Arrow Categories

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Content

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Binary (Boolean valued) relation (Category Rel)

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

Fuzzy relation (Category Rel([0,1]))

\[
\begin{pmatrix}
0.1 & 0.8 & 0.0 \\
1.0 & 0.4 & 0.9 \\
0.0 & 0.2 & 0.1
\end{pmatrix}
\]
$L$-fuzzy relation ($L$ a complete distributive lattice, Category Rel($L$))

\[
\begin{pmatrix}
1 & k & l \\
0 & k & m \\
0 & 1 & l
\end{pmatrix}
\]

\[
L =
\begin{array}{c}
1 \\
l \\
k \\
0
\end{array}
\]

\[
\begin{array}{c}
l \\
m
\end{array}
\]
Dedekind categories

Definition: A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects $A$ and $B$ the collection $\mathcal{R}[A,B]$ is a complete distributive lattice (complete Heyting algebra). Meet, join, the induced ordering, the least and the greatest element are denoted by $\cap, \cup, \subseteq, \sqsubseteq_{AB}, \sqsupseteq_{AB}$, respectively.

2. There is a monotone operation $\sim$ (called converse) such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds

   $$(Q;R) \sim = R \sim ; Q \sim, \quad (Q \sim) \sim = Q.$$ 

3. For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the
modular law holds:

$$Q; R \cap S \sqsubseteq Q; (R \cap Q^\prec; S).$$

4. For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of $S$ and $R$) such that for all $Q : A \to B$ the following holds

$$Q; R \sqsubseteq S \iff Q \sqsubseteq S/R.$$
Definition (Matrix category)

Let $\mathcal{R}$ be a Dedekind category. The category $\mathcal{R}^+$ of matrices with coefficients from $\mathcal{R}$ is defined by:

1. The class of objects of $\mathcal{R}^+$ is the collection of all functions from an arbitrary set $I$ into the class of objects $\text{Obj}_\mathcal{R}$ of $\mathcal{R}$.

2. For every pair $f : I \to \text{Obj}_\mathcal{R}, g : J \to \text{Obj}_\mathcal{R}$ of objects from $\mathcal{R}^+$, a morphism $R : f \to g$ is a function from $I \times J$ into the class of all morphisms $\text{Mor}_\mathcal{R}$ of $\mathcal{R}$ such that $R(i, j) : f(i) \to g(j)$ holds.

3. For $R : f \to g$ and $S : g \to h$ composition is defined by

$$ (R; S)(i, k) := \bigsqcup_{j \in J} R(i, j); S(j, k). $$
4. For $R : f \to g$ conversion defined by

$$R \sim (j, i) := (R(i, j)) \sim.$$ 

5. For $R, S : f \to g$ join and meet are defined by

$$\begin{align*}
(R \sqcup S)(i, j) & := R(i, j) \sqcup S(i, j), \\
(R \sqcap S)(i, j) & := R(i, j) \sqcap S(i, j).
\end{align*}$$

6. The identity, zero and universal elements are defined by

$$\begin{align*}
\mathbb{I}_f(i_1, i_2) & := \begin{cases} \\
\perp_{f(i_1)f(i_2)} : i_1 \neq i_2 \\
\mathbb{I}_{f(i_1)} : i_1 = i_2,
\end{cases} \\
\perp_{fg}(i, j) & := \perp_{f(i)g(j)}, \\
\top_{fg}(i, j) & := \top_{f(i)g(j)}.
\end{align*}$$
Some results

Lemma: $\mathcal{R}^+$ is a Dedekind category.

Corollary: Let $L = (L, \lor, \land, 0, 1)$ be a complete distributive lattice with least element 0 and greatest element 1. Then $L$ is an one-object Dedekind category with identity 1 and composition $\land$ (the residual is given by the pseudo-complement). Consequently, $L^+$ is a Dedekind category, called the full category of $L$-relations.
**Lemma:** The collection of scalar relations on $A$, i.e., the relations $k : A \to A$ with $k \subseteq I_A$ and $\top_{AA}; k = k; \top_{AA}$, constitutes a complete distributive lattice.

**Example:**

\[
\begin{pmatrix}
  k & 0 & 0 \\
  0 & k & 0 \\
  0 & 0 & k
\end{pmatrix}
\]

**Theorem:** There is no formula $\varphi$ in the language of Dedekind categories such that for all lattices $L$ and $L$-relations $R : A \to B$ we have

\[ L^+ \models \varphi[R] \iff R \text{ is 0-1 crisp.} \]
Goguen categories

**Definition:** A Goguen category \( \mathcal{G} \) is a Dedekind category with \( \bot_{AB} \neq \top_{AB} \) for all objects \( A \) and \( B \) together with two operations \( \uparrow \) and \( \downarrow \) satisfying the following:

1. \( R^\uparrow, R^\downarrow : A \to B \) for all \( R : A \to B \).

2. \((\uparrow, \downarrow)\) is a Galois correspondence, i.e., \( R^\uparrow \sqsubseteq S \iff R \sqsubseteq S^\downarrow \) for all \( R, S : A \to B \).

3. \((R^\cap; S^\downarrow)^\uparrow = R^\uparrow \cap; S^\downarrow \) for all \( R : B \to A \) and \( S : B \to C \).

4. If \( \alpha \neq \bot_{AA} \) is a nonzero scalar then \( \alpha^\uparrow = \mathbb{I}_A \).
\[
\begin{pmatrix}
1 & k & l \\
0 & k & m \\
0 & 1 & l \\
\end{pmatrix}
\]

\[
L =
\begin{pmatrix}
1 & l \\
0 & m \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & k & l \\
0 & k & m \\
0 & 1 & l \\
\end{pmatrix} 
\uparrow =
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & k & l \\
0 & k & m \\
0 & 1 & l \\
\end{pmatrix} 
\downarrow =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]
4. For all functions $f$ so that $f(\bigcup M) = \prod_{\alpha \in M} f(\alpha)$ for all sets of scalars and $f(\alpha)^\uparrow = f(\alpha)$ for all scalars the following equivalence holds

$$R \sqsubseteq \bigsqcup_{\alpha: A \rightarrow A} \alpha; f(\alpha) \iff (\alpha \setminus R)^\downarrow \sqsubseteq f(\alpha)$$

for all scalars $\alpha$.

\[
\begin{pmatrix}
  l & 0 & 0 \\
  0 & l & 0 \\
  0 & 0 & l
\end{pmatrix}
\setminus
\begin{pmatrix}
  1 & k & l \\
  0 & k & m \\
  0 & 1 & l
\end{pmatrix}
\downarrow
\begin{pmatrix}
  1 & k & 1 \\
  0 & k & m \\
  0 & 1 & 1
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 1 \\
  0 & 0 & 0 \\
  0 & 1 & 1
\end{pmatrix}
\]
Some results

**Theorem:** Let $L$ be a complete distributive lattice. Then $L^+$ together with the operations

\[
R^\uparrow(x, y) := \begin{cases} 
1 & \text{iff } R(x, y) \neq 0 \\
0 & \text{iff } R(x, y) = 0 
\end{cases},
\]

\[
R^\downarrow(x, y) := \begin{cases} 
1 & \text{iff } R(x, y) = 1 \\
0 & \text{iff } R(x, y) \neq 1
\end{cases},
\]

is a Goguen category. Furthermore, for a relation $R$ in $L^+$ we have $R^\uparrow = R$ iff $R$ 0-1 crisp.
Lemma: For each pair of objects $A$ and $B$ the set of scalar elements on $A$ resp. on $B$ are isomorphic lattices.

Lemma: Let $G$ be a Goguen category and $R : A \to B$ be a relation. Then we have

1. $\bigsqcup_{\alpha \text{ scalar}} \alpha_A; (\alpha_A \setminus R)^\downarrow = R,$
2. $\bigsqcup_{\alpha_A \text{ scalar } \alpha_A \neq \bot_A} (\alpha_A \setminus R)^\downarrow = R^\uparrow.$
Theorem (Pseudo-representation Theorem): Every Goguen category $\mathcal{G}$ is isomorphic to the category of antimorphisms mapping the scalars of $\mathcal{G}$ to the crisp relations of $\mathcal{G}$.

Corollary: A Goguen category is representable iff its subcategory of crisp relations is representable.
Further results/studies of Goguen categories

1. Definability of norm-based operations;

2. Validity of certain formulae in the subcategory of crisp relations;

3. Applications in computer science, e.g., fuzzy controller;

4. ...
**Arrow categories**

**Definition:** An arrow category $\mathcal{A}$ is a Dedekind category with $\top_{AB} \neq \bot_{AB}$ for all objects $A$ and $B$ together with two operations $\uparrow$ and $\downarrow$ satisfying the following:

1. $R^\uparrow, R^\downarrow : A \rightarrow B$ for all $R : A \rightarrow B$.
2. $(\uparrow, \downarrow)$ is a Galois correspondence.
3. $(R^\sim; S^\downarrow)^\uparrow = R^\uparrow \sim; S^\downarrow$ for all $R : B \rightarrow A$ and $S : B \rightarrow C$.
4. $(Q \cap R^\downarrow)^\uparrow = Q^\uparrow \cap R^\downarrow$ for all $Q, R : A \rightarrow B$.
5. If $\alpha_A \neq \bot_{AA}$ is a non-zero scalar then $\alpha_A^\uparrow = \mathbb{I}_A$. 
Lemma: For each pair of objects $A$ and $B$ the set of scalar elements on $A$ resp. on $B$ are isomorphic lattices.

Lemma: Let $\mathcal{A}$ be an arrow category and $R : A \to B$ be a relation. Then we have

1. $\bigsqcup \alpha_A; (\alpha_A \setminus R)^\downarrow \subseteq R$, $\alpha$ scalar

2. $\bigsqcup \alpha_A (\alpha_A \setminus R)^\downarrow \subseteq R^\uparrow$. $\alpha_A$ scalar $\alpha_A \neq \bot_A$
Example 1:
Example 2:
Arrow categories with cuts

**Definition:** An arrow category with cuts $\mathcal{A}$ is an arrow category so that

$$R \subseteq \bigsqcup \alpha_A; (\alpha_A \setminus R)^\downarrow$$

for all relations $R : A \rightarrow B$ holds.
Example

\[ L \]

\[
\begin{align*}
x_0 & \\
x_1 & \\
x_2 & \\
x_\infty & \\
0 & 
\end{align*}
\]

\[ \mathcal{R} \]

\[
\begin{align*}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix} & = \top \\
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix} & = \mathbb{I} \\
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\end{pmatrix} & = \bot
\end{align*}
\]
Thank you for your attention.