The reconstruction of a subclass of domino tilings from two projections

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Abstract

We present a new way of studying the classical and still unsolved problem of the reconstruction of a domino tiling from its row and column projections. After giving a simple greedy strategy for solving the problem from one projection, we introduce the concept of degree of a domino tiling. We generalize an algorithm for the reconstruction of domino tilings of degree two from two projections, to domino tilings of degree three and four.

Keywords: Domino tilings; Discrete Tomography; Reconstruction algorithm; Computational complexity

1. Introduction

In chemistry, a classical way of modelling two-dimensional crystalline structures, where dimer connections between atoms are present, is to consider complete domino tilings. A domino is a rectangle of dimension $2 \times 1$ or $1 \times 2$, and a complete domino tiling of a surface is a way of covering it with dominoes such that no holes and non-overlaps are present. Relevant enumeration results about domino tilings have been achieved starting from the works by Kasteleyn [4], and by Temperley and Fisher [8] who, independently, found a formula for counting the domino tilings of an $m \times n$ rectangle (with $m$ and $n$ even).

Recently, a methodology explained in [6] allows an internal quantitative analysis of a dimer structure by sending X-rays through it along a fixed set of directions and then
by measuring the decreasing rate of their intensities. Such measurements are called the projections of the structures along the set of directions. The obtained data give information about the disposition of the dimers, and so efforts can be addressed toward the new target of a faithful reconstruction of the structure itself. In particular, Picouleau focused his attention on the reconstruction of domino tilings of rectangles from the row and column projections (see [7]): he considered horizontally decreasing tilings, namely tilings whose vectors of row projections are weakly decreasing sequences, and he furnished an algorithm which reconstructs them in polynomial time. Starting from this result, one can naturally define the concept of complexity degree of a domino tiling, which is obtained by partitioning each tiling into strips-like subtilings of minimal height, and then by taking the greatest of these values. In this paper, we observe that the horizontally decreasing property of a domino tiling used by Picouleau is equivalent to the requirement of having complexity degree two, and we succeed in generalize his reconstruction algorithm to domino tilings of complexity degree three and four. However, the general problem of the reconstruction of a domino tiling from two projections is still open.

A possible strategy for proving the NP-completeness of deciding whether a domino tiling is consistent with a couple of vectors of projections is shown in [1]. It requires one to map an instance $I$ of the NP-complete problem 4-colors (see [2]), into a domino tiling displaying particular internal structures. The reduction can be performed by substituting to each color in $I$ an ad hoc chosen subtiling. This strategy has produced good results for tilings composed by pieces different from dominoes, and it seems very promising for domino tilings too. Our work, however, prevents the mapping of the colors into “too simple” domino tiling structures, i.e. having complexity degrees less than or equal to four.

This paper is organized as follows: In Section 2 we give some general definitions and notations, in particular we introduce the notion of complexity degree of a domino tiling, and we give a polynomial greedy strategy for reconstructing a domino tiling from one projection; in Section 3 we introduce the problem of the reconstruction of a domino tiling from two projections, we give a general strategy for its solution, and then we solve the problem for domino tilings of degree three. Finally, in Section 4, we generalize the results of the previous section to domino tilings of degree four.

2. Definitions, notations and a simple result

Our environment is the cartesian plane, where we define cell as a unitary square having integral vertices. A horizontal domino is the union of two horizontally adjacent cells, and a vertical domino is the union of two vertically adjacent cells. A rectangular domino tiling, in the sequel simply tiling, is a complete tiling of a rectangle made with horizontal and vertical dominoes. We indicate with $T_{m \times n}$ the set of all the tilings of a rectangle of dimension $m \times n$, i.e. having $m$ rows and $n$ columns.

For each element $T \in T_{m \times n}$, let $r_i$ and $c_j$ be the number of dominoes that cover at least one cell in the $i$th row and one cell in the $j$th column of $T$, respectively, with $1 \leq i \leq m$ and $1 \leq j \leq n$. We define the vectors $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ to be the row and column projections of $T$. Furthermore, we say that $T$ is consistent with $R$ and $C$. It is often more useful to consider the vector $R^h = (r_1^h, \ldots, r_m^h)$, whose generic element $r_i^h = n - r_i$ is
still independent from the chosen tiling $T$ and it stores the number of horizontal dominoes which intersect the $i$th row of $T$. In a similar fashion we define the vector $C^h$.

As an example, the projections of the tiling in Fig. 1 are

$$R = (6, 6, 5, 7, 7, 6), \quad R^h = (2, 2, 3, 1, 1, 2),$$

$$C = (4, 5, 6, 6, 5, 4, 4), \quad C^h = (1, 3, 5, 5, 3, 1, 1).$$

A vertical domino covering positions $(i, j)$ and $(i + 1, j)$ is said to start on line $i$ and end on line $i + 1$. So, we associate the generic row $i$ of $T$ to the couple $(s_i, e_i)$, whose entries indicate the number of its starting and ending dominoes, respectively. It is easy to observe that each couple is computed independently from $T$, by using vector $R$, and equations

$$(s_1, e_1) = (2r_1 - n, 0), \quad s_{i+1} = 2r_{i+1} - n - s_i \quad \text{and} \quad e_{i+1} = s_i,$$

for each $1 \leq i < m$. Let us finally define the vector $H = ((s_1, e_1), \ldots, (s_m, e_m))$.

In Fig. 1 we have $H = ((4, 0), (0, 4), (2, 0), (4, 2), (2, 4), (4, 2), (0, 4))$.

The vector $H$ contains some crucial information about the set of tilings with which it is consistent, and it shows a possible way of decomposing them into “simpler” parts, i.e. into complete subtilings having fewer rows. More precisely, let $T$ be consistent with $H$. If there exist two elements $(s_i, e_i), (s_{i'}, e_{i'}) \in H$ such that $i \leq i'$, $e_i = 0$ and $s_{i'} = 0$, then we define (horizontal) strip of height $i' - i + 1$ the complete tiling composed by the rows $i, i + 1, \ldots, i'$ of $T$. Obviously, each tiling can be considered as a single strip of height $m$.

The vector $STR = (str_1, \ldots, str_k)$, with $k \leq m$, is defined to be the decomposition of the tiling $T$ into strips if:

- $str_1, \ldots, str_k$ are strips of $T$;
- each row of $T$ belongs to a strip of $STR$;
- $str_1, \ldots, str_k$ are consecutive strips of $T$;
- for each $1 \leq i, j \leq k$, $i \neq j$, $str_i$ and $str_j$ do not share any row of $T$;
- each strip of $STR$ cannot be further decomposed into strips.

In the sequel, each element of $STR$ will be denoted with the sequence of its row projections. As an example the vector $STR = ((6, 6), (5, 7, 7, 6))$ is the decomposition into
strips of the tiling of Fig. 1. We define the complexity degree of the tiling $T$, $\text{deg}(T)$, as follows:

- if $T$ cannot be decomposed into two or more strips, then $\text{deg}(T) = m$;
- if $\text{STR} = (\text{str}_1, \ldots, \text{str}_k)$ is the decomposition of $T$ into strips, then
  \[
  \text{deg}(T) = \max\{\text{deg}(\text{str}_1), \ldots, \text{deg}(\text{str}_k)\}.
  \]

Since the complexity degree is independent from the chosen tiling $T$, then it can be defined on vector $R$ as well. In sequel we will refer to the complexity both of $T$ and $R$. The complexity degree of the tiling in Fig. 1 is five.

The structure of the paper intends to underline once more the possibility of a dramatic increase in the difficulty of the reconstruction of discrete tiling configurations when a priori information are added, i.e. the knowledge of their projections.

2.1. Reconstruction from one projection: an easy task

A natural greedy algorithm can be defined in order to reconstruct a tiling $T \in \mathcal{T}_{m \times n}$ from the vector $R$ of its row projections:

**RECONSTRUCTION ($\mathcal{T}_{m \times n}, R$)**

**Input:** a vector $R = (r_1, \ldots, r_m)$ and an integer $n$.

**Output:** a tiling $T \in \mathcal{T}_{m \times n}$ consistent with $R$, if it exists, else return FAILURE.

**Procedure:**

**Step 1:** create the partial tiling $T$ of dimension $m \times n$ which contains no domino pieces, and initialize all its entries to a blank value;

**Step 2:** for $1 \leq i \leq m$:

1. place $n - r_i$ horizontal dominoes in the leftmost blank positions of row $i$ of $T$;
   if possible, else return FAILURE;

2. in each blank position of row $i$, place a (starting) vertical domino;

**Step 3:** return $T$ as output.

The following example shows a run of the reconstruction procedure:

**Example 1.** Let us consider instance $I$ of RECONSTRUCTION ($\mathcal{T}_{m \times n}, R$) such that

$n = 9$ and $R = (6, 8, 7, 6, 7, 6)$

and let us follow the computation of one of its solution, say $T$:

**Step 1:** the partial tiling $T$ containing no dominoes is created and all its entries are initialized to a blank value.

**Step 2:** for each $1 \leq i \leq 6$, $n - r_i$ horizontal dominoes are placed in the leftmost blank positions of row $i$ of $T$, then each remaining blank position is tiled with a vertical domino.
Fig. 2 shows the tiling $T$ after Step 2, for each $1 \leq i \leq 6$; 

**Step 3:** the complete tiling $T$ is returned as solution of $I$.

The correctness of RECONSTRUCTION ($\mathcal{T}_{m \times n}, R$) is an immediate consequence of the following two simple lemmas:

**Lemma 2.** For each $1 \leq i \leq m$, the reconstruction strategy allows to fill the $i$th row of $T$ with the maximum admitted number of horizontal dominoes, i.e., $\lfloor (n - e_i)/2 \rfloor$ ($e_i$ being the number of ending vertical dominoes of row $i$).

**Theorem 3.** The following statements are equivalent:

1. RECONSTRUCTION ($\mathcal{T}_{m \times n}, R$) does not return FAILURE;
2. there is a tiling consistent with $R$;
3. it holds that

$$0 \leq r^h_i \leq \lfloor (n - e_i)/2 \rfloor \quad \text{and} \quad s_m = 0,$$

for $1 \leq i \leq m$.

The proof is immediate.

**Theorem 4.** The problem RECONSTRUCTION ($\mathcal{T}_{m \times n}, R$) can be solved in pseudo-polynomial time.
3. Reconstruction from two projections: a hard task

3.1. Previous results

The authors of [3,7] found a polynomial time algorithm for reconstructing particular subclasses of $\mathcal{T}_{n \times m}$ from the row and column projections.

In particular, in [3] it was proved how to tile a rectangle with dominoes when these two vectors of projections are known and homogeneous (all the entries of each of them have the same value).

In [7], Picouleau generalized the previous result by considering horizontally decreasing tilings, i.e. tilings whose row projections form a weakly decreasing sequence. He proved that an horizontally decreasing tiling can be divided into strips of height at most two, i.e. it has degree two, and then he furnished a one-to-one polynomial correspondence between horizontally decreasing tilings and horizontal domino packings.\footnote{A domino packing is a partial tiling, usually of a rectangle, which involves only horizontal dominoes. The row and column projections count the number of dominoes on each row and column, respectively.} Fig. 3 helps to understand such a correspondence: each strip of the tiling is associated with a row of the packing by mapping a vertical domino into a single cell, and filling the remaining positions of the packing with horizontal dominoes. Since the reconstruction of horizontal domino packings from two projections can be performed in polynomial time by using a standard greedy technique, the same result holds for horizontally decreasing tilings (see [7, Theorem 2, p. 443]).

3.2. The reconstruction of tilings of degree two

Some results which generalize those of [7] are presented hereafter, without solving the general problem of the reconstruction of a tiling from two projections. Let us indicate with $\mathcal{T}_{m \times n}^{(d)}$ the class of the tilings of dimension $m \times n$ and complexity degree $d$.

The algorithm presented in Paragraph 5 of [7], which reconstructs an horizontally decreasing tiling from the projections $R$ and $C$, can be applied to all tilings which have complexity...
degree two:

**RECONSTRUCTION** \((\mathcal{T}_{n \times m}^{(2)}, (R, C))\)

*Input*: two vectors \(R = (r_1, \ldots, r_m)\) and \(C = (c_1, \ldots, c_n)\).

*Output*: a tiling \(T \in \mathcal{T}_{m \times n}^{(2)}\) consistent with \(R\) and \(C\), if it exists, else return \text{FAILURE}.

*Procedure*:

*Step 1*: compute the decomposition \(\text{STR} = (\text{str}_1, \ldots, \text{str}_k)\) of \(R\) into strips of height at most two;

*Step 2*: apply the algorithm described Paragraph 5 of [7] for the reconstruction of a domino tiling from two projections with input \(R\) and \(C\), and let \(T\) be its output, if it exists, else return \text{FAILURE}; Give \(T\) as output.

The correctness of the procedure immediately follows from Theorem 2 of [7, p. 443].

### 3.3. A general strategy

Now we present a general strategy to solve \(\text{RECONSTRUCTION} (\mathcal{T}_{n \times m}^{(k)}, (R, C))\), i.e. the problem of reconstructing a tiling of arbitrary degree \(k\) from the two vectors of row and column projections \(R\) and \(C\), and then we successfully apply it to the cases \(k = 3\) and \(k = 4\).

The strategy consists in the following three steps process whose input are an integer \(k\) and vectors \(R\) and \(C\), and whose output is a tiling of degree \(k\) consistent with \(R\) and \(C\), if it exists. Let \(k > 2\) and \(k' < k\):

**RECONSTRUCTION** \((\mathcal{T}_{n \times m}^{(k)}, (R, C))\)

*Step 1*: define a polynomial time transformation, say \(\text{Reduction}(k \rightarrow k')\), which maps \(R\) and \(C\) into two vectors \(R'\) and \(C'\), such that \(R'\) has degree \(k' < k\);

*Step 2*: apply **RECONSTRUCTION** \((\mathcal{T}_{n \times m}^{(k')}, (R', C'))\);

*Step 3*: define a polynomial time transformation, say \(\text{Expansion}(k' \rightarrow k)\), which transforms the tiling reconstructed in Step 2, into a tiling of degree \(k\) consistent with \(R\) and \(C\).

In this paper we set \(k' = k - 1\), and we examine the cases \(k = 3\) and \(k = 4\). In particular, for both of them:

**Reduction** \((k \rightarrow k')\) uses a polynomial correspondence between tilings of degree \(k\) and tilings of degree \(k - 1\), indicated by \(\phi^{(k)}\), in order to transform the couple of vectors \(R\) and \(C\) into \(R'\) and \(C'\), respectively. Since the transformation is independent from the chosen starting tiling, it had to be based on specific properties of the class of tilings of degree \(k\), as we will point out in the sequel;

**Expansion** \((k' \rightarrow k)\) changes back a tiling of degree \(k - 1\) into a tiling of degree \(k\), by inverting in some sense, the transformation \(\phi^{(k)}\). This process works directly on the tiling of degree \(k - 1\) by adding or moving domino pieces, with the aim of satisfying the constraints imposed by \(R\) and \(C\).

### 3.4. The reconstruction of tilings of degree three

A simple lemma leads us to the definition both of **Reduction** \((3 \rightarrow 2)\) and **Expansion** \((2 \rightarrow 3)\), and consequently to the reconstruction of tilings of degree three:
Lemma 5. Let \( \text{str} \) be a tiling of dimension \( 3 \times n \) consistent with the projections \( R = (r_1, r_2, r_3) \) and \( C = (c_1, \ldots, c_n) \). There exist two transformations \( \phi^{(3)} \) and \( \psi^{(3)} \) such that

1. \( \phi^{(3)} : \mathcal{T}_{3 \times n} \rightarrow \mathcal{T}_{2 \times n} \), and \( \phi^{(3)}(\text{str}) \) is consistent with \( R' = (r_2, r_2) \) and \( C' = (c_1 - 1, \ldots, c_n - 1) \);
2. \( \psi^{(3)} : A^{(3)} \subseteq \mathcal{T}_{2 \times n} \rightarrow \mathcal{T}_{3 \times n} \), where \( A^{(3)} = \{ \phi^{(3)}(\text{str}) : \text{str} \in \mathcal{T}_{3 \times n} \} \), and \( \psi^{(3)}(\phi^{(3)}(\text{str})) \) is consistent with \( R \) and \( C \);

Proof. We define the transformation \( \phi^{(3)} \) as follows: let \( \phi^{(3)}(\text{str}) = \text{str}' \), and, for each \( j, 1 \leq j \leq n \):

- If \( \text{str}[2, j] \) is covered with an horizontal starting or ending domino, then we cover the two positions \( \text{str}'[1, j] \) and \( \text{str}'[2, j] \) with two horizontal starting or ending dominoes, respectively.
- If \( \text{str}[2, j] \) is covered with a starting or ending vertical domino, then we cover the two positions \( \text{str}'[1, j] \) and \( \text{str}'[2, j] \) with a vertical domino.

It is easy to check that the defined tiling exists, it is complete, and it is consistent with \( R' \) and \( C' \).

Now we define the transformation \( \psi^{(3)} \) as follows: let \( \psi^{(3)}(\text{str}') = \text{str}'' \), and \( j_0 \) be the column, where lies the \( 2(r_2 - r_3) \)th vertical domino of \( \text{str}' \). For each \( j, 1 \leq j \leq n \):

- If \( j \leq j_0 \) then \( \text{str}''[1, j] \) and \( \text{str}''[2, j] \) are covered with the same type of dominoes as \( \text{str}'[1, j] \) and \( \text{str}'[2, j] \). The position \( \text{str}''[3, j] \) is covered with an horizontal starting domino if \( j \) is odd, with an horizontal ending domino otherwise.
- If \( j > j_0 \) then \( \text{str}''[2, j] \) and \( \text{str}''[3, j] \) are covered with the same type of dominoes as \( \text{str}'[1, j] \) and \( \text{str}'[2, j] \). The position \( \text{str}''[1, j] \) is covered with an horizontal starting domino if \( j \) is odd, with an horizontal ending domino otherwise.

To prove that the tiling \( \text{str}'' \) is well defined, it is sufficient that \( 2(r_2 - r_3) \) vertical dominoes are present in \( \text{str}' \). Since the number of vertical dominoes in the 2nd row of \( \text{str}' \) is \( r_2 - (n - r_2) = 2r_2 - n \), and \( 2r_3 \geq n \), then \( 2r_2 - n \geq 2(r_2 - r_3) \), as desired (see Fig. 4).

Furthermore, it is simple to check that \( \text{str}'' \) is complete.

The projection of the first row of \( \text{str}'' \) can be computed by adding the \( 2(r_2 - r_3) \) vertical dominoes that, by construction, are contained in the first \( j_0 \) columns of \( \text{str}'' \), to the horizontal dominoes whose number is \( n/2 - (r_2 - r_3) \), so that we obtain the value \( (r_2 - r_3) + n/2 = r_1 \).

The projections of row 2 and row 3 can be computed as well, and their values are \( r_2 \) and \( r_3 \). So \( \text{str}'' \) is consistent with \( R \). Its consistency with \( C \) follows by construction (see Fig. 4).

\( \square \)

Using the transformations \( \phi^{(3)} \) and \( \psi^{(3)} \), we can naturally define:

**Reduction(3 \rightarrow 2)**

**Input:** a couple of vectors \( R = (r_1, \ldots, r_m) \) and \( C = (c_1, \ldots, c_n) \).

**Output:** a couple of vectors \( R' \) and \( C' \) such that \( R' \) has degree 2.

**Procedure:**

**Step 1:** compute the decomposition \( \text{STR} = (\text{str}_1, \ldots, \text{str}_k) \) of \( R \) into strips, and let \( k^{(3)} \) be the number of those of degree three;
Step 2: create the vector $R' = R$, and for each strip $str_i$ of degree three, substitute in $R'$ its projections with the two row projections of the strip $\varphi^{(3)}(str_i)$. At the end of the substitutions $R'$ has its dimension reduced to $m' = m - k^{(3)}$;

Step 3: return $R'$ and $C' = (c_1 - k^{(3)}, \ldots, c_n - k^{(3)})$.

**Expansion (2 → 3)**

**Input:** a tiling $T'$ of degree 2 consistent with $R'$ and $C'$.

**Output:** a tiling $T$ of degree three consistent with $R$ and $C$.

**Procedure:**

- **Step 1:** create the tiling $T = T'$ and decompose it into strips;
- **Step 2:** apply the transformation $\psi^{(3)}$ to each strip which has been previously modified by REDUCTION(3 → 2);
- **Step 3:** Return tiling $T$ as output.

The details of the two procedures are trivial, and so omitted.

The correctness of the algorithm which uses the two procedures in order to solve RECONSTRUCTION ($\mathcal{F}^{(3)}_{m \times n}$, $(R, C)$) follows from Lemma 5, so it holds:

**Theorem 6.** Let $R$ and $C$ be two integral vectors. The reconstruction of a tiling of degree three having $R$ and $C$ as vectors of row and column projections can be performed in polynomial time.

**Example 7.** The reconstruction of a tiling $T$ consistent with the vectors of projections $R = (6, 8, 8, 8, 9, 7, 7, 8, 7)$ and $C = (8, 9, 7, 8, 8, 7, 7, 6, 8, 8, 9, 9)$:
Fig. 5. The reconstruction of $T'$, matrix (a), and $T$, matrix (b).

**REDUCTION** $(3 \rightarrow 2)$

*Step 1:* from vector $R$ compute the vector $STR = ((6), (8, 8), (8, 9, 7), (7, 8, 7))$;

*Step 2:* apply $\varphi^{(3)}$ to the last two strips of $R$, and obtain $R' = (6, 8, 8, 9, 9, 8, 8)$;

*Step 3:* compute $C' = (6, 7, 5, 6, 6, 5, 4, 6, 6, 7, 7)$.

**RECONSTRUCTION** $(\mathcal{F}_{7 \times 12}, (R', C'))$: let $T'$ be its output, as shown in Fig. 5(a).

**EXPANSION** $(2 \rightarrow 3)$

*Step 1* and *Step 2*: create matrix $T = T'$ and apply $\psi^{(3)}$ on its last two strips, as shown in Fig. 5(b);

*Step 3*: return matrix $T$, consistent with $R$ and $C$, as output.

4. Tilings of degree four

Also for tilings of degree four, specific properties of their internal structure both allow the definitions of Reduction $(4 \rightarrow 2)$ and Expansion $(2 \rightarrow 4)$, and prevent their generalization to tilings of higher degrees.

**Lemma 8.** Let $str$ be a tiling of dimension $4 \times n$. It can be divided into $k \leq m$, subtilings $B_1, \ldots, B_k$, say blocks, of maximum dimensions $4 \times b_1, \ldots, 4 \times b_k$ such that $b_1 + b_2 + \cdots + b_k = n$ and each block contains only zero vertical dominoes or two vertical dominoes, one in its first and one in its last column (if a column contains two vertical dominoes, then it constitutes a block whose first and last columns are considered coincident).

**Proof.** We proceed by induction on the number $v$ of vertical dominoes in $str$, after observing that $v$ cannot assume odd values.

*Base $v = 0$: $str$ is composed by a single block $B_1$ of horizontal dominoes of length $n$. 

*Induction step:* Assume that the statement holds for any tiling of degree less than $4$. Consider a tiling $str$ of degree four. Let $v$ be the number of vertical dominoes in $str$. By the induction hypothesis, $str$ can be divided into $k$ subtilings $B_1, \ldots, B_k$ of maximum dimensions $4 \times b_1, \ldots, 4 \times b_k$. If $v < 4$, then $str$ is already a valid tiling. If $v = 4$, then $str$ is a valid tiling by construction. If $v > 4$, then $str$ contains at least one vertical domino in its first or last column. Without loss of generality, assume that $str$ contains at least one vertical domino in its first column. Let $B_1$ be the first block of $str$ containing a vertical domino. Then $B_1$ must be a block of horizontal dominoes of length $n$, and the remaining vertices can be tiled by applying Lemma 8 to the remaining tiling $str' = str - B_1$. If $v = 4$, then we are done. If $v > 4$, then we can repeat the process on $str'$ until we obtain a tiling $str''$ of degree less than $4$, which can be tiled by applying the induction hypothesis.
Step \( v \to v + 2 \): let column \( j \) contain the \((v + 1)\)th vertical domino of \( str \) (the vertical dominoes are counted from left to right). By inductive hypothesis, the first \( j - 1 \) columns of \( str \) form a complete subtiling which can be divided into blocks, so there are no horizontal dominoes ending in column \( j \). Since \( str \) is complete, there exists a column \( j \leq j' \leq n \) which contain a vertical domino, i.e. the \((v + 2)\)th one, and which does not contain any starting horizontal domino. So, columns from \( j \) to \( j' \) form a complete sub-tiling, i.e. a block, and the same holds from column \( j' \) to column \( n \) (if \( j < n \)), where no vertical dominoes are present (see Fig. 6). □

In \( str \), the vertical dominoes which start in rows 1, 2 and 3 are called high, central and low dominoes, respectively. As a direct consequence of Lemma 8, it holds that a tiling of dimension \( 4 \times n \) has an even number of central dominoes, and the same holds for the sum of its high and low dominoes. Furthermore, we have:

**Lemma 9.** For each even integer number \( b \), there exists a block of dimension \( 4 \times b \) containing two vertical dominoes of the same kind, i.e. high, central or low ones.

**Proof (Sketch).** We construct a \( 4 \times b \) block \( B \) containing two high dominoes (if we choose to construct a \( 4 \times b \) block containing two central or low dominoes the process is analogous) by simply placing a vertical domino in positions \( B[1, 1] \) and \( B[1, 2] \), and the second one in positions \( B[b, 1] \) and \( B[b, 2] \). Then we fill the remaining positions with horizontal dominoes. □

The property of tilings of degree four stated in Lemmas 8 and 9 allows us to define two transformations which act on a generic \( 4 \times n \) tiling by lowering and rising back its degree without changing its row and column projections. So, in the same fashion of what previously stated for tilings of dimension \( 3 \times n \) in Lemma 5, it holds:

**Lemma 10.** Let \( str \) be a \( 4 \times n \) tiling consistent with \( R = (r_1, r_2, r_3, r_4) \) and \( C = (c_1, \ldots, c_n) \). There exist two transformations \( \phi^{(4)} \) and \( \psi^{(4)} \) such that

1. \( \phi^{(4)} : \mathcal{T}_{4 \times n} \rightarrow \mathcal{T}_{4 \times n} \), and \( \phi^{(4)}(str) \) is consistent with \( R' \) and \( C' \), where \( C' = C \) and \( R' = (r_1', r_2', r_3', r_4') \) is defined as follows:
   - if \( r_1 \geq r_4 \) then \( r_1' = r_2' = r_2 \) and \( r_3' = r_4' = r_4 \),
   - else \( r_1' = r_2' = r_1 \) and \( r_3' = r_4' = r_3 \).
(2) $\psi^{(4)} : A^{(4)} \subseteq T_{4 \times n} \to T_{4 \times n}$, where $A^{(4)} = \{\varphi^{(4)}(\text{str}) : \text{str} \in T_{4 \times n}\}$, and $\psi^{(4)}(\varphi^{(4)}(\text{str}))$ is consistent with $R$ and $C$.

**Proof.** Let $r_1 \geq r_4$ (if $r_1 < r_4$ a similar proof holds), and let $B_1, \ldots, B_k$ be the division of $\text{str}$ into $k$ blocks having dimensions $4 \times b_1, \ldots, 4 \times b_k$, respectively, as stated in Lemma 8. Obviously it holds that $r_2 \geq r_1$.

We define the transformation $\varphi^{(4)}$, whose input is $\text{str}$ and whose output is a $4 \times n$ tiling $\text{str}'$, as follows:

\[
\varphi^{(4)}(\text{str}) \\
\text{Step 1: create the tiling } \text{str}' = \text{str}; \\
\text{Step 2: for each block } j \text{ of } \text{str}', \text{ with } 1 \leq j \leq k, \\
\quad \text{if } B_j \text{ contains two central dominoes then change } B_j \text{ with} \\
\quad \quad \text{a block of the same dimension and having two high dominoes;} \\
\quad \quad \text{else leave the block } B_j \text{ unchanged; } \\
\text{Step 3: return } \text{str}' \text{ as output.}
\]

Since the number of columns in a block containing two central dominoes is even, Lemma 9 assures that it can be replaced by a block of the same dimension and containing two high dominoes. As a consequence $\text{str}'$ is well defined and complete.

Furthermore, $\text{str}'$ is consistent with $R'$, since each replacement of a block containing two central dominoes with one containing two high dominoes lowers the projection of its third row by one and rises the projection of its first row by one, so exactly $r_2 - r_1$ dominoes are added to the first row and subtracted from the third row.

It is also immediate to check that the replacements of the blocks do not affect the column projections of $\text{str}'$, so $\text{str}'$ is also consistent with $C' = C$ (see Fig. 7). Finally, at the end of the process $\text{str}'$ does not contain any central domino, so it has degree two.

Now we define the transformation $\psi^{(4)}$ whose input is $\text{str}' \in A^{(4)}$, and whose output is a $4 \times n$ tiling $\text{str}''$. Let $\Lambda = r_2 - r_1$, and let $B'_1, \ldots, B'_k$ be the division of $\text{str}'$ into blocks having dimensions $4 \times b_1, \ldots, 4 \times b_k$, respectively:

\[
\psi^{(4)}(\text{str}') \\
\text{Step 1: create the tiling } \text{str}'' = \text{str}'; \\
\text{Step 2: for each } j, 1 \leq j \leq k, \\
\quad \text{if } B'_j \text{ contains two upper dominoes and } \Lambda > 0 \text{ then change } B'_j \text{ with a block of the} \\
\quad \quad \text{same dimension and having two central dominoes, and decrease } \Lambda \text{ by one; } \\
\quad \quad \text{else leave the block } B'_j \text{ unchanged; } \\
\text{Step 3: return } \text{str}'' \text{ as output.}
\]

Again, Lemma 9 assures that $\text{str}''$ is well defined, it is complete, and it is consistent with $C$ and $R$, since exactly $r_2 - r_1$ dominoes are added to the third row and subtracted from the first row (see Fig. 7).
Using the transformations $\varphi^{(4)}$ and $\psi^{(4)}$, and in the same way as it was done for tilings of degree three, we define REDUCTION($4 \rightarrow 3$) and Expansion($3 \rightarrow 4$) whose correctness is assured by Lemma 10. So it holds:

**Theorem 11.** Let $R$ and $C$ be two integral vectors. The reconstruction of a tiling of degree four having $R$ and $C$ as vectors of row and column projections can be performed in polynomial time.

The following example shows the reconstruction algorithm for a tiling of degree four.

**Example 12.** The reconstruction of a tiling $T$ consistent with the vectors of projections $R = (8, 8, 7, 9, 8, 7, 9, 10, 8)$ and $C = (5, 7, 9, 8, 7, 8, 7, 8, 7, 6)$:

1. REDUCTION($4 \rightarrow 3$)
2. Step 1: from vector $R$ compute the vector $\text{STR} = (8, 8, 7, 9, 8, 7, 9, 10, 8)$;
3. Step 2 and Step 3: apply $\varphi^{(4)}$ to the last strip of $R$, and obtain the degree three vector $R' = (8, 8, 7, 9, 8, 7, 9, 10, 10)$. Let $C' = C$;
4. RECONSTRUCTION($\mathcal{F}^{(3)}_{9 \times 12}$), $(R', C')$: let $T'$ be its output, as shown in Fig. 8(a);
5. EXPANSION($3 \rightarrow 4$)
6. Step 1 and Step 2: create matrix $T = T'$ and apply $\psi^{(4)}$ on its last two strips, as shown in Fig. 8(b);
7. Step 3: return matrix $T$, consistent with $R$ and $C$, as output.
5. Final comments

Specific properties of tilings of degree less than or equal four have been used in their reconstruction process. Now our researches will focus on the solution of the general problem by following, at the same time, two distinct ways: from one side, we try to generalize these properties to tilings of degree greater than four, in order to obtain a strategy which allows us to quickly reconstruct an element of $\mathcal{T}_{m \times n}$, from the other side we search for a proof which states the non polynomiality of the reconstruction problem when high degree tilings are involved. Our first attempt was to extend the result holding for tilings of degree three to those of degree five: it is always possible to find a line of horizontal dominoes going through a strip of degree five, but unfortunately we cannot delete it and automatically compute the row projections of the obtained strip of degree four. Obviously an exhaustive search for such projections will lead to an exponential time process.

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References

