Overview and Results

Bayesian nonparametric models of sparse and exchangeable random graphs

François Caron   Emily B. Fox

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Presenter: Victor Veitch
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Exchangeable random arrays are either empty or dense and thus not appropriate for most real applications.

Big Picture Question
How can we salvage a useful notion of exchangeability for graphs?
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Main Ideas

Point Process $\leftrightarrow$ Random Graph
Set up a correspondence between random graphs and random discrete measures (point processes)

Symmetry
- Natural notion of exchangeability of point processes
- Use associated representation theorem to study random graphs
**Completely Random Measures**

- \( W = \sum_{i=1}^{\infty} w_i \delta_{\theta_i} \) random measure
- \( w_i \) sociability parameter
- \( \theta_i \) embedding of node \( i \) in \( \mathbb{R}^+ \)

**Point Process**

- \( Z = \sum_i \sum_j z_{ij} \delta(\theta_i, \theta_j) \)
- \( z_{ij} = 1 \) if there is a link between \( \theta_i, \theta_j \)
- \( z_{ij} = f(w_i, w_j) \)

**Figure:** Edge between \( \theta_i \) and \( \theta_j \) represented by points at \( (\theta_i, \theta_j) \) and \( (\theta_j, \theta_i) \)
### Directed Multigraph

- \( D = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} n_{ij} \delta(\theta_i, \theta_j) \)
- \( n_{ij} \) number of directed edges from \( \theta_i \) to \( \theta_j \)

### Given \( W \sim CRM(\rho, \lambda) \)

- \( D \mid W \sim PP(W \times W) \) on \( \mathbb{R}_+^2 \)
- informally, \( n_{ij} \) are generated as Poisson(\( w_iw_j \))

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**Figure**: (Restricted) atomic measure \( D \) to directed multigraph to corresponding undirected graph
Undirected Graphs

Hierarchical Model

\[ W = \sum_{i=1}^{\infty} w_i \delta_{\theta_i} \quad W \sim \text{CRM} (\rho, \lambda) \]

\[ D = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} n_{ij} \delta(\theta_i, \theta_j) \quad D|W \sim \text{PP} (W \times W) \]

\[ Z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \min(n_{ij} + n_{ji}, 1) \delta(\theta_i, \theta_j) \]

Observation

The distribution of the random graph is determined by the distribution of \( W \).
Completely Random Measures

\( W \) a CRM if

For any countable collection \( A_1, A_2 \ldots \) of measurable sets

- random variables \( W(A_1), W(A_2), \ldots \) are independent
- \( W(\bigcup_j A_j) = \sum_j W(A_j) \)
- the distribution of \( W([t,s]) \) depends only on \( t - s \)

Important Facts

- \( W = \sum_{i=1}^{\infty} w_i \delta_{\theta_i} \) almost surely
- For any measurable \( A \) the Laplace transform may be written as

\[
\mathcal{L} [W(A)] = \exp \left( - \int_{\mathbb{R}_+ \times A} \left[ 1 - \exp(-tw) \right] \rho(dw) \lambda(d\theta) \right)
\]
CRM $W$ is characterized by measure $\rho$ on $\mathbb{R}^+$ such that
$$\int_0^\infty (1 - e^{-w}) \rho (dw) < \infty$$

$$\int_0^\infty \rho (dw) = \infty \iff \text{number of jumps is infinite in any interval } [0, T]$$

Infinite number of jumps $\iff$ infinite number of nodes (almost all with degree 0)

This (possibly non-parametric) model for random graphs has a representation in terms of $\rho$.
Finite Size Samples

Truncation

- Aldous-Hoover construction: truncate at finite number of nodes $n$
- Random measure construction: truncate $\mathbb{R}_+$ to interval $[0, \alpha]$
  - Define $W^*_\alpha = W([0, \alpha])$
  - Number of directed edges: $D^*_\alpha | W^*_\alpha \sim \text{Poisson} \left( W^*_\alpha^2 \right)$
Exchangeability of Random Measures

Point process $Z$ on $\mathbb{R}_+^2$

- $\pi, \sigma$ permutations of $\mathbb{N}$
- $A_i = [h(i - 1), hi] \ h > 0$
- $Z$ is exchangeable if and only if $Z(A_i \times A_j) \overset{d}{=} Z(A_{\pi(i)} \times A_{\sigma(j)})$

Point process defining the graph construction is exchangeable

- $W(A_i) \overset{d}{=} W(A_{\sigma(i)})$
- $D(A_i \times A_j) \sim \text{Poisson}(W(A_i) \times W(A_j))$
Kallenberg Representation Theorem

Construction of CRM:

- \((\theta_i, \vartheta_i)\) a unit rate Poisson process on \(\mathbb{R}_+ \times \mathbb{R}_+\)
- \(L(x) = \int_x^\infty \rho(dw)\)
- \(w_i \equiv L^{-1}(\vartheta_i)\) then \(W = \sum w_i \delta_{\theta_i}\) is CRM with \(\rho(dw) d\theta\)

Kallenberg Representation:

- \(Z = \sum_{i,j} f(\vartheta_i, \vartheta_j, \zeta_{\{i,j\}}) \delta_{\theta_i, \theta_j}\) (transformed Poisson Processes)
- \(f(\vartheta_i, \vartheta_j, \zeta_{\{i,j\}}) = \begin{cases} 1 & \zeta_{\{i,j\}} \leq M(\vartheta_i, \vartheta_j) \\ 0 & \text{ow} \end{cases}\)
- \(M(\vartheta_i, \vartheta_j) = \begin{cases} 1 - \exp\left(-2L^{-1}(\vartheta_i) L^{-1}(\vartheta_j)\right) & \vartheta_i \neq \vartheta_j \\ 1 - \exp\left(-L^{-1}(\vartheta_i)^2\right) & \vartheta_i = \vartheta_j \end{cases}\)
Figure: Model construction based on Kallenberg representation. (left) Unit rate Poisson process. (right) Graphical representation of $L^{-1}$
**Example 1: Poisson Process**

### Representations

- \( \rho (dw) = \delta_{w_0} (dw) \) so \( \int_0^\infty \rho (dw) < \infty \)
- \( L(x) = \begin{cases} 
1 & x < w_0 \\
0 & \text{ow}
\end{cases} \)

### Construction for fixed \( \alpha \)

- sample \( n \sim \text{Poisson} (\alpha) \), number of nodes
- set \( z_{ij} = z_{ji} = 1 \) with probability \( 1 - \exp (-2w_0^2) \)
- this is Erdős-Renyi conditional on \( n \)
Example 2: Compound Poisson Process

### Representations
- \( \rho (dw) = h(w) \, dw \) with \( \int_0^\infty h(w) \, dw = 1 \)
- \( L(x) = 1 - H(x) \)

### Construction for fixed \( \alpha \)
- Sample \( n \sim \text{Poisson} (\alpha) \), number of nodes
- Set \( z_{ij} = z_{ji} = 1 \) with probability \( M(U_i, U_j) \), \( U_i \) uniform
- \( M(U_i, U_j) = 1 - \exp \left(-2H^{-1}(U_i)H^{-1}(U_j)\right) \)
- Aldous-Hoover representation, conditional on \( n \)
Example 3: Generalized Gamma Process

Representations

\[ \rho(dw) = \frac{1}{\Gamma(1-\sigma)} w^{1-\sigma} \exp(-\tau w) dw, \quad \sigma \in [0,1) \quad \tau \geq 0 \]

\[ L(x) = \begin{cases} 
\frac{\tau^\sigma \Gamma(-\sigma, \tau x)}{\Gamma(1-\sigma)} & \tau > 0 \\
\frac{x^{-\sigma}}{\Gamma(1-\sigma) \sigma} & \tau = 0
\end{cases} \]

Features

- CRM has infinite number of jumps in any interval
- Exact sampling is possible via urn process
- The network growth is not dense
**Power Law**

Let $N_{\alpha,j}$ number of nodes in directed graph $D$ with $j$ outgoing or ingoing edges, then

$$
\frac{N_{\alpha,j}}{N_{\alpha,1}} \rightarrow \frac{\sigma \Gamma (j - \sigma)}{\Gamma (1 - \sigma) \Gamma (j + 1)}, \quad \alpha \rightarrow \infty
$$

$$
\sim \frac{\sigma}{\Gamma (1 - \sigma)} j^{-1 - \sigma}, \quad j \rightarrow \infty
$$

**Sparsity**

Let $E_\alpha$ the number of edges in the undirected graph. Then for $0 < \varepsilon < \sigma$

$$
E_\alpha = O \left( N_\alpha^{2 - \sigma + \varepsilon} \right)
$$

almost surely as $\alpha \rightarrow \infty$
Main points

- Correspondence between point processes and random graphs
- Exchangeability of random measures of $\mathbb{R}_+^2$ gives tractable representation
- There are random graphs in this family that are asymptotically sparse