Φ_2. The resulting equations are relations in R_i, m_i, d and φ describing the relative position of T_1 and T_2.

We assume $d \neq 0$ and by the above assumptions neither $s_\varphi = 0$ nor $c_\varphi = 0$. Therefore A_1 and A_2 are not parallel. Skipping the trivial relations we arrive at the following seven equations

$$m_2+m_1=0,\tag{12}$$

$$m_2-m_1=0,\tag{13}$$

$$s_\varphi^2(m_2^2-m_1^2)+(R_1^2-R_2^2)c_\varphi^2=0,\tag{14}$$

$$c_\varphi^2(m_2^2-m_1^2)+(R_2^2-R_1^2)s_\varphi^2=0,\tag{15}$$

$$2d^2-(R_1^2+R_2^2+m_1^2+m_2^2)s_\varphi^2c_\varphi^2=0,\tag{16}$$

$$2c_\varphi^2(m_1^2-m_2^2)+(R_1^2-R_2^2)s_\varphi^2=0.\tag{17}$$

Eqs. (12) and (13) imply $m_1 = m_2 = 0$. From Eq. (14) we conclude $R_1 = R_2$ or $R_1 = -R_2$. The latter equation does not matter, since the radii of spine curves are always assumed to be positive and Eqs. (15) and (17) are satisfied automatically. Eq. (16) now reads $d^2 - R_i^2 s_\varphi^2 c_\varphi^2 = 0$ and is equivalent to

$$2d = \pm R_i \sin 2\varphi. \tag{18}$$

Thus we can state the following theorem:

Theorem 3 *The configuration of two tori with skew axes ($d \neq 0$, $2\varphi \neq 0, \pi$) and infinitely many common normals is characterized by the relations*

$$m_1 = m_2 = 0, \quad R_1 = R_2 \quad \text{and} \quad 2d = \pm R_i \sin 2\varphi.$$

Fig. 6 shows axes and spine curves of tori as described in Theorem 3. In Fig. 5 a pair of tori with sixty of infinitely many common normals is shown.

When T_1 and T_2 are in a configuration described in Theorem 3, then the equation (11) of the quartic ruled surface simplifies to

$$(s_\varphi^2 x^2 - c_\varphi^2 y^2)^2 + (s_\varphi^2 - c_\varphi^2)\left((x^2 - y^2)z^2 + R^2(c_\varphi^4 y^2 - s_\varphi^4 x^2) - 2Rxyz(c_\varphi^2 - s_\varphi^2)\right) = 0. \tag{19}$$

An example of this type of quartic ruled surface can be seen in Fig. 6.

Let us consider the special case where $2d = R_i$ or equivalently $2\varphi = \pi/2$ holds. Consequently A_1 and A_2 are orthogonal. Then the equation of the algebraic ruled surface $\Phi = \Phi_1 = \Phi_2$ of degree four becomes $(x-y)^2(x+y)^2 = 0$. Obviously Φ splits into a pair of pencils of lines each of multiplicity two.

Eq. (18) allows a reformulation of Theorem 3. Assume that the axis A_1 and the spine curve's radius R_1 of T_1 are fixed. Then we can characterize the set of axis of all tori T_2 with $R_2 = R_1$ which share a one-parameter family of normals:

Theorem 4 *Let A_1 be the axis and R be the radius of the spine curve of a torus T_1. Up to rotations about A_1 the axes A_2 of tori T_2 with spine curve radius R sharing infinitely many common normals with a torus T_1 are rulings of a quartic ruled surface of Sturm type 7.*

Proof: Assume $A_1 = (t, 0, 0)$. Thus the axis A_2 of a torus T_2 that has infinitely many common normals with T_1 is given by $A_2(\varphi, t) = (tc_\varphi, ts_\varphi, 1/2Rs_\varphi)$, with $(t, \varphi) \in \mathbb{R} \times [0, 2\pi]$. This parametrizes a quartic ruled surface of Sturm type 7. The rulings coresponding to $\varphi = 0, \pi, 2\pi$ have to be excluded. □

Further we can state the following theorem:

Theorem 5 *For a given torus T there exists a 4-parameter manifold $\mathcal{T}$ of tori each of them having infinitely many common normals with T. The distance of and the angle enclosed by the axes of T and $T' \in \mathcal{T}$ are related by (18).*

Proof: Consider a given torus T with given axis A and spine curve radius R. If we prescribe the distance $2d \neq 0$ of the axis A' of $T' \in \mathcal{T}$ the angle 2φ between A and A' is determined according to Eq. (18). So the distance of the axes can be choosen freely and the angle is determined uniquely. The radii r and r' of the respective meridian curves can also be chosen freely. The configuration of T and T' can be rotated about T's axis without changing the number of common normals, which gives the fourth parameter. □

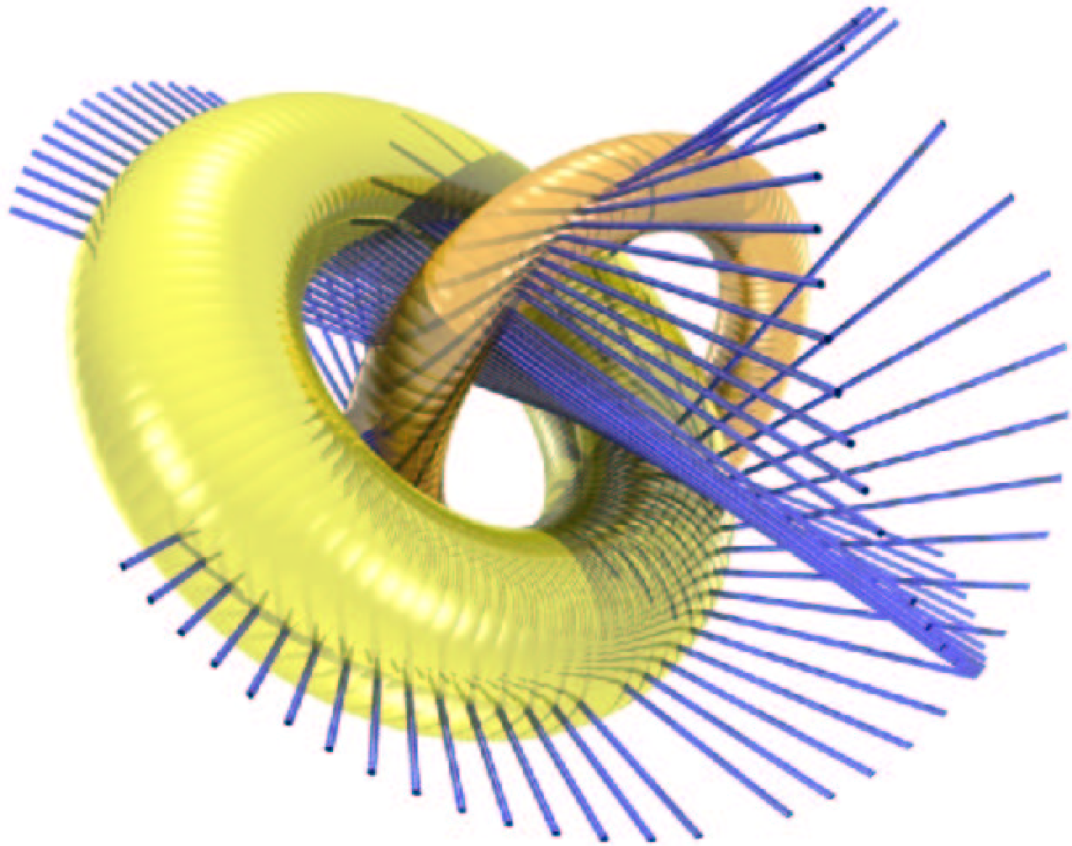
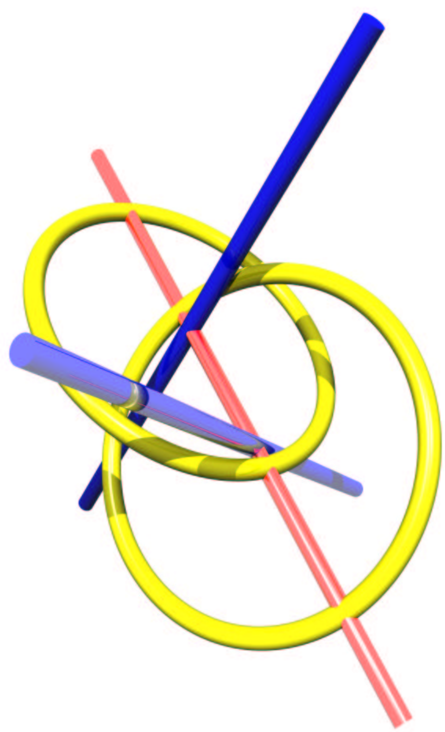


Figure 5: Tori with infinitely many common normals



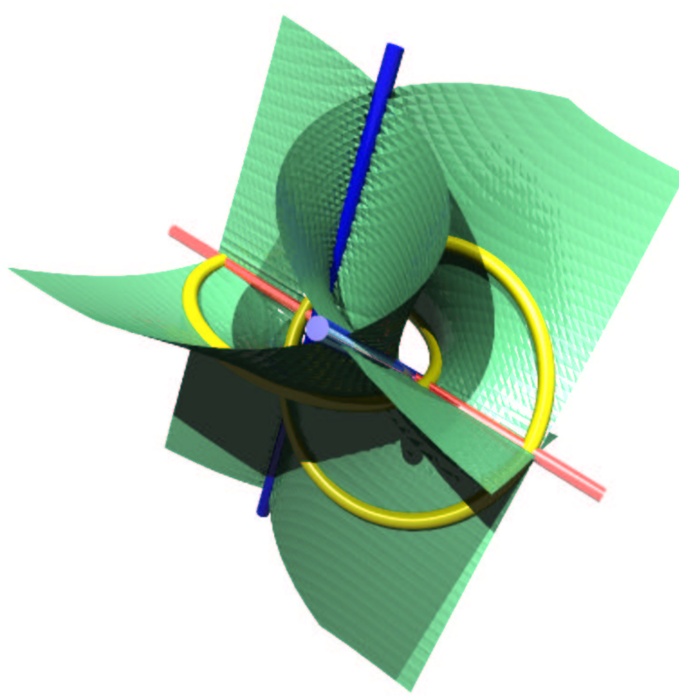


Figure 6: Left: Axes (blue) and spine curves (yellow). Right: Quartic ruled surface (green) whose generators intersect both the axes (blue) and the spine curves (yellow)

The generators of the quartic ruled surface Φ defined by (19) carring the continuum of common normals of T_1 and T_2 meet the spine curves s_1 and s_2 orthogonally. This is equivalent to $\langle s_1 - s_2, \dot{s_1}\rangle - \langle s_1 - s_2, \dot{s_2}\rangle = 0$. Integration yields $\langle s_1 - s_2, s_1 - s_2\rangle = \text{const.}$ and consequently on each common normal the two spine curves enclose a segment of the same length. Thus we have

Theorem 6 *In the particular position of Theorem 3 the spine curve s_2 is entirely contained in a torus with spine curve s_1, and vice versa.*

Thus s_2 is a Villarceau-circle of a torus T_1 with spine curve s_1. For Villarceau sections and generalizations the reader may be referred to [1, 11, 14]. On the other hand we can shrink T_1 such that it degenerates to s_1 and simultaneously we can blow up T_2 such that s_1 becomes one of its Villarceau-circles. Both cases can be seen as borderline cases of tori with infintely many common normals. This is illustrated in Fig. 7.

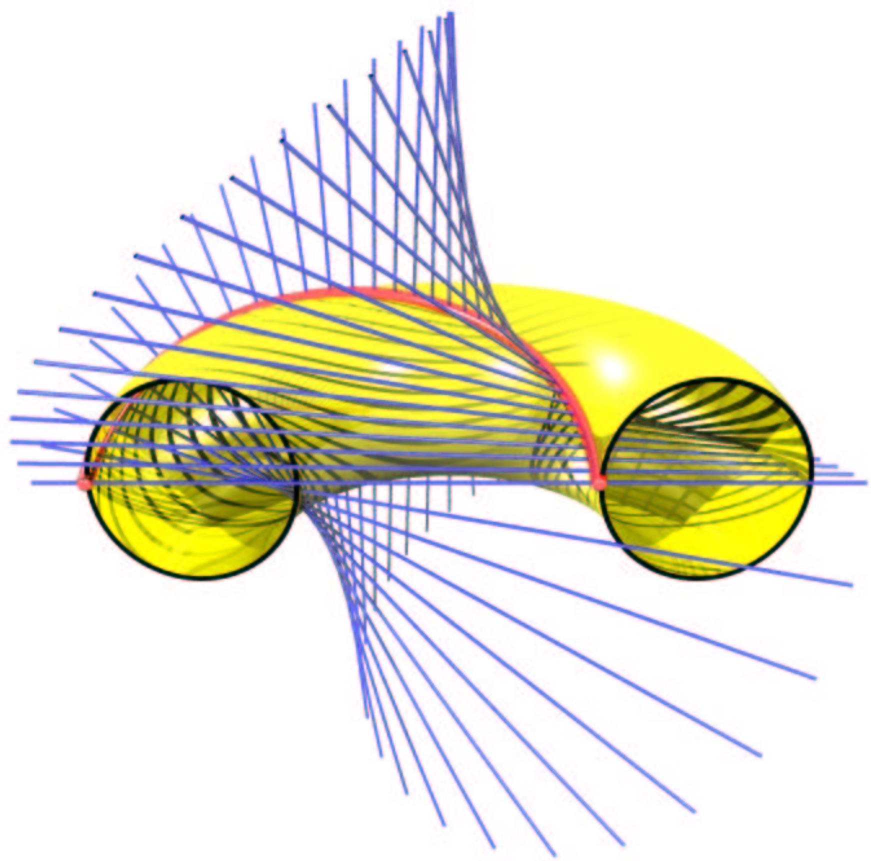


Figure 7: Normals of a torus along a Villarceau-circle

The right choice of radii of meridian curves of T_1 and T_2, respectively, leads to tori in line contact. In this case the meridian radii r_1 and r_2 sum up to $2d$.

Since the curve of intersection of two tori is of degree eight and both surfaces share the absolute conic[3] the curve of contact is of degree four. In general it is not a circle with multiplicity two since Φ given by (19) contains only the two circles s_1 and s_2, respectively. The common curve of the two touching tori becomes a Villarceau-circle only in the limiting case where one torus shrinks to a circle.

[3]This conic in the ideal plane $x_0 = 0$ of the projectively closed Euclidean three-space obeys the equation $x_1^2 + x_2^2 + x_3^2 = 0$.

5.2. Coplanar axes

5.2.1. Axes are not parallel

In case of coplanar axes A_1 and A_2 we find four common normals of T_1 and T_2 in the plane $\mu = A_1 \vee A_2$. Their Plücker coordinates can be given explicitly. We use the abbreviations $\Delta_m := m_2 - m_1$, $\Sigma_m := m_1 + m_2$, $\Delta_R := R_2 - R_1$, and $\Sigma_R := R_1 + R_2$ and find

$$\begin{array}{llll}(\Delta_m s_\varphi + \Sigma_R s_\varphi, & \Delta_R c_\varphi - \Sigma_m c_\varphi, & 0;0,0, & (m_1R_2 - m_2R_1)c_{2\varphi} - (m_1m_2 + R_1R_2)s_{2\varphi}),\\ (\Delta_m s_\varphi - \Delta_R s_\varphi, & -\Sigma_R c_\varphi - \Sigma_m c_\varphi, & 0;0,0, & -(m_1R_2 + m_2R_1)c_{2\varphi} - (m_1m_2 - R_1R_2)s_{2\varphi}),\\ (\Delta_m s_\varphi + \Delta_R s_\varphi, & \Sigma_R c_\varphi - \Sigma_m c_\varphi, & 0;0,0, & (m_1R_2 + m_2R_1)c_{2\varphi} - (m_1m_2 - R_1R_2)s_{2\varphi}),\\ (\Delta_m s_\varphi - \Sigma_R s_\varphi, & -\Delta_R c_\varphi - \Sigma_m c_\varphi, & 0;0,0, & -(m_1R_2 - m_2R_1)c_{2\varphi} - (m_1m_2 + R_1R_2)s_{2\varphi}),\end{array}$$

which are always real and connect the two intersection points $P_1, P_2 \in s_1$ and $Q_1, Q_2 \in s_2$ of either s_i and μ with each other. The remaining four common normals pass through the intersection point $A_1 \cap A_2$. They are the common generators of two cones of revolution given by

$$\begin{aligned} K_1: &\ (R_1^2c_\varphi^2 - m_1^2s_\varphi^2)(x^2+y^2) - m_1^2c_\varphi^2z^2 + 2s_\varphi c_\varphi xy(m_1^2+R_1^2) = 0,\\ K_2: &\ (R_2^2c_\varphi^2 - m_2^2s_\varphi^2)(x^2+y^2) - m_2^2c_\varphi^2z^2 - 2s_\varphi c_\varphi xy(m_2^2+R_2^2) = 0,\end{aligned}$$

which connect $A_1 \cap A_2$ with the spines curves s_1 and s_2, respectively. Since K_1 and K_2 are concentric cones of revolution they share a pair of conjugate complex lines and a further pair of lines. The latter pair can be a pair of real lines, a single double line, or a pair of conjugate complex lines. Therefore we can expect to find four or six real common normals of T_i depending on the number of common real generators of K_i, respectively. In case of the single double line K_1 and K_2 are in line contact along this line. Thus one of the four common normals of T_1 and T_2 is of multiplicity three.

If the spine curves s_i intersect μ in a common point, s_1 and s_2 are in first order contact. In this case the above mentioned cones touch along a generator. Then the two tori T_i have a pencil of common normals and one common normal N not contained in the pencil. One line in the pencil of common normals is the line of contact of K_1 and K_2 and has thus multiplicity three. In this case the radii of the spine curves have to satisfy $m_1^2 + R_1^2 = m_2^2 + R_2^2$. It is not necessary that T_i share a meridian to have a pencil of common normals.

Finally we observe that in case $m_1 = m_2 = 0$ there are four different real common normals with multiplicity one. The common perpendicle of A_1 and A_2 appears as one common normal with multiplicity four. In this case we have five common normals.

5.2.2. Axes are parallel

Now we assume A_i and A_2 are parallel. In this case one has to distinguish two subcases:

1. *The spine curves lie in different planes:* It is a special situation of the above mentioned case of coplanar axes. We choose a Cartesian coordinate system such that T_1's center coincides with the origin and $A_1 = (1,0,0;0,d,0)$, $A_2 = (1,0,0;0,-d,0)$. T_1 may be centered at $(0,0,d)$ and T_2 may be centered at $(m,0,-d)$. The Plücker coordinates of the four common normals which are always real are given by

$$(m,0,\Sigma_R - 2d;0,m(d-R_1),0), \quad (m,0,-\Sigma_R - 2d;0,m(d+R_1),0), \tag{20}$$
$$(m,0,\Delta_R - 2d;0,m(d+R_1),0), \quad (m,0,-\Delta_R - 2d;0,m(d-R_1),0). \tag{21}$$

Table 1: Number of common normals of two tori T_1 and T_2 depending on distance $2d$ and angle $2\varphi < \pi$ of axes, radii R_1, R_2 of spine curves and position m_1, m_2 of centers

$2d$	2φ	R_1, R_2, m_1, m_2	*# real solutions*	*shape*
skew axes				
$\neq 0$	$\neq 0$		up to 8	–
$\neq 0$	$\neq 0$	$2d = R_i \sin 2\varphi$, $m_i = 0$	∞^1	quartic ruled surface
$\neq 0$	$\pi/2$	$2d = R_i \sin 2\varphi$, $m_i = 0$	$2\infty^1$	two two-fold pencils
coplanar axes (not parallel)				
0	$\neq 0$	$R_1^2 + m_1^2 \neq R_2^2 + m_2^2$	4, 6	– [4]
0	$\neq 0$	$R_1^2 + m_1^2 = R_2^2 + m_2^2$	∞^1	pencil + line [5]
0	$\neq 0$	$m_1 = m_2 = 0$	5	– [6]
parallel axes				
$\neq 0$	$0, \pi$	$m \neq 0$, $\lvert R_1 \pm R_2\rvert \neq 2d$	4,6	–
$\neq 0$	$0, \pi$	$m \neq 0$, $\lvert R_1 \pm R_2\rvert = 2d$	5	– [7]
$\neq 0$	$0, \pi$	$m = 0$, $\lvert R_1 \pm R_2\rvert \neq 2d$	1, 3	– [8]
$\neq 0$	$0, \pi$	$m = 0$, $\lvert R_1 \pm R_2\rvert = 2d$	∞^1	pencil [9]
identical axes				
0	0	$m \neq 0$, $R_1 \neq R_2$	$2\infty^1$	two cones of revolution
0	0	$m \neq 0$, $R_1 = R_2$	$2\infty^1$	cone + cylinder of revolution
0	0	$m = 0$, $R_1 \neq R_2$	$4\infty^1$	four-fold pencil
0	0	$m = R_1 = R_2 = 0$	∞^2	–

The remaining four common normals appear as common generators of two cylinders K_1 and K_2 of revolution. Like in the previous case the cylinders share a pair of conjugate complex lines, and a further pair of lines all of them being common normals of T_1 and T_2. The latter pair is a pair of real lines, a single double line, or a pair of conjugate complex lines, if and only if

$$D := (R_1^2 - R_2^2)^2 - 8d^2(R_1^2 + R_2^2 - 2d^2)$$

is greater, equal, or less than zero. In the present case we can find four or six real common normals of T_1 and T_2, respectively, if $D \neq 0$. In analogy to the previous case we can also find four common normals T_i, where one of them has multiplicity 3. This occurs exactly if $D = 0$, i.e., the cylinders touch along this common generator.

2. *The carrier planes of the spine curves coincide:* The four common normals from (21) become one common normal with multiplicity four. In case of $D > 0$ we have three real

[4] In case of four common normals there can be one line with multiplicity three, if the above-mentioned cones touch along a generator.

[5] The pencil contains a line with multiplicity three. The further line is not contained in the pencil.

[6] One line is of multiplicity three.

[7] See Footnote 6.

[8] There is one common normal of multiplicity four.

[9] See Footnotes 5, 8.

common normals one of them having multiplicity four. The case $D < 0$ leads to one real common normal. In case of $D = 0$ the spine curves are in first order contact similar to the case above and there is a pencil of common normals of T_i containing one normal of multiplicity four.

The cases where pencils of common normals appear can be charactrized by

$$|R_1 - R_2| = 2d \quad \text{or} \quad |R_1 + R_2| = 2d.$$

5.3. Coinciding axes

Now we have $A_1 = A_2$ and denote the distance of the centers of T_i by m. There are four different cases:

1. *$m \neq 0$ and $R_1 \neq R_2$:* The four common normals of T_1 and T_2 in any of the common meridian planes of the tori. So there exist two cones of revolution whose generators are common normals of either T_i, i.e., there are two one-parameter families of common normals.
2. *$m \neq 0$ and $R_1 = R_2$:* This case is a special case of the previous one. One of the cones of revolution becomes a cylinder of revolution.
3. *$m = 0$ and $R_1 \neq R_2$:* Consider one of the common meridian planes. The four common normals of T_i in this plane coincide and become thus one common normal with multiplicity four. Since this is the fact for all the common meridian planes, the two tori have a pencil of common normals. This pencil is of multiplicity four. Each of these lines is a diameter of the concentric spine curves.
4. *$m = 0$ and $R_1 = R_2$:* The two tori share the congruence of surface normals. In case of $r_1 = r_2$, i.e., $T_1 = T_2$ this is also true.

The results are summarized in Table 1.

6. Conclusion and future research

Line geometry and the methods used in the present paper can obviously be used to compute the common normals of arbitrary algebraic surfaces. The discussion of coinciding ruled surfaces traced out by the normals of two tori was successful since it was of relatively low degree. Dealing with surfaces of higher order the algebraic congruences of their normals become much more complicated.

Unfortunately, (algebraic) surfaces in general do not have two spine curves like the torus. The only algebraic surfaces with two spine curves are Dupin cyclides, including the torus and cylinder and cone of revolution. These types of surfaces do not differ from the viewpoint of Laguerre-Geometrie, see [7, 8]. They are used in CAGD in order to model pipes and connections of pipes with different radii, see [7, 8, 9, 10].

Dupin cyclides are pipe surfaces with two focal conics (ellipse + hyperbola or parabola + parabola) for their spine curves. The normals of a dupin cyclide D_1 meet the ellipse and hyperbola (or the two parabolas). Common normals of two Dupin cyclides D_1 and D_2 meet the two focal conics of both D_1 and D_2. Thus in general one can expect to find up to 16 common normals of D_1 and D_2, respectively.

The discussion of configurations of D_1 and D_2 with infinitely many common normals would start at a list of trivial cases. The non-trivial cases appear if the set of rulings of two algebraic ruled surfaces Φ_1 and Φ_2 of degree eight coincide. Φ_1 is the ruled surface determined by the directrices of D_1 and one of the directrices of D_2. The surface D_2 is determined by the remaining director curves.

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