

Loop Quantization of Polarized Gowdy Model on T^3 : Kinematical States and Constraint Operators

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Abstract

In this second paper on loop quantization of Gowdy model, we introduce the kinematical Hilbert space on which appropriate holonomies and fluxes are well represented. The quantization of the volume operator and the Gauss constraint is straightforward. Imposition of the Gauss constraint can be done on the kinematical Hilbert space to select subspace of gauge invariant states. We carry out the quantization of the Hamiltonian constraint making specific choices. Alternative choices are briefly discussed. It appears that to get spatial correlations reflected in the Hamiltonian constraint, one may have to adopt the so called ‘effective operator viewpoint’.

PACS numbers: 04.60.Pp, 04.60.Kz, 98.80.Jk

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I. INTRODUCTION

Classical General Relativity is known to be incomplete and it is widely believed that its quantum version will address the incompleteness issue. The main and distinctive feature of general relativity is that the space-time geometry is dynamical and responsive to matter/energy distribution. Keeping this feature as central and manifest, a *background independent* quantization strategy has been developed over the past couple of decades and is referred to as Loop Quantum Gravity (LQG) [1, 2]. While some of the novel features of this background independent quantization have been revealed generically, eg discreteness of *Riemannian geometry*, its role in singularity resolution has been seen only in the simplified homogeneous cosmologies [3].

These models are obtained by restricting to field configurations invariant under the action of various groups and go under the name of *symmetry reduced models*. For the homogeneous models, one is left with only a finite number of degrees of freedom already at the kinematical level i.e. before imposition of constraints. The example of spherically symmetric models have infinitely many degrees of freedom at the kinematical level but still only a finitely many ones in the gravitational sector (vacuum model). The next simplest example is that of polarized Gowdy model on T^3 [4] which has infinitely many degrees of freedom also at the physical level. As discussed in [5], this model has solutions with curvature singularity, solutions which have bounded curvature but are still “singular” in the sense of having a Cauchy horizon, exhibits a form of BKL scenario, has an open algebra of constraints and is possibly relevant to see homogenization at late times. One can expect to learn important lessons by confronting the background independent quantization strategy to this model.

A few words on available quantizations may be useful. These models have been quantized in the canonical framework in terms of the metric variables as well as the complex Ashtekar variables. The first attempts of quantization, were carried out in ADM variables in [6]. Another approach which has been followed is based on an interesting property of the model. After a suitable (partial) gauge fixing, these models can be described by (modulo a remaining global constraint) a “point particle” degree of freedom and by a scalar field ϕ . This equivalence was used in the quantization carried out in [7] and [8]. However in these quantizations, the evolution turned out to be non-unitary and in [9] a new parametrization was introduced which implemented unitary evolution in quantum theory.

Canonical quantization of *unpolarised* Gowdy T^3 model in terms of the complex Ashtekar variables has been given in [10] and [11]. More recently a hybrid quantization wherein the homogeneous modes are loop quantized while the inhomogeneous ones are Fock quantized, has been proposed [12], claiming that loop quantization of the homogeneous modes suffices to resolve the Gowdy singularity.

In this paper, we specifically focus on the loop quantization of the Gowdy model. The methods and steps used here follow closely those used in LQG and are to be viewed as first steps towards constructing a background independent quantum theory of the Gowdy model. Analogous steps have been carried out in the case of spherical symmetry [13, 14]. Two issues we do not address (but comment on in the last section) are: (i) viewing Gowdy model as a sector of the full theory and (ii) retrieving the homogeneous Bianchi I model from this midisuperspace model.

The classically reduced Gowdy model has all the ingredients of the full general relativity: it is a generally covariant field theory on $\mathbb{R} \times S^1$, its basic fields are 0-forms and connection 1-forms, it has the three sets of first class constraints – Gauss, diffeo and Hamiltonian. It is simpler than the full 1 + 3 dimensional theory in that its graphs will be 1-dimensional, its gauge group is Abelian ($U(1)$) and flux or triad representation exist (so the volume operator is simpler). It differs from the full theory in that certain limits available in the full theory are not available here. For example, in the full theory one gets back the classical expression of the constraints in the limit of shrinking the tetrahedra to their base points (continuum limit). This also shrinks the loops appearing in the (edge) holonomies, thereby ensuring that the exponents in the holonomies can be taken to be small. In the reduced theory, however, we have point holonomies and the exponents are not necessarily small in the continuum limit. (When the exponents *are* components of extrinsic curvature, they are indeed small in the classical regime as is the case in the present context.) Nevertheless, the strategies of background independent quantization continue to be available and are discussed below in detail.

In section II, we will introduce the background independent basic variables and construct the kinematical Hilbert space, define the Volume operator and solve the Gauss constraint to determine the gauge invariant spin network states. In section III we will carry out the regularization of the Hamiltonian constraint. We make specific choices for the partitions as well as for transcribing the expressions in terms of the basic variables. Section IV is

devoted to the action of the Hamiltonian constraint on basis states. Section V contains a discussion of ambiguities in the transcriptions as well as in the choices of partitions. These have a bearing on incorporating the spatial correlations in the classical constraint (spatial derivatives) also in the quantum operator.

II. QUANTUM THEORY

A. Preliminary remarks

In the connection formulation of general relativity, the basic canonical variables are a real, $SU(2)$ connection $A := A_a^i \tau_i dx^i$ and a densitized triad $E := \tau^i E_i^a \partial_a$ with the Poisson bracket given by $\{A_a^i(x), E_j^b(y)\} = (8\pi G_{\text{Newton}}) \gamma \delta_a^b \delta_j^i \delta^3(x, y)$. There are three sets of constraints which can be conveniently presented in matrix notation as follows. Introduce:

$$\begin{aligned} \kappa &:= 8\pi G_{\text{Newton}} \quad , \quad \tau_i := -i\sigma_i/2 \quad , \quad \tau_i \tau_j = -(1/4)\delta_{ij}\mathbb{I} + (1/2)\epsilon_{ijk}\tau_k, \\ A_a &:= A_a^i \tau_i \quad , \quad E^a := E_i^a \tau^i \quad , \quad F_{ab} := \partial_a A_b - \partial_b A_a + [A_a, A_b] . \end{aligned} \quad (1)$$

Then,

$$G(x) := G_i \tau^i = \frac{1}{\kappa\gamma} [\partial_a E^a + [A_a, E^a]] \quad (2)$$

$$C_a(x) = \frac{1}{\kappa\gamma} [(-2)\text{Tr}(F_{ab}E^b - A_a G)] \quad (3)$$

$$\begin{aligned} H(x) = \frac{1}{\kappa} (|\det E_i^a|)^{-1/2} [(-\text{Tr})(F_{ab}[E^a, E^b]) \\ - 2(1 + \gamma^2) (\text{Tr}(E^a K_a)\text{Tr}(E^b K_b) - \text{Tr}(E^a K_b)\text{Tr}(E^b K_a))] . \end{aligned} \quad (4)$$

For the polarized Gowdy model, the connection and triad variables get restricted to satisfy $E_3^x = E_3^y = E_1^\theta = E_2^\theta = 0$, $A_x^3 = A_y^3 = A_\theta^1 = A_\theta^2 = 0$. These can then be expressed in the form [15]:

$$\tau_x(\theta) := \cos\beta(\theta) \tau_1 + \sin\beta(\theta) \tau_2 \quad , \quad \tau_y(\theta) := -\sin\beta(\theta) \tau_1 + \cos\beta(\theta) \tau_2 \quad (5)$$

$$A(\theta) := \tau_3 \mathcal{A}(\theta) d\theta + \left\{ \tau_x(\theta) X(\theta) + \tau_y(\theta) \tilde{X}(\theta) \right\} dx + \left\{ \tau_y(\theta) Y(\theta) + \tau_x(\theta) \tilde{Y}(\theta) \right\} dy \quad (6)$$

$$E(\theta) := \tau_3 \mathcal{E}(\theta) \partial_\theta + \tau_x(\theta) E^x(\theta) \partial_x + \tau_y(\theta) E^y(\theta) \partial_y \quad (7)$$

In the above, we have essentially defined $\sum_{i=1,2} E_i^x \tau^i := E^x \tau_x$, $\sum_{i=1,2} E_i^y \tau^i := E^y \tau_y$ and demanded that $\tau_x^2 = -(1/4) = \tau_y^2$. It follows that $[\tau_x, \tau_y] = \tau_3$ iff polarization condition, $\sum_{i=1,2} E_i^x E_i^y = 0$, holds. This allows us to identify E^x, E^y as the magnitudes

of the two dimensional vectors \vec{E}^x, \vec{E}^y and introduce an angular coordinate β so that $E_1^x := E^x \cos \beta, E_2^x := E^x \sin \beta, E_1^y := -E^y \sin \beta, E_2^y := E^y \cos \beta$. From these, the definitions of the β -dependent τ matrices follows. The matrices $E_i^a(\theta)$ are now “diagonal” for each θ . This fact together with the properties of β -dependent τ matrices, simplifies the computations. In particular, the co-triad e , the spin connection Γ and the extrinsic curvature $K := \gamma^{-1}(A - \Gamma)$ are obtained as,

$$e = \tau_3 \frac{\sqrt{E}}{\mathcal{E}} d\theta + \frac{\sqrt{E}}{E^x} \tau_x dx + \frac{\sqrt{E}}{E^y} \tau_y dy \quad , \quad E := E^x E^y |\mathcal{E}| \quad (8)$$

$$\Gamma = \tau_3 \Gamma_\theta^3 d\theta + \tau_y \Gamma_x dx + \tau_x \Gamma_y dy \quad \text{where,}$$

$$\Gamma_\theta^3 = -\partial_\theta \beta \quad , \quad \Gamma_x := \frac{1}{2} \frac{E_3^\theta}{E^x} \partial_\theta \ln \left(\frac{E_3^\theta E^y}{E^x} \right) \quad , \quad \Gamma_y := \frac{1}{2} \frac{E_3^\theta}{E^y} \partial_\theta \ln \left(\frac{1}{E_3^\theta} \frac{E^y}{E^x} \right) ; \quad (9)$$

$$\gamma K = \tau_3 (\mathcal{A} + \partial_\theta \beta) d\theta + (\tau_x X + \tau_y (\tilde{X} - \Gamma_x)) dx + (\tau_y Y + \tau_x (\tilde{Y} - \Gamma_y)) dy \quad (10)$$

The preservation of the polarization condition or equivalently diagonal form of the extrinsic curvature K_a^i requires, $\tilde{X} = \Gamma_x$, $\tilde{Y} = \Gamma_y$.

Thus, the basic variables are $X, Y, \mathcal{A}, \eta := \beta$ and $E^x, E^y, \mathcal{E}, P^n$ with the Poisson brackets of the form $\{X, E^x\} = (2G_{\text{Newton}}/\pi)\gamma\delta(\theta - \theta')$. We have relabelled β by η for conformity with the notation of [15] (modulo a factor of 2).

Putting $\kappa' := \kappa/(4\pi^2)$, the constraints take the form,

$$G := G_3 = \frac{1}{\kappa' \gamma} [\partial_\theta \mathcal{E} + P^n] \quad (11)$$

$$C_\theta = \frac{1}{\kappa' \gamma} [E^x \partial_\theta X + E^y \partial_\theta Y - \mathcal{A} \partial_\theta \mathcal{E} + P^n \partial_\theta \eta] \quad (12)$$

$$H = -\frac{1}{\kappa'} \frac{1}{\sqrt{E}} \left[\frac{1}{\gamma^2} (X E^x Y E^y + \mathcal{A} \mathcal{E} (X E^x + Y E^y) + \mathcal{E} \partial_\theta \eta (X E^x + Y E^y)) \right. \\ \left. - E^x \Gamma_x E^y \Gamma_y \right] + \frac{1}{2\kappa'} \partial_\theta \left\{ \frac{2\mathcal{E} (\partial_\theta \mathcal{E})}{\sqrt{E}} \right\} - \frac{\kappa'}{4} \frac{G^2}{\sqrt{E}} - \frac{\gamma}{2} \partial_\theta \left(\frac{G}{\sqrt{E}} \right) \quad (13)$$

It is obvious from these definitions that X, Y, \mathcal{E}, η are scalars while $E^x, E^y, \mathcal{A}, P^n$ are scalar densities of weight 1. The Gauss constraint shows that \mathcal{A} transforms as a $U(1)$ connection while η is *translated* by the gauge parameter. All other variables are gauge invariant.

B. Basic States

The configuration variable \mathcal{A} is a $U(1)$ connection 1-form, so we integrate it along an edge (an arc along the S^1) and by taking its exponential we define the (edge) holonomy

variable valued in $U(1)$: $h_e^{(k)}(\mathcal{A}) := \exp(i\frac{k}{2} \int_e \mathcal{A})$, $k \in \mathbb{Z}$. The integer label k denotes the representation and the factor of $1/2$ is introduced for later convenience. The Hilbert space can be constructed via projective families labelled by closed, oriented graphs in S^1 . The graphs are just n arcs with n vertices. Associated with each arc is an edge holonomy in the representation k . For a given graph γ , consider functions ψ of n group elements $h_{e_i}^{(k_i)}(\mathcal{A})$ and define an inner product using the Haar measure on $U(1)$. The projective methods then allow one to obtain the Hilbert space as a completion of the projective limits of the graph Hilbert spaces.

The configuration variables $X, Y \in \mathbb{R}$ and $\eta \in \mathbb{R}/\mathbb{Z}$ are scalars and hence no smearing is needed. For these we define the point holonomies (at points v): $h_v^{(\mu)}(X) := \exp(i\frac{\mu}{2}X(v))$, $h_v^{(\nu)}(Y) := \exp(i\frac{\nu}{2}Y(v))$, $h_v^\lambda(\eta) := \exp(i\lambda\eta(v))$. Again, the factor of $1/2$ is introduced for later convenience. A similar factor is *not* introduced for the η holonomy since η is already an angle. The X, Y point holonomies are interpreted as unitary representation of the compact, Abelian group \mathbb{R}_{Bohr} which is the Bohr compactification of the additive group of real numbers, \mathbb{R} ¹. The representation labels μ, ν take values in \mathbb{R} . By contrast, η is an angle variable, so the corresponding point holonomy is valued in $U(1)$. The representation label then takes only integer values, $\lambda \in \mathbb{Z}$. The corresponding Hilbert spaces are constructed again via projective families – now labelled by finite sets of points which can be taken to be the vertices of the graphs used in the previous paragraph.

The kinematical Hilbert space for the model is thus a tensor product of the Hilbert spaces constructed for \mathcal{A}, X, Y, η variables. A convenient orthonormal basis for this is provided by the “charge network functions” which are labelled by a close, oriented graph G with n edges e and n vertices v , a $U(1)$ representation k_e for each edge, a $U(1)$ representation $\lambda_v \in \mathbb{Z}$ for each vertex and \mathbb{R}_{Bohr} representations μ_v, ν_v for each vertex:

$$T_{G, \vec{k}, \vec{\mu}, \vec{\nu}, \vec{\lambda}}(\mathcal{A}, X, Y, \eta) := \prod_{e \in G} k_e(h^{(e)}) \prod_{v \in V(G)} \mu_v(h_v(X)) \nu_v(h_v(Y)) \lambda_v(h_v(\eta)) \quad (14)$$

¹ The functions $\{\exp(i\mu X), \mu \in \mathbb{R}\}$ form a separating set of functions to separate points in \mathbb{R} . These are also characters of the topological group \mathbb{R} . Their finite linear combinations give almost periodic functions of X . From these one constructs a commutative C^* algebra with unity. The spectrum of this algebra happens to be the Bohr compactification, \mathbb{R}_{Bohr} , of the topological group \mathbb{R} . Its (unitary) irreducible representations are one dimensional and are labelled by real numbers. The point holonomies are the representatives. The Haar measure on this compact group can be presented as: $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T$. With this measure, the Hilbert space of functions on the group is defined via the inner product: $\langle f, g \rangle := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dX f^*(X)g(X)$.

$$= \prod_{e \in G} \exp \left(i \frac{k_e}{2} \int_e \mathcal{A} \right) \prod_{v \in V(G)} \left(\exp \left(i \frac{\mu_v}{2} X \right) \exp \left(i \frac{\nu_v}{2} Y \right) \exp \left(i \lambda_v \eta \right) \right)$$

where $V(G)$ represents the set of vertices belonging to the graph G . The functions with any of the labels different, are orthogonal – in particular two graphs must coincide for non-zero inner product.

These basis states provide an orthogonal decomposition for the kinematical Hilbert space when all the representation labels are non-zero.

C. Flux Operators

The conjugate variables are represented as $E^x(\theta) \sim -i\gamma\ell_P^2 \frac{\delta h_\theta(X)}{\delta X(\theta)} \frac{\partial}{\partial h_\theta(X)}$, where $\ell_P^2 := \kappa' \hbar$. The flux variables corresponding to E^x, E^y, P^η are defined by integrating these densities on an interval \mathcal{I} of the circle, eg $\mathcal{F}_{x,\mathcal{I}} := \int_{\mathcal{I}} E^x, \mathcal{F}_{y,\mathcal{I}} := \int_{\mathcal{I}} E^y$. \mathcal{E} being a scalar, is already a suitable variable. Their actions on the basis functions (14) are:

$$\hat{\mathcal{E}}(\theta) T_{G,k,\mu,\nu,\lambda} = \frac{\gamma\ell_P^2}{2} \frac{k_{e^+(\theta)} + k_{e^-(\theta)}}{2} T_{G,k,\mu,\nu,\lambda} \quad (15)$$

$$\int_{\mathcal{I}} \hat{E}^x T_{G,k,\mu,\nu,\lambda} = \frac{\gamma\ell_P^2}{2} \sum_{v \in V(G) \cap \mathcal{I}} \mu_v T_{G,k,\mu,\nu,\lambda} \quad (16)$$

$$\int_{\mathcal{I}} \hat{E}^y T_{G,k,\mu,\nu,\lambda} = \frac{\gamma\ell_P^2}{2} \sum_{v \in V(G) \cap \mathcal{I}} \nu_v T_{G,k,\mu,\nu,\lambda} \quad (17)$$

$$\int_{\mathcal{I}} \hat{P}^\eta T_{G,k,\mu,\nu,\lambda} = \gamma\ell_P^2 \sum_{v \in V(G) \cap \mathcal{I}} \lambda_v T_{G,k,\mu,\nu,\lambda} \quad (18)$$

where \mathcal{I} is an interval on S^1 . The $e^\pm(\theta)$ refer to the two oriented edges of the graph G , meeting at θ if there is a vertex at θ or denote two parts of the same edge if there is no vertex at θ . The k labels in such a case are the same. The \cap allows the case where a vertex may be an end-point of the interval \mathcal{I} . In such a case, there is an additional factor of $\frac{1}{2}$ for its contribution to the sum. This follows because

$$\int_a^b dx \delta(x - x_0) = \begin{cases} 1 & \text{if } x_0 \in (a, b); \\ \frac{1}{2} & \text{if } x_0 = a \text{ or } x_0 = b; \\ 0 & \text{if } x_0 \notin [a, b]. \end{cases} \quad (19)$$

Note that classically the triad components, E^x, E^y are positive. The fluxes however can take both signs since they involve integrals which depend on the orientation.

This completes the specification of the kinematical Hilbert space together with the representation of the basic background independent variables. Next we turn to the construction of certain operators.

D. Construction of more general Operators

The diffeomorphism covariance requires all operators of interest (constraints and observables) are integrals of expressions in terms of the basic operators. Secondly, operators of interests also involve products of elementary operators at the same point (same θ) and thus need a “regularization”. As in LQG, the general strategy to define such operators is: (i) replace the integral by a Riemann sum using a “cell-decomposition” (or partition) of S^1 ; (ii) for each cell, define a regulated expression choosing suitable ordering of the basic operators and evaluate the action on basis states; (iii) check “cylindrical consistency” of this action in (ii) so that the (regulated) operator can be densely defined on the kinematical Hilbert space via projective limit; (iv) finally one would like to remove the regulator. One would like to do this in such a manner that the constructed limiting operator has the same properties under the diffeomorphism. To achieve this, usually one has to restrict the cell-decomposition in relation to a graph.

In the present case of one dimensional spatial manifold, both the cell-decomposition and the graphs underlying the basis states are characterised by finitely many points and the arcs connecting the consecutive points. As in LQG [2], the products of elementary variables are regulated by using a “point splitting” and then expressing the fields in terms of the appropriate holonomies and fluxes both of which need at most edges and at each point there are precisely two edges (in the full theory one needs edges as well as close loops and there can be an arbitrary number of these). A regulator, for each given graph G , then consists of a family of partitions, Π_ϵ^G , such that for each ϵ , the partition is such that each vertex of G is contained in exactly one cell². There is also a choice of representation labels $k_0, \mu_0, \nu_0, \lambda_0$ made which can be taken to be the same for all ϵ . The regulated expressions constructed depend on ϵ and are such that one recovers the classical expressions in the limit of removing the regulator ($\epsilon \rightarrow 0$). There are of course infinitely many such regulators. A diffeomorphism

² This is one possible natural choice of a class of partitions adapted to a graph. The vertices of G always lie in the interior of the cells and some cells have no vertices. We discuss this further in the last section.

covariant regulator is one such that if under a diffeomorphism the graph $G \rightarrow G'$, then the corresponding partitions also transform similarly. Since each Π_G can also be thought of as being defined by a set of points such that each vertex is flanked by two points (between two consecutive points, there need not be any vertex), any orientation preserving diffeomorphism will automatically preserve the order of the vertices and cell boundaries. Every sufficiently refined partition then automatically becomes a diffeomorphism covariant regulator. This is assumed in the following.

As in LQG, the issue of cylindrical consistency is automatically sorted out by referring to the orthogonal decomposition of \mathcal{H}_{kin} i.e. by specifying the action of the operators on basis states with all representation labels being non-zero.

With these preliminaries, we proceed to define some operators.

E. Volume Operator

In the classical expression for the Hamiltonian constraint, powers of $E := |\mathcal{E}|E^xE^y$, occur in the same manner as in the full theory. It is therefore natural to consider the expression for the volume of a region $\mathcal{I} \times T^2$ and construct the corresponding operator. With the canonical variables chosen, the volume involves only the conjugate momenta whose quantization is already done. The volume operator written in terms of the basic operators :

$$\begin{aligned}
\mathcal{V}(\mathcal{I} \times T^2) &= \int_{\mathcal{I} \times T^2} d^3x \sqrt{g} \\
&= 4\pi^2 \int_{\mathcal{I}} d\theta \sqrt{|\mathcal{E}|E^xE^y} \\
\mathcal{V}_\epsilon(\mathcal{I}) &\approx \sum_{i=1}^n \int_{\theta_i}^{\theta_i+\epsilon} \sqrt{|\mathcal{E}|E^xE^y} \\
&\approx \sum_{i=1}^n \epsilon \sqrt{|\mathcal{E}(\bar{\theta}_i)|} \sqrt{E^x(\bar{\theta}_i)} \sqrt{E^y(\bar{\theta}_i)} \\
&\approx \sum_{i=1}^n \sqrt{|\mathcal{E}|} \sqrt{\epsilon E^x} \sqrt{\epsilon E^y} \\
&\approx \sum_{i=1}^n \sqrt{|\mathcal{E}|(\bar{\theta}_i)} \sqrt{\left| \int_{\theta_i}^{\theta_i+\epsilon} E^x \right|} \sqrt{\left| \int_{\theta_i}^{\theta_i+\epsilon} E^y \right|} \tag{20}
\end{aligned}$$

The right hand side is expressed in terms of flux variables, so the regulated volume operator can be defined as:

$$\hat{V}_\epsilon(\mathcal{I}) := \sum_{i=1}^n \sqrt{|\hat{\mathcal{E}}(\bar{\theta}_i)|} \sqrt{\left| \int_{\mathcal{I}_i} E^x \right|} \sqrt{\left| \int_{\mathcal{I}_i} E^y \right|} \quad (21)$$

Clearly this is diagonal in the basis states. and its action on a basis state $T_{G, \vec{k}, \vec{\mu}, \vec{\nu}, \vec{\lambda}}$ gives the eigenvalue,

$$V_{\vec{k}, \vec{\mu}, \vec{\nu}, \vec{\lambda}} = \frac{1}{\sqrt{2}} \left(\frac{\gamma l_P^2}{2} \right)^{3/2} \sum_{v \in \mathcal{I} \cap V(G)} \left(|\mu_v| |\nu_v| |k_{e^+(v)} + k_{e^-(v)}| \right)^{\frac{1}{2}} \quad (22)$$

Remarks:

(1) In the above, \mathcal{I}_i denotes the i^{th} cell of the partition and $\bar{\theta}_i$ denotes a point in that cell – it need not be an end-point of the interval. We have also assumed the “length of the intervals” to be same and equal to ϵ . This corresponds to a “cubic” partition and is chosen for convenience only. We will always use such partitions in all the operators below.

(2) Although we could restrict to $\mu_v, \nu_v > 0$, it will be more convenient (eg in the Hamiltonian constraint below) to allow both signs (corresponding to the orientation of the interval). The eigenvalues of the volume operator then must have explicit absolute values. We have thus used the absolute value operators defined from the flux operators.

(3) For a given graph, the partition (of \mathcal{I}) is so chosen that each vertex is included in one and only one interval \mathcal{I}_i . For those intervals which contain no vertex of the graph, there is no contribution to the summation since flux operators have this property. Hence, the sum collapses to contributions only from the vertices, independent of the partition. The action is manifestly independent of ϵ and even though the number of intervals go to infinity as $\epsilon \rightarrow 0$, the action remains *finite* and well defined.

Because of this property of the fluxes, we can choose the $\bar{\theta}_i$ point in a cell to coincide with a vertex of a graph if \mathcal{I}_i contains a vertex or an arbitrary point if \mathcal{I}_i does not contain a vertex. Such a choice will be understood in the following.

(4) For intervals $\mathcal{I} \neq S^1$, a choice of diffeo-covariant regulator retains the $v \in \mathcal{I} \cap V(G)$ and hence the action is diffeo-invariant. The eigenvalues are also manifestly independent of “location labels” of the states. For the total volume (which is diffeo-invariant), the operator is manifestly diffeo-invariant.

F. Gauss Constraint

Consider the Gauss constraint (11):

$$\begin{aligned}
G_3 &= \int_{S^1} d\theta (\partial_\theta \mathcal{E} + P^\eta) \\
&\approx \sum_{i=1}^n \int_{\theta_i}^{\theta_i + \epsilon} (\partial_\theta \mathcal{E} + P^\eta) \\
&\approx \sum_{i=1}^n \left[\int_{\mathcal{I}_i} P^\eta + \mathcal{E}(\theta_i + \epsilon) - \mathcal{E}(\theta_i) \right] \tag{23}
\end{aligned}$$

$$\hat{G}_3^\epsilon := \sum_{i=1}^n \left[\widehat{\int_{\mathcal{I}_i} P^\eta + \hat{\mathcal{E}}(\theta_i + \epsilon) - \hat{\mathcal{E}}(\theta_i)} \right] \tag{24}$$

Again, this is easily quantized with its action on a basis state $T_{G, \vec{k}, \vec{\mu}, \vec{\nu}, \vec{\lambda}}$ giving the eigenvalue,

$$\gamma l_P^2 \sum_{v \in V(G)} \left[\lambda_v + \frac{k_{e^+}(v) - k_{e^-}(v)}{2} \right] \tag{25}$$

Notice that in the limit of infinitely fine partitions, for a given graph, if there is a vertex $v \in \mathcal{I}_i$, then there is *no vertex* in the adjacent cells. As a result, $\mathcal{E}(\theta_{i+1})$ gives the $k_{e^+}(v)/2$ and $-\mathcal{E}(\theta_i)$ gives $-k_{e^-}(v)/2$, since θ_i divides the same edge and so does θ_{i+1} .

Once again, the eigenvalues are manifestly independent of ϵ and the action is diffeoinvariant. Imposition of Gauss constraint can be done simply by restriction to basis states with labels satisfying $\lambda_v = -(k_{e^+}(v) - k_{e^-}(v))/2, \forall v \in V(G)$. Since $\lambda_v \in \mathbb{Z}$, the difference in the k labels at each vertex must be an *even* integer. We will assume these restrictions on the representation labels and from now on deal with gauge invariant basis states. The label $\vec{\lambda}$ will be suppressed and terms proportional to the Gauss law constraint in the Hamiltonian constraint will also be dropped.

Substituting for λ_v for each of the vertices and rearranging the holonomy factors, one can write the gauge invariant basis states are explicitly given by,

$$T_{G, \vec{k}, \vec{\mu}, \vec{\nu}} = \prod_{e \in G} \exp \left\{ i \frac{k_e}{2} \int_e (\mathcal{A}(\theta) - \partial_\theta \eta) \right\} \prod_{v \in V(G)} \left(\exp \left\{ i \frac{\mu_v}{2} X(v) \right\} \exp \left\{ i \frac{\nu_v}{2} Y(v) \right\} \right). \tag{26}$$

We have also used, $\eta(v^+(e)) - \eta(v^-(e)) = \int_e \partial_\theta \eta$, where $v^\pm(e)$ denote the tip and tail of the edge e .

III. HAMILTONIAN CONSTRAINT

The Hamiltonian constraint is a more complicated object. Let us write (14) as a sum of a kinetic term and a potential term,

$$H := -\frac{1}{\kappa'}[H_K + H_P] \quad (27)$$

$$H_K := \frac{1}{\gamma^2} \int_{S^1} d\theta N(\theta) \frac{1}{\sqrt{E}} [X E^x Y E^y + (\mathcal{A} + \partial_\theta \eta) \mathcal{E} (X E^x + Y E^y)] \quad (28)$$

$$H_P := - \int_{S^1} d\theta N(\theta) \frac{1}{\sqrt{E}} \left[-\frac{1}{4} (\partial_\theta \mathcal{E})^2 + \frac{(\mathcal{E})^2}{4} \left(\frac{\partial_\theta E^x}{E^x} - \frac{\partial_\theta E^y}{E^y} \right)^2 \right] \\ - \int_{S^1} d\theta N(\theta) \frac{1}{2} \partial_\theta \left[\frac{2\mathcal{E} (\partial_\theta \mathcal{E})}{\sqrt{E}} \right] \quad (29)$$

In the above we have used the expressions of Γ_x and Γ_y and suppressed the terms dependent on the Gauss constraint which will drop out on gauge invariant basis states. Only H_K depends on the configuration variables and all terms have two powers of momenta in the numerator and the \sqrt{E} in the denominator whose vanishing is a potential problem.

The kinetic term has a structure similar to the Euclidean term in the full theory $\sim EEF/\sqrt{q}$ (but it is *not* the simplification of the Euclidean term of LQG). This will be treated in a manner similar to the full theory, using appropriate holonomies in the form $h_i h_j h_i^{-1} h_j^{-1} h_k \{h_k^{-1}, V\}$. The remaining terms are functions of momenta alone and the $E^{-1/2}$ is treated using Poisson bracket of the volume with suitable holonomy,

Although the holonomies defined before, eg in the basis states (14), are all phases (Abelian gauge theory), it is convenient to introduce their $SU(2)$ valued analogues using the η dependent τ matrices defined in eq. (1) and in eq. (5). Thus,

$$h_\theta(\mathcal{I}) := \exp \left(\tau_3 \int_{\mathcal{I}} \mathcal{A}(\theta') d\theta' \right) = \cos \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) + 2\tau_3 \sin \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) \\ h_x(\theta) := \exp(\mu_0 X(\theta) \tau_x(\theta)) = \cos \left(\frac{\mu_0}{2} X(\theta) \right) + 2\tau_x(\theta) \sin \left(\frac{\mu_0}{2} X(\theta) \right) \\ h_y(\theta) := \exp(\nu_0 Y(\theta) \tau_y(\theta)) = \cos \left(\frac{\nu_0}{2} Y(\theta) \right) + 2\tau_y(\theta) \sin \left(\frac{\nu_0}{2} Y(\theta) \right) \quad (30)$$

Each of the sin, cos are well defined on the kinematical Hilbert space (this was the reason for the factors of 1/2 in the definitions of the holonomies in the basis states) and therefore so are the above $SU(2)$ -valued holonomies. The interval \mathcal{I} will typically be a cell of a partition, $(\theta_i, \theta_i + \epsilon)$. The parameters μ_0, ν_0 are the chosen and fixed representations of \mathbb{R}_{Bohr} , $k_0 = 1$ is the fixed representation of the $U(1)$, while ϵ is a small parameter which will also play

the role of the regulator parameter. Let us also define, the volume function labelled by an interval \mathcal{I} and a point θ inside the interval:

$$V(\mathcal{I}, \theta) := \sqrt{|\mathcal{E}(\theta)| \left| \int_{\mathcal{I}} E^x \right| \left| \int_{\mathcal{I}} E^y \right|} \quad (31)$$

For brevity of notation we will suppress the label θ and denote the above volume function as $V(\mathcal{I})$.

Consider expression of the form $\text{Tr}(h_i h_j h_i^{-1} h_j^{-1} h_k \{h_k^{-1}, \sqrt{E}\})$ for distinct i, j, k taking values θ, x, y . For small values of $X, Y, \int_{\mathcal{I}} \mathcal{A}$, the holonomies can be expanded in a power series. Because of the trace, it is enough to expand each holonomy up to 1st order. The surviving terms are quadratic terms arising from products of the linear ones and a linear term coming from h_k . If one interchanges the $i \leftrightarrow j$ holonomies, the linear term retains the sign while the quadratic one changes the sign. Thus taking the difference of the two traces, leaves us with only the quadratic terms which are exactly of the form needed in H_K . Explicitly,

$$\begin{aligned} \text{Tr} \left[\left\{ h_x(\theta) h_y(\theta) h_x^{-1}(\theta) h_y^{-1}(\theta) - h_y(\theta) h_x(\theta) h_y^{-1}(\theta) h_x^{-1}(\theta) \right\} h_\theta(\mathcal{I}) \{h_\theta^{-1}(\mathcal{I}), V(\mathcal{I})\} \right] \\ \approx \left(-\frac{\kappa' \gamma}{2} \mu_0 \nu_0 \right) \frac{\epsilon X(\theta) Y(\theta) E^x(\theta) E^y(\theta)}{\sqrt{E(\theta)}} \end{aligned} \quad (32)$$

$$\begin{aligned} \text{Tr} \left[\left\{ h_y(\theta) h_\theta(\mathcal{I}) h_y^{-1}(\theta + \epsilon) h_\theta^{-1}(\mathcal{I}) - h_\theta(\mathcal{I}) h_y(\theta + \epsilon) h_\theta^{-1}(\mathcal{I}) h_y^{-1}(\theta) \right\} h_x(\theta) \{h_x^{-1}(\theta), V(\mathcal{I})\} \right] \\ \approx \left(-\frac{\kappa' \gamma}{2} \mu_0 \nu_0 \right) \frac{\epsilon Y(\theta) (\mathcal{A}(\theta) + \partial_\theta \eta(\theta)) E^y(\theta) \mathcal{E}(\theta)}{\sqrt{E(\theta)}} \end{aligned} \quad (33)$$

$$\begin{aligned} \text{Tr} \left[\left\{ h_\theta(\mathcal{I}) h_x(\theta + \epsilon) h_\theta^{-1}(\mathcal{I}) h_x^{-1}(\theta) - h_x(\theta) h_\theta(\mathcal{I}) h_x^{-1}(\theta + \epsilon) h_\theta^{-1}(\mathcal{I}) \right\} h_y(\theta) \{h_y^{-1}(\theta), V(\mathcal{I})\} \right] \\ \approx \left(-\frac{\kappa' \gamma}{2} \mu_0 \nu_0 \right) \frac{\epsilon (\mathcal{A}(\theta) + \partial_\theta \eta(\theta)) X(\theta) \mathcal{E}(\theta) E^x(\theta)}{\sqrt{E(\theta)}} \end{aligned} \quad (34)$$

In equations (33) and (34) \mathcal{I} is the interval between θ and $\theta + \epsilon$. The derivatives of η arise from the position dependence of the τ_x, τ_y matrices which satisfy:

$$\begin{aligned} \tau_x(\theta + \epsilon) - \tau_x(\theta) &\approx \epsilon \partial_\theta \tau_x = \epsilon \partial_\theta \eta \tau_y(\theta) \\ \tau_y(\theta + \epsilon) - \tau_y(\theta) &\approx \epsilon \partial_\theta \tau_y = -\epsilon \partial_\theta \eta \tau_x(\theta) \end{aligned} \quad (35)$$

In the above, we have also used:

$$\begin{aligned}
h_x(\theta)\{h_x(\theta)^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\mu_0\tau_x\frac{\mathcal{E}(\theta)\int_{\mathcal{I}}E^y}{V(\mathcal{I})}\approx -\frac{\kappa'\gamma}{2}\mu_0\tau_x\frac{E^y(\theta)\mathcal{E}(\theta)}{\sqrt{E(\theta)}} \\
h_y(\theta)\{h_y(\theta)^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\nu_0\tau_y\frac{\mathcal{E}(\theta)\int_{\mathcal{I}}E^x}{V(\mathcal{I})}\approx -\frac{\kappa'\gamma}{2}\nu_0\tau_y\frac{E^x(\theta)\mathcal{E}(\theta)}{\sqrt{E(\theta)}} \\
h_\theta\{h_\theta^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\tau_3\frac{\int_{\mathcal{I}}E^x\int_{\mathcal{I}}E^y}{V(\mathcal{I})}\approx -\frac{\kappa'\gamma}{2}\epsilon\tau_3\frac{E^x(\theta)E^y(\theta)}{\sqrt{E(\theta)}} \tag{36}
\end{aligned}$$

$$\int_{\mathcal{I}}\mathcal{A}\approx\epsilon\mathcal{A}(\theta)\quad,\quad\int_{\mathcal{I}}E^x\approx\epsilon E^x(\theta)\quad,\quad\int_{\mathcal{I}}E^y\approx\epsilon E^y(\theta) \tag{37}$$

In the quantization of the H_P, H_T , we also use the following identities repeatedly (in the form LHS/RHS = 1):

$$\begin{aligned}
\mathcal{Z}(\mathcal{I}) &:= \epsilon^{abc}\text{Tr}\left[h_a\{h_a^{-1}, V(\mathcal{I})\}h_b\{h_b^{-1}, V(\mathcal{I})\}h_c\{h_c^{-1}, V(\mathcal{I})\}\right] \\
&= \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3\mu_0\nu_0V(\mathcal{I}) \tag{38} \\
\mathcal{Z}_\alpha(\mathcal{I}) &:= \epsilon^{abc}\text{Tr}\left[h_a\{h_a^{-1}, (V(\mathcal{I}))^\alpha\}h_b\{h_b^{-1}, (V(\mathcal{I}))^\alpha\}h_c\{h_c^{-1}, (V(\mathcal{I}))^\alpha\}\right] \\
&= \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3\mu_0\nu_0\alpha^3(V(\mathcal{I}))^{3\alpha-2} \\
&= \alpha^3(V(\mathcal{I}))^{3(\alpha-1)}\mathcal{Z}(\mathcal{I}) \tag{39}
\end{aligned}$$

These are essentially versions of the identity $1 = \left(\frac{|\det(e_a^i)|}{\sqrt{E}}\right)^n$ [16]. Having noted the ingredients common to the quantization of the different pieces of the Hamiltonian constraint, we turn to each one in explicit details.

A. Quantization of H_K

Choosing a partition of S^1 with a sufficiently large number of n points at $\theta_i, i = 1, \dots, n, \theta_n = 2\pi, \epsilon = \theta_{i+1} - \theta_i$, we write the integral as a sum,

$$\begin{aligned}
H_K &\approx \frac{1}{\gamma^2}\sum_{i=1}^n\epsilon N(\bar{\theta}_i)\frac{1}{\sqrt{E(\bar{\theta}_i)}}[XE^xYE^y + (\mathcal{A} + \partial_\theta\eta)\mathcal{E}(XE^x + YE^y)](\bar{\theta}_i) \tag{40} \\
&= \frac{1}{\gamma^2}\sum_{i=1}^nN(\bar{\theta}_i)\frac{1}{\sqrt{\epsilon^2E(\bar{\theta}_i)}}[X(\epsilon E^x)Y(\epsilon E^y) + \epsilon(\mathcal{A} + \partial_\theta\eta)\mathcal{E}(X(\epsilon E^x) + Y(\epsilon E^y))] \\
&= \frac{1}{\gamma^2}\sum_{i=1}^nN(\bar{\theta}_i)\frac{1}{V(\mathcal{I}_i)}\left[X(\bar{\theta}_i)\left(\int_{\mathcal{I}_i}E^x\right)Y(\bar{\theta}_i)\left(\int_{\mathcal{I}_i}E^y\right) + \right. \\
&\quad\left.\left(\int_{\mathcal{I}_i}\mathcal{A} + \partial_\theta\eta\right)\mathcal{E}(\bar{\theta}_i)\left\{X(\bar{\theta}_i)\int_{\mathcal{I}_i}E^x + Y(\bar{\theta}_i)\int_{\mathcal{I}_i}E^y\right\}\right] \tag{41}
\end{aligned}$$

From the equations (32, 33, 34), one sees immediately that for small values of the extrinsic curvature components ($\sim X, Y$, classical regime) and sufficiently refined partition ($\epsilon \ll 1$, continuum limit), the i^{th} term in the sum can be written in terms of the traces of the $SU(2)$ valued holonomies. In other words, the expression in terms of holonomies and fluxes, does go over to the classical expression in the classical regime and can be promoted to an operator by putting hats on the holonomies and fluxes and replacing Poisson brackets by $(i\hbar)^{-1}$ times the commutators. Here, there are possibilities for choosing the ordering of various factors and in this paper, we will make the “standard choice” of putting the holonomies on the left. Thus, the regulated quantum operator corresponding to the kinetic piece is (suppressing the hats on the holonomies and using $\ell_{\text{P}}^2 := \kappa'\hbar$),

$$\begin{aligned} \hat{H}_K^{\text{reg}} := & i \frac{2}{\ell_{\text{P}}^2 \gamma^3} \frac{1}{\mu_0 \nu_0} \sum_{i=1}^n N(\bar{\theta}_i) \text{Tr} \left(\right. \\ & \left. \{h_x h_y h_x^{-1} h_y^{-1} - h_y h_x h_y^{-1} h_x^{-1}\} h_{\theta}(\mathcal{I}_i) \left[h_{\theta}^{-1}(\mathcal{I}_i), \hat{V}(\mathcal{I}_i) \right] \right. \\ & + \left\{ h_y h_{\theta}(\mathcal{I}_i) h_y^{-1} (\bar{\theta}_i + \epsilon) h_{\theta}^{-1}(\mathcal{I}_i) - h_{\theta_i}(\mathcal{I}_i) h_y (\bar{\theta}_i + \epsilon) h_{\theta}^{-1}(\mathcal{I}_i) h_y^{-1} \right\} h_x \left[h_x^{-1}, \hat{V}(\mathcal{I}_i) \right] \\ & \left. + \left\{ h_{\theta}(\mathcal{I}_i) h_x (\bar{\theta}_i + \epsilon) h_{\theta}^{-1}(\mathcal{I}_i) h_x^{-1} - h_x h_{\theta}(\mathcal{I}_i) h_x^{-1} (\bar{\theta}_i + \epsilon) h_{\theta}^{-1}(\mathcal{I}_i) \right\} h_y \left[h_y^{-1}, \hat{V}(\mathcal{I}_i) \right] \right) \end{aligned} \quad (42)$$

In the above equation, the point holonomies without an explicit argument, are at $\bar{\theta}_i$.

At this point it is convenient to define the following families of operators:

$$\begin{aligned} \hat{\mathcal{O}}_{\alpha}^x(\mathcal{I}, \theta) := & \left[\cos \left(\frac{1}{2} \mu_0 X(\theta) \right) \hat{V}^{\alpha}(\mathcal{I}) \sin \left(\frac{1}{2} \mu_0 X(\theta) \right) - \right. \\ & \left. \sin \left(\frac{1}{2} \mu_0 X(\theta) \right) \hat{V}^{\alpha}(\mathcal{I}) \cos \left(\frac{1}{2} \mu_0 X(\theta) \right) \right] \\ \hat{\mathcal{O}}_{\alpha}^y(\mathcal{I}, \theta) := & \left[\cos \left(\frac{1}{2} \mu_0 Y(\theta) \right) \hat{V}^{\alpha}(\mathcal{I}) \sin \left(\frac{1}{2} \mu_0 Y(\theta) \right) - \right. \\ & \left. \sin \left(\frac{1}{2} \mu_0 Y(\theta) \right) \hat{V}^{\alpha}(\mathcal{I}) \cos \left(\frac{1}{2} \mu_0 Y(\theta) \right) \right] \\ \hat{\mathcal{O}}_{\alpha}^{\theta}(\mathcal{I}, \theta) := & \left[\cos \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) \hat{V}^{\alpha}(\mathcal{I}) \sin \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) - \right. \\ & \left. \sin \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) \hat{V}^{\alpha}(\mathcal{I}) \cos \left(\frac{1}{2} \int_{\mathcal{I}} \mathcal{A} \right) \right] \end{aligned} \quad (43)$$

In the above, θ is a point in the interval \mathcal{I} and $\alpha > 0$ is the power of the volume operator. Again for simplicity of notation we will suppress the θ labels in the above operators.

The operator form of $\mathcal{Z}_{\alpha}(\mathcal{I})$ can be obtained as:

$$\hat{\mathcal{Z}}_{\alpha}(\mathcal{I}) := \epsilon^{abc} \text{Tr} \left(\hat{h}_a \left[\hat{h}_a^{-1}, \hat{V}(\mathcal{I})^{\alpha} \right] \hat{h}_b \left[\hat{h}_b^{-1}, \hat{V}(\mathcal{I})^{\alpha} \right] \hat{h}_c \left[\hat{h}_c^{-1}, \hat{V}(\mathcal{I})^{\alpha} \right] \right) \quad (44)$$

$$= -12 \hat{\mathcal{O}}_\alpha^x(\mathcal{I}) \hat{\mathcal{O}}_\alpha^y(\mathcal{I}) \hat{\mathcal{O}}_\alpha^\theta(\mathcal{I}) \quad (45)$$

Using the expressions for the holonomies in terms of the ‘‘trigonometric’’ operators given in the eq. (30), and evaluating the traces etc, one can see that,

$$\begin{aligned} \hat{H}_K^{\text{reg}} = & -i \frac{4}{\ell_P^2 \gamma^3} \frac{1}{\mu_0 \nu_0} \sum_{i=1}^n N(\bar{\theta}_i) \left[\{ \sin(\mu_0 X(\bar{\theta}_i)) \sin(\nu_0 Y(\bar{\theta}_i)) \} \times \mathcal{O}_1^\theta(\mathcal{I}_i) + \right. \\ & \left\{ 2 \sin\left(\frac{1}{2} \nu_0 Y(\bar{\theta}_i + \epsilon)\right) \cos\left(\frac{1}{2} \nu_0 Y(\bar{\theta}_i)\right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i\right) \right\} \times \mathcal{O}_1^x(\mathcal{I}_i) + \\ & \left. \left\{ 2 \sin\left(\frac{1}{2} \mu_0 X(\bar{\theta}_i + \epsilon)\right) \cos\left(\frac{1}{2} \mu_0 X(\bar{\theta}_i)\right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i\right) \right\} \times \mathcal{O}_1^y(\mathcal{I}_i) \right] \end{aligned}$$

In the above $\Delta_i := \eta(\bar{\theta}_i) - \eta(\bar{\theta}_i + \epsilon)$ and is outside the integral.

B. Quantization of H_P

All the three terms of H_P are functions of the momenta (triad) only. These have to be expressed in terms of fluxes and holonomies alone. Furthermore, the power(s) of momenta in the denominators will make the action on some states to be singular. The first part is easy to take care of thanks to the density weight 1. For the second part we use the by now familiar procedure of using the identities (38, 39). Due to the spatial dimension being 1, it is easier to convert triads in terms of fluxes directly, without explicitly doing any point-splitting (one could of course do that if so desired [16]).

The terms in the H_P will be manipulated in the following steps: (i) introduce sufficient number, $k > 0$, of positive powers of $1 = 16(3(\kappa'\gamma)^3 \mu_0 \nu_0)^{-1} \mathcal{Z}(\mathcal{I})/V(\mathcal{I})$ and express \mathcal{Z} in terms of \mathcal{Z}_α . This introduces further powers of the volume; (ii) choose $\alpha(k)$ such that explicit multiplicative factors of the volume become 1 and further choose k sufficiently large so that $\alpha(k) > 0$ is obtained. Note that the choice of $k > 0$ constitutes a quantization ambiguity. Now the expression can be promoted to an operator. Here are the details.

The first term of H_P :

$$\begin{aligned} - \int_{S^1} N(\theta) \frac{1}{\sqrt{E(\theta)}} \left[-\frac{1}{4} (\partial_\theta \mathcal{E})^2 \right] & \approx + \frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \epsilon \frac{(\partial_\theta \mathcal{E}(\bar{\theta}_i))^2}{\sqrt{E(\bar{\theta}_i)}} = \frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\epsilon \partial_\theta \mathcal{E}(\bar{\theta}_i))^2}{\epsilon \sqrt{E(\bar{\theta}_i)}} \\ & = \frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i))^2}{\sqrt{\mathcal{E}(\bar{\theta}_i)} \int_{\mathcal{I}_i} E^x \int_{\mathcal{I}_i} E^y} (1)^k \quad (46) \end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i))^2}{V(\mathcal{I}_i)} \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0} \right)^k \left(\frac{\mathcal{Z}(\mathcal{I}_i)}{V(\mathcal{I}_i)} \right)^k \quad (47) \\
&= \frac{1}{4} \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0} \right)^k \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i))^2}{V(\mathcal{I}_i)} \left(\frac{\mathcal{Z}_\alpha(\mathcal{I}_i)}{\alpha^3 (V(\mathcal{I}_i))^{(3\alpha-2)}} \right)^k \\
&= \frac{1}{4} \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0 \alpha^3} \right)^k \sum_{i=1}^n N(\bar{\theta}_i) (\mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i))^2 (\mathcal{Z}_\alpha(\mathcal{I}_i))^k \Big|_{\alpha := \frac{2}{3} - \frac{1}{3k}}
\end{aligned}$$

In the last line we have chosen $\alpha := \frac{2}{3} - \frac{1}{3k}$ which removes the explicit factors of the volume. The choice of $k > 0$ is limited by $\alpha > 0$ (being a power of the volume appearing in \mathcal{Z}_α). Some convenient choices would be $k = 1$ ($\alpha = 1/3$), $k = 2$ ($\alpha = 1/2$) etc. For all such choices, the above expression can be promoted to a well defined operator.

The second term of H_P :

To begin with one observes that E^y/E^x is a scalar, $\partial_\theta \ln(E^y/E^x)$ is a scalar density. This term is then manipulated as:

$$-\frac{1}{4} \int_{S^1} N(\theta) \frac{(\mathcal{E}(\theta))^2}{\sqrt{E(\theta)}} \left(\frac{\partial_\theta E^x}{E^x} - \frac{\partial_\theta E^y}{E^y} \right)^2 = -\frac{1}{4} \int_{S^1} N(\theta) \frac{(\mathcal{E}(\theta))^2}{\sqrt{E(\theta)}} \left(\partial_\theta \ln \left(\frac{E^y}{E^x} \right) \right)^2 \quad (48)$$

$$\begin{aligned}
\text{RHS} &\approx -\frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \epsilon \frac{(\mathcal{E}(\bar{\theta}_i))^2}{\sqrt{E(\bar{\theta}_i)}} \left(\partial_\theta \ln \left(\frac{E^y}{E^x}(\bar{\theta}_i) \right) \right)^2 \\
&= -\frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i))^2}{\sqrt{\epsilon^2 E(\bar{\theta}_i)}} \left(\frac{E^x(\bar{\theta}_i)}{E^y(\bar{\theta}_i)} \epsilon \partial_\theta \left(\frac{E^y}{E^x}(\bar{\theta}_i) \right) \right)^2 \\
&= -\frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i))^2}{V(\mathcal{I}_i)} \left[\frac{E^x(\bar{\theta}_i)}{E^y(\bar{\theta}_i)} \left\{ \frac{E^y}{E^x} \Big|_{\bar{\theta}_i+\epsilon} - \frac{E^y}{E^x} \Big|_{\bar{\theta}_i} \right\} \right]^2 \\
&= -\frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i))^2}{V(\mathcal{I}_i)} \left[\frac{\int_{\mathcal{I}_i} E^x}{\int_{\mathcal{I}_i} E^y} \left\{ \frac{\int_{\mathcal{I}_{i+1}} E^y}{\int_{\mathcal{I}_{i+1}} E^x} - \frac{\int_{\mathcal{I}_i} E^y}{\int_{\mathcal{I}_i} E^x} \right\} \right]^2 \quad (49)
\end{aligned}$$

Now we have the fluxes in the denominator which can be defined exactly as the inverse triad operators of LQC [17]. To be explicit, denoting the fluxes as $\mathcal{F}_{x,\mathcal{I}} := \int_{\mathcal{I}} E^x$, $\mathcal{F}_{y,\mathcal{I}} := \int_{\mathcal{I}} E^y$.

$$\begin{aligned}
\mathcal{F}_{x,\mathcal{I}}^{-1} &= \left(\frac{1}{\kappa'\gamma l} \right)^{\frac{1}{1-l}} \{X(v), \mathcal{F}_{x,\mathcal{I}}^l\}^{\frac{1}{1-l}} \quad , \quad l \in (0, 1) \\
&= \left(\frac{2i}{\kappa'\gamma l \mu_0} \right)^{\frac{1}{1-l}} \left(h_v^{(\mu_0/2)}(X) \{h_v^{(-\mu_0/2)}(X), \mathcal{F}_{x,\mathcal{I}}^l\} \right)^{\frac{1}{1-l}} \quad (50)
\end{aligned}$$

and similarly for $\mathcal{F}_{y,\mathcal{I}}^{-1}$. These can be promoted to a well defined operator. Continuing with the equation above,

$$\begin{aligned}
\text{RHS} &= -\frac{1}{4} \sum_{i=1}^n N(\bar{\theta}_i) \frac{(\mathcal{E}(\bar{\theta}_i))^2}{V(\mathcal{I}_i)} \left[\mathcal{F}_{y,\mathcal{I}_i}^{-1} \mathcal{F}_{x,\mathcal{I}_i} \left(\mathcal{F}_{x,\mathcal{I}_{i+1}}^{-1} \mathcal{F}_{y,\mathcal{I}_{i+1}} - \mathcal{F}_{x,\mathcal{I}_i}^{-1} \mathcal{F}_{y,\mathcal{I}_i} \right) \right]^2 \\
&= -\frac{1}{4} \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0 \alpha^3} \right)^k \sum_{i=1}^n N(\bar{\theta}_i) (\mathcal{E}(\bar{\theta}_i))^2 \times \\
&\quad \left[\mathcal{F}_{y,\mathcal{I}_i}^{-1} \mathcal{F}_{x,\mathcal{I}_i} \left(\mathcal{F}_{x,\mathcal{I}_{i+1}}^{-1} \mathcal{F}_{y,\mathcal{I}_{i+1}} - \mathcal{F}_{x,\mathcal{I}_i}^{-1} \mathcal{F}_{y,\mathcal{I}_i} \right) \right]^2 (\mathcal{Z}_\alpha(\mathcal{I}_i))^k \Big|_{\alpha=\frac{2}{3}-\frac{1}{3k}}
\end{aligned} \tag{51}$$

where, in the last step, we have manipulated,

$$\begin{aligned}
\frac{1}{V(\mathcal{I}_i)} &= \frac{1}{V(\mathcal{I}_i)} (1)^k = \frac{1}{V(\mathcal{I}_i)} \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0} \right)^k \left(\frac{\mathcal{Z}(\mathcal{I}_i)}{V(\mathcal{I}_i)} \right)^k \\
&= \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0 \alpha^3} \right)^k (\mathcal{Z}_\alpha(\mathcal{I}_i))^k \Big|_{\alpha=\frac{2}{3}-\frac{1}{3k}}
\end{aligned} \tag{52}$$

The choice of α would be same as that in the first term.

The third term of H_P :

$$\begin{aligned}
H_T &= - \int_{S^1} N(\theta) \partial_\theta \left[\frac{\mathcal{E} \partial_\theta \mathcal{E}}{\sqrt{E(\theta)}} \right] \approx - \sum_{i=1}^n N(\bar{\theta}_i) \epsilon \partial_\theta \left[\frac{\mathcal{E}(\bar{\theta}_i) \partial_\theta \mathcal{E}}{\sqrt{E(\bar{\theta}_i)}} \right] \\
\text{RHS} &= - \sum_{i=1}^n N(\bar{\theta}_i) \left[\left\{ \frac{\mathcal{E} \epsilon \partial_\theta \mathcal{E}}{\epsilon \sqrt{E}} \right\} \Big|_{\bar{\theta}_i+\epsilon} - \left\{ \frac{\mathcal{E} \epsilon \partial_\theta \mathcal{E}}{\epsilon \sqrt{E}} \right\} \Big|_{\bar{\theta}_i} \right] \\
&= - \sum_{i=1}^n N(\bar{\theta}_i) \left[\frac{\mathcal{E}(\bar{\theta}_i + \epsilon) \{ \mathcal{E}(\bar{\theta}_i + 2\epsilon) - \mathcal{E}(\bar{\theta}_i + \epsilon) \}}{V(\mathcal{I}_{i+1})} - \frac{\mathcal{E}(\bar{\theta}_i) \{ \mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i) \}}{V(\mathcal{I}_i)} \right] \\
&= - \left(\frac{16}{3(\kappa'\gamma)^3 \mu_0 \nu_0 \alpha^3} \right)^k \sum_{i=1}^n N(\bar{\theta}_i) \left[\mathcal{E}(\bar{\theta}_i + \epsilon) \{ \mathcal{E}(\bar{\theta}_i + 2\epsilon) - \mathcal{E}(\bar{\theta}_i + \epsilon) \} (\mathcal{Z}_\alpha(\mathcal{I}_{i+1}))^k \right. \\
&\quad \left. - \mathcal{E}(\bar{\theta}_i) \{ \mathcal{E}(\bar{\theta}_i + \epsilon) - \mathcal{E}(\bar{\theta}_i) \} (\mathcal{Z}_\alpha(\mathcal{I}_i))^k \right] \Big|_{\alpha=\frac{2}{3}-\frac{1}{3k}}
\end{aligned} \tag{53}$$

At this point we have expressed the H_P in terms of the holonomy-flux variables and quantization can be carried out simply by replacing the $(\mathcal{Z}_\alpha)^k \rightarrow (-i/\hbar)^{3k} (\hat{\mathcal{Z}}_\alpha)^k$. This correctly combines with the powers of κ' to give $(\ell_P^2)^{3k}$ in the denominator. The $(\mathcal{Z}_\alpha)^k$, will give $(\ell_P^{3\alpha})^{3k}$ since each factor of volume gives 3α , there are 3 factors of volume in each \mathcal{Z} and there is the overall power of k . Substituting for α one sees that each of the terms in

H_P, H_T has ℓ_P^{-3} apart from the ℓ_P^4 supplied by the factors of momenta/fluxes, thus giving the correct dimensions.

The operators $\mathcal{O}_\alpha^a := [\cos(\dots)\hat{V}^\alpha \sin(\dots) - \sin(\dots)\hat{V}^\alpha \cos(\dots)]$, $a = x, y, \theta$, appear in all the terms and is a function of both holonomies and fluxes. To see that this is actually diagonal in the charge network basis, write the cos and sin operators as sums and differences of the exponentials (i.e. holonomies). It then follows that,

$$\cos(\dots)\hat{V}^\alpha \sin(\dots) - \sin(\dots)\hat{V}^\alpha \cos(\dots) = \frac{1}{2i} \left[e^{-i(\dots)}\hat{V}e^{+i(\dots)} - e^{+i(\dots)}\hat{V}e^{-i(\dots)} \right] \quad (54)$$

It is now obvious that the operators are diagonal and thus commute with all the flux operators. Thus there are no ordering issues in quantization of H_P operators. In the H_K , however, operators of the above type are ordered on the right as in LQG.

IV. ACTION ON STATES

To make explicit the action of the Hamiltonian constraint on the basis states, it is useful to make a couple of observations. Every gauge-invariant *basis* state can be thought of as a collection of m -vertices with a quadruple of labels (k_v^\pm, μ_v, ν_v) , all non-zero. The k_v^\pm denoting the U(1) representations on the two edges meeting at v with $+$ referring to the exiting edge and $-$ to the entering edge. A partition may also be viewed as a graph except that at its “vertices” all representation labels are zero. Secondly, the action of the flux operators labelled by \mathcal{I} , on a basis state is necessarily zero if none of the vertices of the state have an intersection with the label interval. Note that the operator $\mathcal{E}(\theta_i)$, however always has a non-zero action on a basis state. This is because, all graphs are closed and hence there is always an edge (and non-zero label for a basis state) which overlaps with θ_i . The volume operator associated with an interval \mathcal{I} gives a non-zero contribution on a basis state *only if* \mathcal{I} contains a vertex of the graph. Recall that our partition is sufficiently refined so that each cell contains *at most one vertex* (two vertices at the cell boundaries are counted as a single vertex in the interior).

The full Hamiltonian has been written as a sum using a partition of S^1 . Consider the i^{th} term in each of the H_K, H_P . Each of these contains \mathcal{O}_α^a operators either separately (as in H_K) or as a product through the \mathcal{Z}_α (as in H_P). Since these contain the volume operator, it ensures that the action of each of these terms is necessarily zero unless the \mathcal{I}_i contains a

vertex of the basis state. Evidently the action of the full constraint is *finite* regardless of the chosen partition.

The factors of trigonometric operators multiplying the \mathcal{O}_α^a on the left in H_K , can be thought of as “creating new vertices” at the points $\bar{\theta}_i$ of the partition. Notice however that at these new vertices one has either an edge holonomy or *one* of the point holonomies only i.e. the volume operator acting at these vertices will give zero.

Summarizing, thanks to the $\hat{\mathcal{O}}, \hat{\mathcal{Z}}$ operators acting first, only those intervals of a partition will contribute which contain at least one vertex of the graph of a basis state. This immediately implies that in the second term of H_P (eq. 51), only one of the terms in square bracket will contribute. We will return to this later. Let us denote the factor associated with a vertex v of a basis state by $|k_v^\pm, \mu_v, \nu_v\rangle$. Here are the actions of all the 6 terms of the Hamiltonian constraint restricted to the interval containing v :

$$\hat{H}_K^\theta |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma\ell_P^2}}{4\gamma^2\mu_0\nu_0} \left[\sqrt{|\mu_v|\nu_v} \left(\sqrt{|k_v^+ + k_v^- + 1|} - \sqrt{|k_v^+ + k_v^- - 1|} \right) \times \right. \quad (55)$$

$$\left. \sin(\mu_0 X(\bar{\theta}_i)) \sin(\nu_0 Y(\bar{\theta}_i)) \right] |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_K^x |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma\ell_P^2}}{4\gamma^2\mu_0\nu_0} \left[\sqrt{|k_v^+ + k_v^-|\nu_v} \left(\sqrt{|\mu_v + \mu_0|} - \sqrt{|\mu_v - \mu_0|} \right) \times \right. \quad (56)$$

$$\left. 2 \sin\left(\frac{1}{2}\nu_0 Y(\bar{\theta}_i + \epsilon)\right) \cos\left(\frac{1}{2}\nu_0 Y(\bar{\theta}_i)\right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i\right) \right] |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_K^y |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma\ell_P^2}}{4\gamma^2\mu_0\nu_0} \left[\sqrt{|k_v^+ + k_v^-|\mu_v} \left(\sqrt{|\nu_v + \nu_0|} - \sqrt{|\nu_v - \nu_0|} \right) \times \right. \quad (57)$$

$$\left. 2 \sin\left(\frac{1}{2}\nu_0 X(\bar{\theta}_i + \epsilon)\right) \cos\left(\frac{1}{2}\nu_0 X(\bar{\theta}_i)\right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i\right) \right] |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_P^{(1)} |k_v^\pm, \mu_v, \nu_v\rangle = \left[\frac{\sqrt{\gamma\ell_P^2}}{2} \left(\frac{1}{8\mu_0\nu_0\alpha^3} \right)^k \right] \left\{ (k_v^+ - k_v^-)^2 \times \right. \quad (58)$$

$$\left[\{ (|\mu_v + \mu_0|^\alpha - |\mu_v - \mu_0|^\alpha) |\nu_v|^\alpha |k_v^+ + k_v^-|^\alpha \} \times \right.$$

$$\{ |\mu_v|^\alpha (|\nu_v + \nu_0|^\alpha - |\nu_v - \nu_0|^\alpha) |k_v^+ + k_v^-|^\alpha \} \times$$

$$\left. \left. \{ |\mu_v|^\alpha |\nu_v|^\alpha (|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- - 1|^\alpha) \} \right]^k |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_P^{(2)} |k_v^\pm, \mu_v, \nu_v\rangle = \left[-\frac{\sqrt{\gamma\ell_P^2}}{2} \left(\frac{1}{8\mu_0\nu_0\alpha^3} \right)^k \right] \left\{ (k_v^+ + k_v^-)^2 \times \right.$$

$$\left. \left(\widehat{\mathcal{F}}_{x,\mathcal{I}_i}^{-1} \hat{\mathcal{F}}_{x,\mathcal{I}_i}(\mu_v) \right)^2 \left(\widehat{\mathcal{F}}_{y,\mathcal{I}_i}^{-1} \hat{\mathcal{F}}_{y,\mathcal{I}_i}(\nu_v) \right)^2 \right\} \times$$

$$\begin{aligned}
& \left[\left\{ (|\mu_v + \mu_0|^\alpha - |\mu_v - \mu_0|^\alpha) |\nu_v|^\alpha |k_v^+ + k_v^-|^\alpha \right\} \times \right. \\
& \left\{ |\mu_v|^\alpha (|\nu_v + \nu_0|^\alpha - |\nu_v - \nu_0|^\alpha) |k_v^+ + k_v^-|^\alpha \right\} \times \\
& \left. \left\{ |\mu_v|^\alpha |\nu_v|^\alpha (|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- - 1|^\alpha) \right\} \right]^k |k_v^\pm, \mu_v, \nu_v\rangle
\end{aligned} \tag{59}$$

$$\begin{aligned}
\hat{H}_P^{(3)} |k_v^\pm, \mu_v, \nu_v\rangle = & \left[-2\sqrt{\gamma \ell_P^2} \left(\frac{1}{8\mu_0\nu_0\alpha^3} \right)^k \right] \left\{ -k_v^- (k_v^+ - k_v^-) \right\} \times \\
& \left[\left\{ (|\mu_v + \mu_0|^\alpha - |\mu_v - \mu_0|^\alpha) |\nu_v|^\alpha |k_v^+ + k_v^-|^\alpha \right\} \times \right. \\
& \left\{ |\mu_v|^\alpha (|\nu_v + \nu_0|^\alpha - |\nu_v - \nu_0|^\alpha) |k_v^+ + k_v^-|^\alpha \right\} \times \\
& \left. \left\{ |\mu_v|^\alpha |\nu_v|^\alpha (|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- - 1|^\alpha) \right\} \right]^k |k_v^\pm, \mu_v, \nu_v\rangle
\end{aligned} \tag{60}$$

In the above, factors of $N(\bar{\theta})$ are suppressed.

In the first three equations, we have explicitly evaluated only the action of the diagonal operators and kept the holonomies which “create new vertices” as operators acting on $|k_v^\pm, \mu_v, \nu_v\rangle$. In the terms involving α , we have to use $\alpha = \frac{2}{3} - \frac{1}{3k}$. The last square brackets in the last three terms is the action of the $\mathcal{Z}_\alpha(\mathcal{I}_i)$ after the dimensional and numerical factors are collected together in the first square bracket. In $H_P^{(2)}$, the products of inverse flux and flux operators approach 1 only for large values of μ_v, ν_v while for smaller values, these products vanish.

The above actions have to be summed over all the vertices of the graph. These being finite, the action is finite as noted before. There is no explicit appearance of ϵ . Reference to cells enclosing the vertices (eg $\bar{\theta}_i, \mathcal{I}_i$), will again transfer only to the vertices in the limit of infinite refinement. The technical issue of limiting operator on Cyl^* can be done in the same manner as in the full theory eg as in [2].

The above definitions of the quantization of the Hamiltonian constraint constitute a *choice* and there are many choices possible. There is also the issue related to “local degrees of freedom”. In the next section, a preliminary discussion of these features is presented.

V. DISCUSSION

Let us quickly recapitulate where we made various choices. To begin with, we made a cell decomposition with the understanding of taking the limit of infinitely many cells. At this stage, no reference to any state or graph is made. In the regularization of the kinetic term we

used the ‘inverse volume’ and ‘plaquette holonomies’. We could have introduced inverse flux operators and $\hat{\mathcal{E}}$ operators to replace $1/\sqrt{E}$ and also replaced the $X, Y, \int_{\mathcal{I}_i} \mathcal{A}$ by $\sin(\mu_0 X)/\mu_0$ and similarly for the others. Such a replacement would still give the classical expression back, in the limit of small X, Y, ϵ . The quantum operator however would be different. This procedure will also deviate from the full theory. From the point of view of the reduced theory, this is an ambiguity. Also, in the transcription of H_P in terms of holonomies and fluxes certain choices have been made. For example, the second term in the H_P , could have been manipulated in terms of inverse powers of \sqrt{E} instead of introducing inverse flux operators (eg by replacing $1/E^x = \mathcal{E}E^y/(\sqrt{E})^2$). This would lead to $\mathcal{E}^2 [\mathcal{F}_{x,\mathcal{I}_i}\mathcal{F}_{y,\mathcal{I}_{i+1}} - \mathcal{F}_{y,\mathcal{I}_i}\mathcal{F}_{x,\mathcal{I}_{i+1}}]^2$ and lead to $\alpha(k) = 2/3 - 5/(3k)$. In the limit of infinite refinement, each cell will contain *at most* one vertex and the cells adjacent to such a cell will always be empty. Consequently, the second term of H_P , regulated in the above manner will always give a zero action. Over and above these different transcriptions, we also have the ambiguity introduced by the arbitrary positive power k (and $\alpha(k)$) as well as that introduced by the arbitrary power $l \in (0, 1)$ in the definition of inverse flux operators. All these ambiguities refer to the transcription stage.

There are also issues related to the choice of partitions, subsequent $\epsilon \rightarrow 0$ limit and the presence/absence of local degrees of freedom. This is most dramatically brought out by the second term of H_P . Classically, this is the term which reveals spatial correlations in a solution space-time through $\partial_\theta \ln(E^y/E^x)$ [15] and reflect the infinitely many, physical solutions. In the (vacuum) spherically symmetric case, such a term is absent and so are local physical degrees of freedom. We would like to see if there is a quantization of this term which reflects these correlations. The quantization chosen above does not correlate μ, ν labels at different vertices.

In general, given a graph, a partition may be chosen to have (i) every cell containing *at least* one vertex or (ii) every cell containing *exactly* one vertex or (iii) every cell containing *at most* one vertex. In this classification, we assume that a vertex is never a boundary-point of a cell, which is always possible to choose. Infinite refinement is possible only for (iii) which we have been assuming so far. This is the reason that in the contribution from the \mathcal{I}_i cell, the terms referring to \mathcal{I}_{i+1} drop out.

We could introduce a fourth case by requiring; (iv) every vertex to be a boundary point of a cell. Then we would receive contributions from two adjacent cells. However, in $H_P^{(2)}$ (eq. 51), the two terms with labels $\mathcal{I}_{i+1}, \mathcal{I}_i$, both give equal contribution such that the total

is zero! The same would happen in the alternative expression given above. It seems that in either of (iii) or (iv) type partitions, we will either get a zero or a contribution depending only on a single vertex. Note that these are the only partitions which allow infinite refinement ($\epsilon \rightarrow 0$) in a diffeo-covariant manner.

We can give up on the infinite partitions (and $\epsilon \rightarrow 0$ limit) and consider instead case (ii) partitions – each cell contains exactly one vertex (say in the interior). Now the contributions will explicitly depend upon μ, ν labels of adjacent vertices and in this sense, spatial correlations will survive in the constraint operator. An even more restrictive choice would be to choose the partition defined by the graph itself - cells defined by the edges and the boundary points of cells as vertices. In this case, the new vertices created by H_K would be the already present vertices and the constraint equation would lead to a (partial) difference equation among the labels. This case has been considered in the spherical symmetry [14] and corresponds to ‘effective operator viewpoint’ discussed by Thiemann in [1]. The $\epsilon \rightarrow 0$ limit may then be thought to be relevant when states have support on graphs with very large (but finite) number vertices, heuristically for semiclassical states. Whether requiring the constraint algebra to be satisfied on diffeomorphism invariant states chooses/restricts the alternatives and ambiguities remains to be seen and will be explored in the third paper in this series.

It is important to be able to identify the quantum theory of the symmetry reduced model as a ‘sector’ of the full theory. There are more than one ways for such an identification [18] and this is still an open problem. The midi-superspace (inhomogeneous) model is in between the full theory and the mini-superspace (homogeneous) models. This provides an opportunity to explore the identification of the appropriate homogeneous (anisotropic as well as isotropic) models as sectors of the Gowdy model³. However, at present, we do not have any specific results to report.

³ We thank an anonymous referee for pointing this out.

Acknowledgments

Discussions with Alok Laddha are gratefully acknowledged.

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