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A Non-linear System of Parabolic Type PDEs for Epidemic Models

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Abstract

The present paper shows the existence of solutions to a system of reaction-diffusion PDE's

$$L_i[u_i] = f_i(x,t,u_1,u_2) \quad i = 1,2 \text{ en } \overline{\Omega} \times [0,s],$$

with the following initial and boundary conditions

$$B_i[u_i] = h_i(x,t) \quad i = 1,2 \text{ en } \partial\Omega \times [0,s],$$
$$u_1(x,0) = u_0(x) \text{ y } u_2(x,0) = v_0(x) \text{ en } \overline{\Omega}.$$

L_i denote uniformly parabolic operators in $\overline{\Omega} \times [0,s]$, $u_1 = u_1(x,t)$; $u_2 = u_2(x,t)$ represent both the susceptible population and the infective population, respectively; $f_i(x,t,u_1,u_2) = -a_i u_i \mp c_i G(u_2) u_1 + q_i(x,t)$, where a_i, c_i are positive constants that represent the reaction rates and q_i are possible external sources. The function $f_i : \overline{\Omega} \times [0,s] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is regular enough for each $i = 1,2$; Ω is an open, bounded and connected set $\Omega \subset \mathbb{R}^N$, $N \geq 1$; $G(u_2)$ is a functional that will be defined bellow.

1 Introduction

A *small* set of individuals that host an infectious disease (infective population) is introduced into a larger population of individuals that are likely to contract such a disease (susceptible population). Recent studies about this model can be found in [4], [7]. In [4], the basic epidemic model is described by:

$$u_t - \nabla(D_1 \nabla u) = -a_1 u - c_1(G(v))u + q_1(x,t) \text{ in } \overline{\Omega} \times [0,s], \tag{1}$$
$$v_t - \nabla(D_2 \nabla v) = -a_2 v + c_2(G(v))u + q_2(x,t) \text{ in } \overline{\Omega} \times [0,s], \tag{2}$$
$$u(x,0) = u_0(x) \text{ y } v(x,0) = v_0(x) \text{ in } \overline{\Omega}, \tag{3}$$
$$u(x,t) = h_1(x,t) \quad \text{ y } \quad v(x,t) = h_2(x,t) \text{ in } \partial\Omega \times [0,s], \tag{4}$$

where $u(x,t)$ is the susceptible population and $v(x,t)$ is the infective population, D_1 and D_2 are the diffusion coefficients, a_1, a_2, c_1, c_2 represent the reaction rates, q_1 and q_2 are continuous, non-negative functions defined in $\overline{\Omega} \times [0,s]$, are possible external sources. This paper considers a more general problem than that in (1)–(4), given by

$$L_1[u] = -a_1 u - c_1(G(v))u + q_1(x,t) \text{ in } \overline{\Omega} \times [0,s], \tag{5}$$
$$L_2[v] = -a_2 v + c_2(G(v))u + q_2(x,t) \text{ in } \overline{\Omega} \times [0,s], \tag{6}$$
$$u(x,0) \equiv 0 \text{ y } v(x,0) \equiv 0 \text{ in } \overline{\Omega}, \tag{7}$$
$$u(x,t) \equiv 0 \text{ y } v(x,t) \equiv 0 \text{ in } \partial\Omega \times [0,s], \tag{8}$$

where $L_1[u]$ and $L_2[v]$ are uniformly parabolic operators in $D_s = \overline{\Omega} \times [0,s]$. The functional $G(v)$ is defined as follows:

$$G(v)(x,t) = \int_\Omega g(x,y)v(y,t)dy, \tag{9}$$

for every function $v \in C(D_s)$, where g is a continuous positive function defined in $\overline{\Omega} \times \overline{\Omega}$.

2 Definitions and Preliminaries

Let Ω be a bounded domain set $\Omega \subset \mathbb{R}^N$, $N \geq 1$, $[0,s]$ a compact interval in $\mathbb{R}$, $x = (x_1, x_2, \cdots x_N) \in \mathbb{R}^N$, $N \geq 1$ and u is a real-valued function defined in $D_s = \overline{\Omega} \times [0,s]$. Let us denote $u \in C^{\alpha,\alpha/2}(D_s)$ with exponent α, $0 < \alpha < 1$ if the number

$$\overline{H}_\alpha(u) = \sup_{\substack{(x_1,y_1),(x_2,y_2) \in D_s \\ (x_1,y_1) \neq (x_2,y_2)}} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{[|x_1 - x_2|^2 + |t_1 - t_2|]^{\alpha/2}} < \infty.$$

The set $C^{\alpha,\alpha/2}(D_s)$ is a Banach space with norm

$$||u||_\alpha = ||u||_\infty + \overline{H}_\alpha(u) \quad \text{ where } ||u||_\infty = \max_{(x,t) \in D_s} |u(x,t)|. \tag{10}$$

It is readily observable that, if $\overline{H}_\alpha(u) < \infty$ then $||u||_\infty < \infty$. u is denoted as uniformly continuous in D_s with exponent α if $u \in C^{\alpha,\alpha/2}(D_s)$. If $0 < \alpha < \beta < 1$, we have $C^{\beta,\beta/2}(D_s) \subset C^{\alpha,\alpha/2}(D_s)$. Let us write $u \in C^{2+\alpha,1+\alpha/2}(D_s)$ if u, u_{x_i}, $u_{x_ix_j}$, and u_t belongs to $C^{\alpha,\alpha/2}(D_s)$ for $1 \le i,j \le N$. The set $C^{2+\alpha,1+\alpha/2}(D_s)$ is a Banach space with norm defined by

$$||u||_{2+\alpha} = |||u||_\alpha + \sum_{i=1}^{N} ||u_{x_i}||_\alpha + \sum_{i}^{N} \sum_{j=1}^{N} ||u_{x_ix_j}||_\alpha + ||u_t||_\alpha. \tag{11}$$

Similarly $C^{1+\alpha,\alpha/2}(D_s)$ is defined as the set of all functions u defined in D_s, such that u, u_{x_i} and u_t for $1 \le i \le N$ belongs to $C^{\alpha,\alpha/2}(D_s)$; the norm la of u is defined by

$$||u||_{1+\alpha} = |||u||_\alpha + \sum_{i=1}^{N} ||u_{x_i}||_\alpha + ||u_t||_\alpha. \tag{12}$$

From (10), (11), (12) it can be inferred that

$$C^{2+\alpha,1+\alpha/2}(D_s) \subset C^{1+\alpha,\alpha/2}(D_s) \subset C^{\alpha,\alpha/2}(D_s)$$

The differential operators $L_k : C^{2+\alpha,1+\alpha/2}(D_s) \to C^{\alpha,\alpha/2}(D_s)$ are defined for each $k = 1,2$ for

$$L_1[u] \equiv u_t - \left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(\cdot) u_{x_ix_j} + \sum_{i=1}^{N} b_i(\cdot) u_{x_i} + \tilde{c}_1(\cdot) u \right) \tag{13}$$

$$L_2[v] \equiv v_t - \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \overline{a}_{ij}(\cdot) v_{x_ix_j} + \sum_{i=1}^{N} \overline{b}_i(\cdot) v_{x_i} + \tilde{c}_2(\cdot) v \right) \tag{14}$$

, for all $(x,t) \in D_s$ and also for all $u, v \in C^{2+\alpha,1+\alpha/2}(D_s)$, where the coefficients of L_1 and L_2 are functions that belong to $C^{\alpha,\alpha/2}(D_s)$, $a_{ij} = a_{ji}$, $\overline{a}_{ij} = \overline{a}_{ji}$, for $1 \le i,j \le N$, $\tilde{c}_k(\cdot) \le 0$ for $k = 1,2$ in D_s. L_k is a linear, bounded operator for $k = 1,2$, see [3]. The operators (13) and (14) are uniformly parabolic in D_s for $k = 1,2$ see [5], [2], if matrices $(a_{ij}(x,t))$ and $(\overline{a}_{ij}(x,t))$ are positive definite in D_s and there exist constants $M_1 > 0$ and $M_2 > 0$ such that for all $(x,t) \in D_s$ and for all vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \neq 0$, the following inequalities hold

$$\sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}(x,t)\varepsilon_i\varepsilon_j \ge M_1 \sum_{i=1}^{N} \varepsilon_i^2 \quad \text{y} \quad \sum_{j=1}^{N} \sum_{i=1}^{N} \overline{a}_{ij}(x,t)\varepsilon_i\varepsilon_j \ge M_2 \sum_{i=1}^{N} \varepsilon_i^2.$$

It can be said that D_s satisfies property $(\tilde{E})$ if for every point p in $\overline{S}$ there is an $N+1$- neighborhood V and a function $h \in C^{2+\alpha,1+\alpha/2}(V)$ such that $V \cap \overline{S}$, can be represented, for some i, $1 \leq i \leq N$ in the form $x_i = h(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N, t)$. Le us denote $||\cdot||_{2,p}$ and $|||\cdot|||_{2,p}$ the equivalent norms

$$||u||_{2,p} = \left(\int_0^s \left(\int_\Omega (|u|^p + \sum_{i=1}^N |u_{x_i}|^p + \sum_{i=1}^N \sum_{j=1}^N |u_{x_i x_j}|^p + |u_t|^p) dx \right) dt \right)^{1/p}$$

The norm $|||u|||_{2,p}$ is defined as the sum of the norms of u, u_{x_i}, $u_{x_i x_j}$ and u_t in $L^p(D_s)$ for $1 \leq i, j \leq N$, where $||u||_{2,p}$ is the norm of the Sobolev space $W_p^{2,1}(D_s)$ of all functions $u(x,t) \in L^p(D_s)$ that have generalized derivatives with respect to x up to the order 2 inclusive, and up to the order 1 with respect to t belonging to $L^p(D_s)$. The space $W_p^{2,1}(D_s)$ is a Banach space.

For functions $u \in C^{2+\alpha/2, 1+\alpha/2}(D_s)$ we have $||u||_{2,p} < \infty$. The Sobolev inequality states that, if $\alpha \in (0,1)$ and p is large enough, such that $0 < \alpha < 1 - (N+2)/p$, for values of β such that $0 < \alpha < \beta < 1$, then we have $W_p^{2,1}(D_s) \subset C^{1+\beta, \beta/2}(D_s)$ and there is a constant $\overline{C_1} = \overline{C_1}(\Omega, \beta, p)$ such that, if $u \in C^2(\overline{\Omega})$ then $||u||_{1+\beta} \leq \overline{C_1} ||u||_{2,p}$. The method we use to solve the problem (5)–(8) is the super-solutions and sub-solutions method, which is analogous to the super- and sub-solutions method for elliptical problems.
Let us say that a couple $(\overline{u}, \overline{v})$ in $C^2(D_s)$ is a super-solution of (5)–(8) if

$$L_1[\overline{u}] \geq -a_1 \overline{u} - c_1(G(\overline{v}))\overline{u} + q_1(x,t) \text{ in } \overline{D_s}, \tag{15}$$

$$L_2[\overline{v}] \geq -a_2 \overline{u} + c_2(G(\overline{v}))\overline{u} + q_2(x,t) \text{ in } \overline{D_s}, \tag{16}$$

$$\overline{u}(x,t) \geq 0 \quad \text{ y } \quad \overline{v}(x,t) \geq 0 \text{ in } (\overline{\Omega} \times \{0\}) \cup S. \tag{17}$$

Likewise, we say that a couple $(\underline{u}, \underline{v})$ of functions in $C^2(D_s)$ is a sub-solution of (5)–(8) if

$$L_1[\underline{u}] \leq -a_1 \underline{u} - c_1(G(\underline{v}))\underline{u} + q_1(x,t) \text{ in } \overline{D_s}, \tag{18}$$

$$L_2[\underline{v}] \leq -a_2 \underline{v} + c_2(G(\underline{v}))\underline{u} + q_2(x,t) \text{ in } \overline{D_s}, \tag{19}$$

$$\underline{u}(x,t) \leq 0 \quad \text{ and } \quad \underline{v}(x,t) \leq 0 \text{ in } (\overline{\Omega} \times \{0\}) \cup S. \tag{20}$$

In [4], by using standard methods of successive approximations, it is shown that (1)–(4) has a unique solution (u, v) if $f_i = f_i^*$ satisfies the Lipschitz condition

$$||f_i^*(u_1, u_2) - f_i^*(v_1, v_2)|| \leq K_i \left(||u_1 - v_1|| + ||u_2 - v_2|| \right),$$

where $\underline{u_i} \leq u_i \leq \overline{u_i}$ and $\underline{v_i} \leq v_i \leq \overline{v_i}$, for each $i = 1, 2$.
If $\mathbb{E}$ and $\mathbb{F}$ are the Banach spaces defined by $\mathbb{F} = C^{\alpha, \alpha/2}(D_s)$ with norm $||\cdot||_{\mathbb{F}} = ||\cdot||_\alpha$ and $\mathbb{E} = \dot{C}^{2+\alpha, 1+\alpha/2}(D_s) = \{u \in C^{2+\alpha, 1+\alpha/2}(D_s) : u \equiv 0 \text{ en } (\Omega \times \{0\}) \cup S\}$ with norm $||\cdot||_{\mathbb{E}} = ||\cdot||_{2+\alpha}$. It can be shown that, if $M : \mathbb{E} \to \mathbb{F}$ is the linear operator defined by $M[u] = L[u] + du$ then M is continuous, one to one and surjective since operator L also is; therefore $M^{-1} : \mathbb{F} \to \mathbb{E}$ is continuous.

From the theorem of Arzela-Ascoli, the injection $i : \mathbb{E} \to \mathbb{F}$, defined by $i(u) = u$ is a linear, compact operator. Therefore, if $T : \mathbb{F} \to \mathbb{F}$ is the operator defined by $T[u] = ioM^{-1}[u]$ then T is compact, that is, the image $T(S)$ of any set bounded in $\mathbb{F}$ is relatively compact in $\mathbb{F}$. For compactness, it is sufficient to show that the image is a unit closed ball that is relatively compact, or equivalently, for every sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{F}$, such that $||u_n||_{\mathbb{F}} \leq r$, for some $r > 0$ and for every integer $n \geq 1$ there is a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}_{n=1}^{\infty}$ such that $\{T[u_{n_k}]\}_{k=1}^{\infty}$ converges in $\mathbb{F}$. If D_s satisfies property $\tilde{E}$, then, for every $f \in \mathbb{F}$, there is a unique function $u \in \mathbb{E}$, see [5], that fulfills $L[u] = f$ in D_s. Additionally, there are constants $\overline{k}_1 > 0$ (which depend on $\beta \in (0, 1)$, fixed, and $\overline{k}_2 > 0$, such that $||u||_{2+\alpha} \leq \overline{k}_1||L[u]||_{\alpha}$ and $||u||_{1+\beta} \leq \overline{k}_2||L[u]||_{\infty}$. This result implies that the operator $L : \mathbb{E} \to \mathbb{F}$ is one-to-one and surjective, and also that $L^{-1} : \mathbb{F} \to \mathbb{E}$ is continuous. The fixed-point theorem of Shauder, see [1], [3] states that, if D is a non-empty closed bounded convex subset of the Banach space $\mathbb{E}$ and assuming that $T : D \to \mathbb{E}$ is compact $T(D) \subset D$, then T has a fixed point in D, i.e., exist $u^* \in D$ such that $T[u^*] = u^*$. Let us define the partial order relation in the Banach space $\tilde{E} = \mathbb{E} \times \mathbb{E}$, defined by $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, if and only if $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$.

The main result of the present paper is the following theorem:

Theorem 1 *If there is a super-solution $(\overline{u}, \overline{v})$ and a sub-solution $(\underline{u}, \underline{v})$ of (5)–(8) such that $\overline{u}$, $\overline{v}$, $\underline{u}$, $\underline{v}$, belonging to $C^{\alpha,\alpha/2}(D_s) \cap C^2(D_s)$ and $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, then, there exist (u^*, v^*) belonging to $C^{2+\alpha,1+\alpha/2}(D)$, possibly identical, such that $(\underline{u}, \underline{v}) \leq (u^*, v^*) \leq (\overline{u}, \overline{v})$*

Proof. For $u, v \in \mathbb{F} = C^{\alpha,\alpha/2}(D_s)$ let us define

$$(\rho u)(x,t) = \begin{cases} u(x,t) & \text{si } \underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t), \\ \overline{u}(x,t) & \text{if } u(x,t) \geq \overline{u}(x,t), \\ \underline{u}(x,t) & \text{if } u(x,t) \leq \underline{u}(x,t), \end{cases}$$

for all $(x, t) \in \Omega$. For all $u \in \mathbb{F}$, we have $\rho u \in \mathbb{F}$, that is, $\rho : \mathbb{F} \to \mathbb{F}$.

$$(\sigma v)(x,t) = \begin{cases} v(x,t) & \text{if } \underline{v}(x,t) \leq v(x,t) \leq \overline{v}(x,t), \\ \overline{v}(x,t) & \text{if } v(x,t) \geq \overline{v}(x,t), \\ \underline{v}(x,t) & \text{if } v(x,t) \leq \underline{v}(x,t), \end{cases}$$

for all $(x, t) \in D_s$. Likewise, for all $v \in \mathbb{F}$, we have $\sigma v \in \mathbb{F}$. Let us denote

$$M_i[u] = L_i[u] + a_i u \quad i = 1, 2, \quad \mathbb{E}_1 = \{u \in C^{1+\beta,\beta/2}(D) : u \equiv 0 \text{ in } S\},$$

$$||u||_{\mathbb{E}_1} = ||u||_{1+\beta}, \quad \widetilde{\mathbb{E}} = \mathbb{E}_1 \times \mathbb{E}_1, \quad ||(u, v)||_{\widetilde{\mathbb{E}}} = \max\{||u||_{\mathbb{E}_1}, ||v||_{\mathbb{E}_1}\}$$

The space $\widetilde{\mathbb{E}} = \mathbb{E}_1 \times \mathbb{E}_1$ is a Banach space with norm $||(u,v)||_{\tilde{\mathbb{E}}}$.
Let us define the following functions

$$\widetilde{g}_1(x,t,u,v) = g_1(x,t,\rho u,\sigma v) = -c_1 G(\rho v)(\sigma u) + q_1,$$
$$\widetilde{g}_2(x,t,u,v) = g_2(x,t,\rho u,\sigma v) = +c_2 G(\rho v)(\sigma u) + q_2.$$

Let us define operator $T : \widetilde{\mathbb{E}} \to \widetilde{\mathbb{E}}$as follows:

$$T(u,v) = \left(i \circ M_1^{-1}[\tilde{g}_1(x,t,u,v)], i \circ M_2^{-1}[\tilde{g}_2(x,t,u,v)]\right),$$

for all $(u,v) \in \widetilde{\mathbb{E}}$. Since $C^{2+\alpha,1+\alpha/2}(D_s) \subset C^{1+\beta,\beta/2}(D_s)$ for $0 < \alpha < \beta < 1$, then the injection

$$i : C^{2+\alpha,1+\alpha/2}(D_s) \to \mathbb{E}_1$$

is well defined and is a linear, compact operator, which implies that operator $T = ioM^{-1}$ is compact. Let us consider the following problem

$$\begin{cases} M_1[u] & = \tilde{g}_1(x,t,u,v) = -c_1 G(\rho v)(\sigma u) + q_1 \text{ in } D_s, \\ M_2[v] & = \tilde{g}_2(x,t,u,v) = +c_2 G(\rho v)(\sigma u) + q_2 \text{ in } D_s. \end{cases} \tag{21}$$

Due to the existence and uniqueness theorem of Shauder, there exist two constants $k_1 > 0$ y $k_2 > 0$ such that

$$||M_1^{-1}(\tilde{g}_1(x,t,u,v))||_{1+\beta} \le k_1 ||\tilde{g}_1(x,t,u,v)||_\infty \le r, \tag{22}$$
$$||M_2^{-1}(\tilde{g}_2(x,t,u,v))||_{1+\beta} \le k_2 ||\tilde{g}_2(x,t,u,v)||_\infty \le r. \tag{23}$$

Then, there exists $r > 0$ such that $||T(u,v)||_{\widetilde{E}} \le r$ for all $(u,v) \in \widetilde{\mathbb{E}}$. Let us denote $\overline{D}_r(0)$ as the closed ball in $\tilde{\mathbb{E}}$, with center in 0 and radius r.
From (22)–(23), we have $T(\overline{D}_r(0)) \subset \overline{D}_r(0)$, because of this relation and considering the fixed point theorem of Shauder, there exist $(u^*, v^*) \in \overline{D}_r(0)$ such that $T(u^*, v^*) = (u^*, v^*)$, that is

$$M_1[u^*] = \widetilde{g}_1(x,t,u^*,v^*) = -c_1 G(\sigma v^*)(\rho u^*) + q_1 \text{ in } D_s,$$
$$M_2[v^*] = \widetilde{g}_2(x,t,u^*,v^*) = c_2 G(\sigma v^*)(\rho u^*) + q_2 \text{ in } D_s.$$

From the definitions (15)–(17) and (18)–(20) we have

$$M_2[\overline{v} - v^*] \ge c_2[G(\overline{v})\overline{u} - G(\sigma v^*)\rho u^*] \ge 0 \text{ in } D_s \tag{24}$$
$$M_2[v^* - \underline{v}] \ge c_2[G(\rho v^*)\sigma u^* - G(\underline{v})\underline{u}] \ge 0 \text{ in } D_s \tag{25}$$
$$\overline{v} - v^* \ge 0 \text{ y } v^* - \underline{v} \ge 0 \text{ in } S. \tag{26}$$

The inequalities (24)—(26) and the maximum principle for PDE's of parabolic type [5], [2], [1], imply that

$$\max_{\overline{D_s}}(\overline{v} - v^*) = \max_{\partial D_s}(\overline{v} - v^*) \ge 0 \text{ and } \max_{\overline{D_s}}(v^* - \underline{v}) = \max_{\partial D_s}(v^* - \underline{v}) \ge 0$$

Therefore, $\overline{v} - v^* \geq 0$, and $v^* - \underline{v} \geq 0$ then $\overline{v} \geq v^*$ and $v^* \geq \underline{v}$ in D_s, that is, $\overline{v} \geq v^* \geq \underline{v}$ in D_s. Now, let us show that $\underline{u} \leq u^* \leq \overline{u}$ in D_s. To prove that $u^* \leq \overline{u}$ in D_s, let us suppose the opposite, namely that there exist $(x_0, t_0) \in D_s$ such that $u^*(x_0, t_0) > \overline{u}(x_0, t_0)$. Let $Q = \{(x, t) \in D_s : (u^* - \overline{u})(x, t) > 0\} \neq \emptyset$ and A be a non-empty connected component of Q that contains (x_0, t_0), in such a case A is a bounded domain in $\mathbb{R}^{N+1}$. Let us define $w = \overline{u} - u^*$ for all $(x, t) \in A$, $w|_{\partial A} \equiv 0$

$$M_1[w] \geq c_1[G(v^*)\rho u^* - G(\underline{v})\overline{u})] = c_1[G(v^*)\overline{u} - G(\underline{v})\overline{u}) \geq 0 \text{ in } A$$

The maximum principle for PDE's of parabolic type [5] implies that $w \geq 0$ in A, which contradicts the choice of (x_0, t_0), then $u^* \leq \overline{u}$ in D_s. A similar argument shows that $\underline{u} \leq u^*$ in D_s, that is, $\underline{u} \leq u^* \leq \overline{u}$ in $\overline{D}_s$ and $\rho u^* = u^*$ in D_s. Therefore,

$$M_1[u^*] = -c_1 G(\sigma v^*)(\rho u^*) + q_1 \text{ and } M_2[v^*] = c_2 G(\sigma v^*)(\rho u^*) + q_2 \text{ in } D_s$$

Therefore, it can be concluded that $\underline{u} \leq u^* \leq \overline{u}$ and $\underline{v} \leq v^* \leq \overline{v}$ in D_s. ■

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