$(1, \lambda)$-embedded graphs and the acyclic edge choosability*

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Abstract

A $(1, \lambda)$-embedded graph is a graph that can be embedded on a surface with Euler characteristic λ so that each edge is crossed by at most one other edge. A graph G is called α-linear if there exists an integral constant β such that $e(G') \leq \alpha v(G') + \beta$ for each $G' \subseteq G$. In this paper, it is shown that every $(1, \lambda)$-embedded graph G is 4-linear for all possible λ, and is acyclicly edge-$(3\Delta(G) + 70)$-choosable for $\lambda = 1, 2$.

Keywords: $(1, \lambda)$-embedded graph, α-linear graph, acyclic edge choosability.

MSC: 05C10, 05C15.

1 Introduction and basic definitions

In this paper, all graphs considered are finite, simple and undirected. Let G be a graph, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G. Let $e(G) = |E(G)|$ and $v(G) = |V(G)|$. Moreover, for embedded graph G (i.e., a graph that can be embedded on a surface), by $F(G)$ we denote the face set of G. Let $f(G) = |F(G)|$. The girth $g(G)$ of a graph G is the length of the shortest cycle of G. A k-, k^+- and k^--vertex (or face) is a vertex (or face) of degree k, at least k and at most k, respectively. A graph G is called α-linear if there exists an integral constant β such that $e(G') \leq \alpha v(G') + \beta$ for each $G' \subseteq G$. Furthermore, if $\beta \geq 0$, then G is said to be α-nonnegative-linear; and if $\beta < 0$, then G is said to be α-negative-linear. For other undefined concepts we refer the reader to [3].

A mapping c from $E(G)$ to the sets of colors $\{1, \cdots, k\}$ is called a proper edge-k-coloring of G provided that any two adjacent edges receive different colors. A proper edge-k-coloring c of G is called an acyclic edge-k-coloring of G if there are no bichromatic cycles in G under the coloring c. The smallest number of colors such that G has an acyclic edge coloring is called the *acyclic edge chromatic number* of G, denoted by $\chi'_a(G)$. A graph is said to be acyclic edge-f-choosable, whenever we give a list L_e of $f(e)$ colors to each edge $e \in E(G)$, there exists an acyclic

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edge-k-coloring of G, where each element is colored with a color from its own list. If $|L_e| = k$ for edge $e \in E(G)$, we say that G is acyclicly edge-k-choosable. The minimum integer k such that G is acyclicly edge-k-choosable is called the *acyclic edge choice number* of G, denoted by $\chi'_c(G)$.

Acyclic coloring problem introduced in [8] has been extensively studied in many papers. One of the famous conjectures on the acyclic chromatic index is due to Alon, Sudakov and Zaks [2]. They conjectured that $\chi'_a(G) \leq \Delta(G) + 2$ for any graph G. Alon et al.[1] proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph G by using probabilistic arguments. This bound for arbitrary graph was later improved to $16\Delta(G)$ by Molloy and Reed [9] and recently improved to $9.62\Delta(G)$ by Ndreca et al.[10]. In 2008, Fiedorowicz et al.[7] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$ for each planar graph G by applying a combinatorial method. Nowadays, acyclic coloring problem has attracted more and more attention since Coleman et al.[4, 5] identified acyclic coloring as the model for computing a Hessian matrix via a substitution method. Thus to consider the acyclic coloring problems on some other special classes of graphs seems interesting.

A graph is called 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar-graph was introduced by Ringel[11] while trying to simultaneously color the vertices and faces of a planar graph such that any pair of adjacent/incident elements receive different colors. In fact, from a planar graph G, we can construct a 1-planar graph G' with its vertex set being $V(G) \cup F(G)$, and any two vertices of G' being adjacent if and only if their corresponding elements in G are adjacent or incident. Now we generalize this concept to $(1, \lambda)$-embedded graph, namely, a graph that can be embedded on a surface S with Euler characteristic λ so that each edge is crossed by at most one other edge. Actually, a $(1, 2)$-embedded graph is a 1-planar graph. It is shown in many papers such as [6] that $e(G) \leq 4v(G) - 8$ for every 1-planar graph G. Whereafter, to determine whether the number of edges in the class of $(1, \lambda)$-embedded graphs is linear or not linear in the number of vertices for every $\lambda \leq 2$ might be interesting.

In this paper, we first investigate some structures of $(1, \lambda)$-embedded graph G in Section 2 and then give a relationship among the three parameters $e(G)$, $v(G)$ and $g(G)$ of G, which implies that every $(1, \lambda)$-embedded graph is 4-linear for any $\lambda \leq 2$. In Section 3, we will introduce a linear upper bound for the acyclic edge choice number of the classes $(1, \lambda)$-embedded graphs with special given λ.

2 The linearity of $(1, \lambda)$-embedded graphs

Given a "good" graph G (i.e., one for which all intersecting edges intersect in a single point and arise from four distinct vertices), the crossing number, denoted by $cr(G)$, is the minimum possible number of crossings with which the graph can be drawn.

Let G be a $(1, \lambda)$-embedded graph. In the following we always assume that G has been embedded on a surface with Euler characteristic λ so that each edge is crossed by at most one other edge and the number of crossings of G in this embedding is minimum. Thus, G has exactly $cr(G)$ crossings. Sometimes we say such an embedding proper for convenience.

Theorem 2.1. Let G be a $(1, \lambda)$-embedded graph. Then $cr(G) \leq v(G) - \lambda$.

Proof. Suppose G has been properly embedded on a surface with Euler characteristic λ. Then for each pair of edges ab, cd that cross each other at a crossing point s, their end vertices are pairwise distinct. For each such pair, we add new edges ac, cb, bd, da (if it does not exist originally) to close s, then arbitrarily delete one edge ab or cd from G. Denote the resulting graph by G^* and then we have $cr(G^*) = 0$. By Euler's formula $v(G^*) - e(G^*) + f(G^*) = \lambda$ and the well-known relation $\sum_{v\in V(G^*)} d_{V(G^*)}(v) = \sum_{f\in F(G^*)} d_{V(G^*)}(f) = 2e(G^*)$, $f(G^*) \leq 2v(G^*) - 2\lambda$. Since each crossing point s (note that s is not a real vertex in G) lies on a common boundary of two faces of G^* and each face of G^* is incident with at most one crossing point (recall the definition of G^*), we deduce that $2cr(G) \leq f(G^*)$. Since $v(G) = v(G^*)$, we have $cr(G) \leq \frac{f(G^*)}{2} \leq v(G^*) - \lambda = v(G) - \lambda$ in final. □

Theorem 2.2. Let G be a $(1, \lambda)$-embedded graph with girth at least g. Then $e(G) \leq \frac{2g-2}{g-2}(v(G) - \lambda)$.

Proof. Suppose G has been properly embedded on a surface with Euler characteristic λ. Now for each pair of edges ab, cd that cross each other, we arbitrarily delete one from G. Let G' be the resulting graph. One can easily see that $cr(G') = 0$. By Euler's formula $v(G') - e(G') + f(G') = \lambda$ and the relations $v(G') = v(G)$, $e(G') = e(G) - cr(G)$, we have

$$v(G) - e(G) + f(G') = v(G') - e(G') + f(G') - cr(G) = \lambda - cr(G) \tag{2.1}$$

and

$$\sum_{f\in F(G')} d_{G'}(f) = 2e(G') = 2(e(G) - cr(G)) \geq g(G')f(G') \geq g(G)f(G') \geq g \cdot f(G'). \tag{2.2}$$

Now combine equations (2.1) and (2.2) together, we immediately have $e(G) \leq \frac{g}{g-2}(v(G) - \lambda) + cr(G) \leq \frac{2g-2}{g-2}(v(G) - \lambda)$ by Theorem 2.1. □

By Theorem 2.2, the following two corollaries are natural.

Corollary 2.3. Every $(1, \lambda)$-embedded graph is 4-linear for any $\lambda \leq 2$.

Corollary 2.4. Every triangle-free $(1, \lambda)$-embedded graph is 3-linear for any $\lambda \leq 2$.

3 Acyclic edge choosability of $(1, \lambda)$-embedded graphs

In this section we mainly investigate the acyclic edge choosability of $(1, \lambda)$-embedded graphs with special given λ. In [7], Fiedorowicz et al. proved the following two results.

Theorem 3.1. If G is a graph such that $e(G') \leq 2v(G') - 1$ for each $G' \subseteq G$, then $\chi'_a(G) \leq \Delta(G) + 6$.

Theorem 3.2. If G is a graph such that $e(G') \leq 3v(G') - 1$ for each $G' \subseteq G$, then $\chi'_a(G) \leq 2\Delta(G) + 29$.

In fact, these two theorems respectively imply that the acyclic edge chromatic number of 2-negative-linear graph G is at most $\Delta(G) + 6$ and that the acyclic edge chromatic number of 3-negative-linear graph G is at most $2\Delta(G) + 29$.

Note that every triangle-free $(1, \lambda)$-embedded graph is 3-negative-linear for any $1 \leq \lambda \leq 2$ by Theorem 2.2. Hence the following corollary is trivial.

Corollary 3.3. Let G be a triangle-free $(1,\lambda)$-embedded graph with $1 \le \lambda \le 2$. Then $\chi'_a(G) \le 2\Delta(G) + 29$.

The following main theorem in this section is dedicated to giving a linear upper bound for the acyclic edge choice number of 4-negative-linear graphs.

Theorem 3.4. If G is a graph such that $e(G') \le 4v(G') - 1$ for each $G' \subseteq G$, then $\chi'_c(G) \le 3\Delta(G) + 70$.

As an immediately corollary of Theorems 2.2 and 3.4, we have the following result.

Corollary 3.5. Let G be a $(1,\lambda)$-embedded graph with $1 \le \lambda \le 2$. Then $\chi'_c(G) \le 3\Delta(G) + 70$.

Before proving Theorem 3.4, we first show an useful structural lemma.

Lemma 3.6. Let G be a graph such that $e(G) \le 4v(G) - 1$ and $\delta(G) \ge 4$, Then at least one of the following configurations occurs in G:
($C1$) a 4-vertex adjacent to a 19^--vertex;
($C2$) a 5-vertex adjacent to two 19^--vertices;
($C3$) a 6-vertex adjacent to four 19^--vertices;
($C4$) a 7-vertex adjacent to six 19^--vertices;
($C5$) a vertex v such that $20 \le d(v) \le 22$ and at least $d(v) - 3$ of its neighbors are 7^--vertices;
($C6$) a vertex v such that $23 \le d(v) \le 25$ and at least $d(v) - 2$ of its neighbors are 7^--vertices;
($C7$) a vertex v such that $26 \le d(v) \le 28$ and at least $d(v) - 1$ of its neighbors are 7^--vertices;
($C8$) a vertex v such that $29 \le d(v) \le 31$ and all its neighbors are 7^--vertices;
($C9$) a vertex v such that at least $d(v) - 7$ of its neighbors are 7^--vertices and at least one of them is of degree 4.

Proof. Suppose, to the contrary, that none of the nine configurations occurs in G. We assign to each vertex v a charge $w(v) = d(v) - 8$, then $\sum_{v\in V(G)} w(v) = \sum_{v\in V(G)}(d(v) - 8) \le -2$. In the following, we will reassign a new charge denoted by $w'(x)$ to each $x \in V(G)$ according to some discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{v\in V(G)} w'(v) = \sum_{v\in V(G)} w(v) \le -2. \tag{3.1}$$

We next show that $w'(v) \ge 0$ for each $v \in V(G)$, which leads to a desired contradiction. We say a vertex big (resp. small) if it is a 20^+-vertex (resp. 7^--vertex). The discharging rules are defined as follows.

($\mathcal{R}1$) Each big vertex gives 1 to each adjacent 4-vertex.
($\mathcal{R}2$) Each big vertex gives $\frac{3}{4}$ to each adjacent vertex of degree between 5 and 7.

Let v be a 4-vertex. Since ($C1$) does not occur, v is adjacent to four big vertices. So v totally receives 4 by ($\mathcal{R}1$). This implies that $w'(v) = w(v) + 4 = d(v) - 4 = 0$. Similarly, we can also prove the nonnegativity of $w'(v)$ if v is a k-vertex where $5 \le k \le 7$. Let v be a k-vertex where $8 \le k \le 19$. Since v is not involved in the discharging rules, $w'(v) = w(v) = d(v) - 8 \ge 0$. Let v be a k-vertex where $20 \le k \le 22$. If v is adjacent to a 4-vertex, then v is adjacent to at most $d(v) - 8$ small vertices since ($C9$) does not occur. Since v sends each small vertex at most 1 By ($\mathcal{R}1$) and ($\mathcal{R}2$), $w'(v) \ge w(v) - (d(v) - 8) = 0$. If v is adjacent to no 4-vertices, then v sends each small vertex $\frac{3}{4}$ by ($\mathcal{R}2$). Since ($C5$)

does not occur either, v is adjacent at most $d(v)-4$ small vertices. So $w'(v) \geq w(v) - \frac{3}{4}(d(v)-4) = \frac{1}{4}(d(v)-20) \geq 0$. By similar arguments as above, we can also respectively show the nonnegativity of $w'(v)$ if v is a k-vertex where $k \geq 23$. □

***Proof of Theorem* 3.4.** Let K stands for $3\Delta(G)+70$. We prove the theorem by contradiction. Let G be a counterexample to the theorem with the number of edges as small as possible. So there exists a list assignment L of K colors such that G is not acyclicly edge-L-choosable. For each coloring c of G, we define $c(uv)$ to be the color of edge uv and set $C(u) = \{c(uv) | uv \in E(G)\}$ for each vertex u. For $W \subseteq V(G)$, set $C(W) = \bigcup_{w\in W} C(w)$. If $uv \in E(G)$, we let $W_G(v,u)$ stands for the set of neighbors w of v in G such that $c(vw) \in C(u)$. Now, we first prove that $\delta(G) \geq 4$.

Suppose that there is a 3-vertex $v \in V(G)$. Denote the three neighbors of v by x, y and z. Then by the minimality of G, the graph $H = G - ux$ is acyclicly edge-L-choosable. Let c be an acyclic edge coloring of H. We can extend c to uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \backslash (C(x) \cup C(y) \cup C(z)).$$

Since $|C(x)| \leq \Delta(G)-1$, $|C(y)| \leq \Delta(G)$ and $|C(z)| \leq \Delta(G)$, we have $|A(uv)| \geq K - 3\Delta(G) + 1 > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$, a contradiction. Similarly, one can also prove the absences of 1-vertices and 2-vertices in G. Hence $\delta(G) \geq 4$. Then by Lemma 3.6, G contains at least one of the configurations $(C1)$-$(C9)$. In the following, we only show that if one of the configurations $(C4)$, $(C5)$ and $(C9)$ appears, then we would get a contradiction. That is because the proofs are similar and easier for another six cases.

Configuration $(C4)$: Suppose that there is a 7-vertex v who is adjacent to six 19^--vertices, say $x_1, x_2, \cdots, x_6$. Denote another one neighbor of v by x_7. Then by the minimality of G, the graph $H = G - vx_7$ is acyclicly edge-L-choosable. Let c be an acyclic edge coloring of H. Suppose $c(vx_j) \notin C(x_7)$ for some $1 \leq j \leq 6$. Then we can extend c to vx_7 by defining a list of available colors for vx_7 as follows:

$$A(vx_7) = L(vx_7) \backslash \bigcup_{1\leq i\neq j\leq 7} C(x_i).$$

Since $|C(x_i)| \leq \min\{19, \Delta(G)\}$ for every $1 \leq i \leq 6$ and $|C(x_7)| \leq \Delta(G)-1$, we have $|A(vx_7)| \geq K - \min\{\Delta(G) + 94, 6\Delta(G)-1\} = \max\{2\Delta(G)-24, -3\Delta(G)+69\} > 0$. So we can color vx_7 by a color in $A(vx_7) \subseteq L(vx_7)$, a contradiction. Thus we shall assume that $c(vx_j) \in C(x_7)$ for every $1 \leq j \leq 6$. This implies that $|\bigcup_{1\leq i\leq 7} C(x_i)| \leq \Delta(G) - 1 + 6 \times \min\{19, \Delta(G)\} - 6 = \min\{\Delta(G)+107, 7\Delta(G)-7\}$. Now we extend c to vx_7 by defining a list of available colors for vx_7 as follows:

$$A(vx_7) = L(vx_7) \backslash \bigcup_{1\leq i\leq 7} C(x_i).$$

Note that $|A(vx_7)| \geq K - \min\{\Delta(G)+107, 7\Delta(G)-7\} = \max\{2\Delta(G)-37, -4\Delta(G)+77\} > 0$. So we can again color vx_7 by a color in $A(vx_7) \subseteq L(vx_7)$, also a contradiction.

Configuration $(C5)$: If there is a vertex v such that $20 \leq d(v) \leq 22$ and at least $d(v)-3$ of its neighbors are 7^--vertices. Without loss of generality, we assume that $d(v) = 22$ and that v have nineteen 7^--neighbors. Denote another three neighbors of v by x, y and z. Choose one 7^--neighbor, say u, of v. Without loss of generality, we assume that $d(u) = 7$.

Then by the minimality of G, the graph $H = G - uv$ is acyclicly edge-L-choosable. Let c be an acyclic edge coloring of H. Then we can extend c to uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \backslash \{C(u) \cup C(v) \cup C(x) \cup C(y) \cup C(z) \cup C(W_H(v,u)\}.$$

Since c is an acyclic (and thus it is proper), $|W_H(v,u)| \leq d(u)-1 = 6$. Since $|C(x)| \leq \Delta(G)$, $|C(y)| \leq \Delta(G)$, $|C(z)| \leq \Delta(G)$ and $|C(w)| \leq 7$ for each $w \in W_H(v,u)$, we have $|A(uv)| \leq K - (3\Delta(G) + 6 + 6 \times 6 + 21 - 9) > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$, a contradiction.

Configuration ($C9$) If there is a vertex v such that at least $d(v) - 7$ of its neighbors are 7^--vertices and at least one of them is of degree 4, say u. Denote another three neighbors of u by x, y and z. Let $C_1 = \{c(vw) | w \in N_H(v) \textit{ and } d_H(w) > 7\}$ and $C_2 = \{c(vw) | w \in N_H(v) \textit{ and } d_H(w) \leq 7\}$. Then $|C_1| \leq 7$. By the minimality of G, the graph $H = G - uv$ is acyclicly edge-L-choosable. Let c be an acyclic edge coloring of H. Suppose $C(u) \cap C_1 \neq \emptyset$. Without loss of generality, we assume that $c(ux) \in C_1$. Now we erase the color of the edge ux from c and recolor it from the list defined as follows:

$$A(ux) = L(ux) \backslash \{C(x) \cup C(y) \cup C(z) \cup C_1\}.$$

Since $|C(x)| \leq \Delta(G)$, $|C(y)| \leq \Delta(G)$, $|C(z)| \leq \Delta(G)$ and $|C_1| \leq 7$, we have $|A(ux)| \geq K - (3\Delta(G) + 7) > 0$. Note that $A(ux)$ is just a sub-list of the original list given at the beginning of the proof and the new color of ux preserves the acyclicity of the coloring of H. So we can assume that $C(u) \cap C_1 = \emptyset$. In this case, we can extend c to the edge uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \backslash \{C(u) \cup C_1 \cup C_2 \cup C(W_H(v,u))\}.$$

Since $C(u) \cap C_1 = \emptyset$, we have $c(vw) \in C_2$ for each $w \in W_H(v,u)$ and thus $|C(W_H(v,u))| \leq 7d_H(u) = 21$. Since $|C_1 \cup C_2| = d_H(v) \leq \Delta(G) - 1$ and $|C(u)| = 3$, we have $|A(uv)| \geq K - (\Delta(G) + 23) > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$. This contradiction completes the proof of Theorem 3.4. □

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