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TOPOLOGICAL PRESSURE OF THE SET OF GENERIC POINTS FOR $\mathbb{Z}^d$-ACTIONS

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Abstract. We show that, if a continuous $\mathbb{Z}^d$ or $\mathbb{Z}_+^d$-action of a compact metric space has the almost product property (which is weaker than the specification property), then, for any continuous function and any invariant measure, the topological pressure of the set of all generic points coincides with the sum of the metric entropy and the mean of the continuous function.

1. Introduction

Let (X, d) be a compact metric space, $\mathcal{T}$ be a continuous L-action on X with $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$, and μ be a $\mathcal{T}$-invariant Borel probability measure on X. Here $\mathbb{Z}_+$ denotes the set of all non-negative integers $\mathbb{N} \cup \{0\}$. A point $x \in X$ is called a generic point for μ if its empirical measure converges weakly to μ. We denote by G_μ the set of all generic points for μ.

In this paper, we give the natural definition of the topological pressure of non-compact sets for general L-actions in Section 2, which was introduced by Pesin and Pitskel [**11**] for $\mathbb{Z}_+$-action. The aim of this paper is to show that, under some assumption on $\mathcal{T}$, the equation

$$P_{G_\mu}(\mathcal{T}, \varphi) = h_\mu(\mathcal{T}) + \int \varphi \, d\mu \tag{1.1}$$

holds for any $\mathcal{T}$-invariant measure μ and any continuous function $\varphi \colon X \to \mathbb{R}$, where $P_{G_\mu}(\mathcal{T}, \varphi)$ is the topological pressure of G_μ for φ and $h_\mu(\mathcal{T})$ is the metric entropy of $\mathcal{T}$ with respect to μ.

The present paper is largely motivated by the following result of Pfister and Sullivan [**12**, Theorems 3.1 and 3.2]. Let $\varphi \colon X \to \mathbb{R}$ be a continuous function satisfying that $-\varphi$ is an upper and weak lower-energy function for a certain reference measure m. If the ergodic measures of $\mathcal{T}$ are entropy dense and the entropy map of $\mathcal{T}$ is upper semicontinuous, then $\mathcal{T}$ satisfies the (level-two) large deviation principle for m and the rate function $q(\mu) := h_\mu(\mathcal{T}) + \int \varphi \, d\mu$ (if μ is $\mathcal{T}$-invariant). In this case, equation (1.1) states that the rate function $q(\mu)$ can be expressed as the topological pressure of G_μ.

In the case of $L = \mathbb{Z}_+$, equation (1.1) is considered by several authors under some assumptions. Pesin and Pitskel [**11**] proved this for ergodic measures μ. We remark that their result can be extended to higher dimensional actions in a similar way to their proof (see [**10**, Theorem A2.1]). Since the domain of the rate function q is the set of all $\mathcal{T}$-invariant

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measures (not only ergodic measures), we also have to show (1.1) for non-ergodic measures μ. Recently, Pfister and Sullivan [**13**] proved this for non-ergodic measures μ under the assumptions that $L = \mathbb{Z}_+$, $\varphi \equiv 0$ and $\mathcal{T}$ has the g-almost product property. So we also require a higher dimensional version of the g-almost product property for $\mathcal{T}$, which is called the *almost product property* (see Section 2).

Now we state our main result of this paper.

THEOREM 1.1. *Let $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$. If a continuous L-action $\mathcal{T}$ has the almost product property, then, for any $\mathcal{T}$-invariant measure μ and any continuous function $\varphi\colon X \to \mathbb{R}$,*

$$P_{G_\mu}(\mathcal{T}, \varphi) = h_\mu(\mathcal{T}) + \int \varphi \, d\mu.$$

Applying Theorem 1.1 (more precisely, Theorems 3.3(1) and 3.6), we can obtain the following variational principle (for a proof see also [**13**, Proposition 7.1]):

COROLLARY 1.2. *Let $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$. If a continuous L-action $\mathcal{T}$ has the almost product property, then, for any $\alpha \in \mathbb{R}$ and any continuous functions $\varphi, \psi\colon X \to \mathbb{R}$,*

$$P_{K_\alpha^\varphi}(\mathcal{T}, \psi) = \sup\left\{ h_\mu(\mathcal{T}) + \int \psi \, d\mu : \mu \text{ is } \mathcal{T}\text{-invariant with } \int \varphi \, d\mu = \alpha \right\},$$

where

$$K_\alpha^\varphi := \left\{ x \in X : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{h \in \Lambda_n} \varphi(T^h x) = \alpha \right\}, \tag{1.2}$$

$$\Lambda_n := \{h = (h_1, \dots, h_d) \in L : |h_i| < n \ (1 \leq i \leq d)\}, \tag{1.3}$$

and λ_n denotes the cardinality of the set Λ_n.

We give the definitions of the topological pressure for non-compact sets and the almost product property for L-actions in Section 2, and prove Theorem 1.1 in Section 3. We refer to [**6**] and [**7**] for ergodic theory of higher dimensional actions.

2. Preliminaries

2.1. Notation and basic notions

Let (X, d) be a compact metric space. A family of continuous transformations $\mathcal{T} := \{T^h\colon X \to X\}_{h\in L}$ is called a *continuous L-action*, with $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$, if $\mathcal{T}$ satisfies $T^{h+k} = T^h \circ T^k$ $(h, k \in L)$ and T^0 is the identity map. For $k \in L$ and $\Lambda \subset L$, we set $\Lambda + k := \{h + k : h \in \Lambda\}$. For $n \in \mathbb{Z}_+$ let

$$\Lambda_n := \{h = (h_1, \dots, h_d) \in L : |h_i| < n \ (1 \leq i \leq d)\} \text{ and } \lambda_n := \sharp\Lambda_n,$$

where $\sharp A$ denotes the cardinality of the set A.

Example 2.1. A typical example of a continuous L-action is a (full) shift. Let $A^L := \{(\omega_h)_{h\in L} : \omega_h \in A,\ \ h \in L\}$ with a finite set $A = \{0, \dots, r-1\}$. Then the space A^L is compact with the product topology. For $\theta > 1$ and $\omega, \omega' \in A^L$, let $n(\omega, \omega') := \min\{k : \omega_h =$

ω'_h $(h \in \Lambda_{k-1})$ and $\omega_h \neq \omega'_h$ (for some $h \in \Lambda_k \setminus \Lambda_{k-1}$)}. If we define

$$d_\theta(\omega, \omega') := \theta^{-\lambda_{n(\omega,\omega')}},$$

for $\omega, \omega' \in A^L$, then d_θ is a compatible metric for A^L.

For $h \in L$, we define the shift action $\sigma^h : A^L \to A^L$ as

$$(\sigma^h(\omega))_k := \omega_{h+k} \quad (k \in L).$$

Then, $\mathcal{T} = \{\sigma^h\}_{h \in L}$ is a continuous L-action on A^L.

Let $C(X, \mathbb{R})$ be the Banach space of continuous real-valued functions of X with the sup norm $\|\cdot\|_\infty$, and let $\mathcal{M}(X)$ be the set of all Borel probability measures on X with the weak topology. Since $C(X, \mathbb{R})$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \ldots\}$ that is dense in $C(X, \mathbb{R})$. For $\mu, \nu \in \mathcal{M}(X)$, we define

$$D(\mu, \nu) := \sum_{n=1}^{\infty} \frac{|\int \varphi_n \, d\mu - \int \varphi_n \, d\nu|}{2^{n+1} \|\varphi_n\|_\infty}.$$

Then D is a compatible metric for $\mathcal{M}(X)$ and $(\mathcal{M}(X), D)$ is compact. It is easy to see that $D(\mu, \nu) \leq 1$ for any $\mu, \nu \in \mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$ and $\epsilon > 0$, we set $\mathbb{B}(\mu, \epsilon) := \{\nu \in \mathcal{M}(X) : D(\mu, \nu) \leq \epsilon\}$. We denote by $\mathcal{M}_{\mathcal{T}}(X)$ the set of all $\mathcal{T}$-invariant (i.e., T^h-invariant for each $h \in L$) Borel probability measures on X.

For any $x \in X$ and any finite subset $\Lambda \subset L$, the *empirical measure* $\mathcal{E}_\Lambda(x)$ of x is defined by

$$\mathcal{E}_\Lambda(x) := \frac{1}{\sharp\Lambda} \sum_{h \in \Lambda} \delta_{T^h x},$$

where δ_y stands for the δ-measure at the point y. For simplicity, we use the notation $\mathcal{E}_n(x)$ instead of $\mathcal{E}_{\Lambda_n}(x)$. A point $x \in X$ is called *generic* with respect to an invariant measure $\mu \in \mathcal{M}_{\mathcal{T}}(X)$ if, for any $\varphi \in C(X, \mathbb{R})$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{h \in \Lambda_n} \varphi(T^h x) = \int \varphi \, d\mu$$

holds. We denote by G_μ the set of all generic points for μ. By the definition, $G_\mu = \{x \in X : \lim_{n \to \infty} D(\mathcal{E}_n(x), \mu) = 0\}$.

2.2. *Topological pressure for non-compact sets*

The generalization of the topological pressure of non-compact or non-invariant sets was introduced by Pesin and Pitskel [**11**] for $\mathbb{Z}_+$-action. In this subsection we consider the case of higher dimensional L-actions.

Firstly we recall the classical topological pressure defined by Ruelle [**14**]. Let (X, d) be a compact metric space and $\mathcal{T}$ be a continuous L-action on X with $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \geq 1$ (in the rest of this paper, we always assume these conditions). For $n \in \mathbb{N}$, $x \in X$ and $\epsilon > 0$, we set

$$B_n(x, \epsilon) := \Big\{ y \in X : \max_{h \in \Lambda_n} d(T^h x, T^h y) \leq \epsilon \Big\}.$$

A subset $E \subset X$ is called (n, ϵ)-*separated* if, for any $x, y \in E$ with $x \neq y$, $y \notin B_n(x, \epsilon)$ holds. Then, for $\varphi \in C(X, \mathbb{R})$, the *classical topological pressure* of φ is defined as

$$P(\mathcal{T}, \varphi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{\lambda_n} \log \sup_E \sum_{x \in E} e^{S_n \varphi(x)},$$

where $S_n\varphi(x) := \sum_{h \in \Lambda_n} \varphi(T^h x)$ and the supremum is taken over all (n, ϵ)-separated sets E.

Now we define the topological pressure of any subset Z and any continuous function $\varphi \in C(X, \mathbb{R})$ for $\mathcal{T}$. For $s \in \mathbb{R}$, $M \in \mathbb{N}$ and $\epsilon > 0$, we set

$$M_\varphi(Z, s, M, \epsilon) := \inf_\Gamma \left\{ \sum_{B_{n_i}(x_i, \epsilon) \in \Gamma} e^{-s\lambda_{n_i} + \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i}\varphi(y)} \right\},$$

where the infimum is taken over all $\Gamma := \{B_{n_i}(x_i, \epsilon)\}$, which is a finite or countable cover of Z. Since the quantity $M_\varphi(Z, s, M, \epsilon)$ does not decrease with M, we can define

$$M_\varphi(Z, s, \epsilon) := \lim_{M \to \infty} M_\varphi(Z, s, M, \epsilon).$$

Then we can easily see that the following critical value exists:

$$P_Z(\mathcal{T}, \varphi, \epsilon) := \inf\{s : M_\varphi(Z, s, \epsilon) = 0\} = \sup\{s : M_\varphi(Z, s, \epsilon) = \infty\}.$$

Finally we set

$$P_Z(\mathcal{T}, \varphi) := \lim_{\epsilon \to 0} P_Z(\mathcal{T}, \varphi, \epsilon),$$

which is called the *topological pressure* of Z for φ. The topological pressure of Z corresponding to the function $\varphi \equiv 0$ is called the *topological entropy* of Z, which we denote by $h_{\text{top}}(\mathcal{T}, Z)$.

The topological pressure of non-compact sets for L-actions also satisfies most of the basic facts of that for $\mathbb{Z}_+$-action found in [**10**]. In particular, we can easily see that the following properties hold.

PROPOSITION 2.1. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$. Then the following hold:*

(1) $P_X(\mathcal{T}, \varphi) = P(\mathcal{T}, \varphi)$;

(2) $P_{Z_1}(\mathcal{T}, \varphi) \leq P_{Z_2}(\mathcal{T}, \varphi)$, *if* $Z_1 \subset Z_2 \subset X$; *and*

(3) *if* $X = A^L$, *then we have*

$$h_{\text{top}}(\mathcal{T}, Z) = \log \theta \cdot HD_\theta(Z) \quad \text{for } Z \subset A^L,$$

where $HD_\theta(Z)$ stands for the Hausdorff dimension of Z with respect to the metric d_θ defined in Example 2.1.

Proof. Parts (1) and (2) are clear by the definition. For $m \in \mathbb{N}$ and $\omega \in A^L$ we define a *cylinder set* as

$$C_m(\omega) := \{\omega' \in A^L : \omega_h = \omega'_h,\ h \in \Lambda_m\}.$$

By the choice of the metric d_θ of A^L, it is easy to see that the s-Hausdorff outer measure of $Z \subset A^L$ can be expressed as

$$H(Z, s) = \lim_{\delta \to 0} \inf_\Gamma \left\{ \sum_{C_{m_i}(\omega_i) \in \Gamma} \operatorname{diam}(C_{m_i}(\omega_i))^s \right\},$$

where the infimum is taken over all $\Gamma := \{C_{m_i}(\omega_i)\}$, which is a finite or countable cover of Z with $\sup_i \operatorname{diam} C_{m_i}(\omega_i) < \delta$ and diam U denotes the diameter of a subset U.

Let $\epsilon > 0$ be sufficiently small and choose $n \in \mathbb{N}$ as $\theta^{-\lambda_{n+1}} \le \epsilon < \theta^{-\lambda_n}$. Then it also follows from the choice of the metric d_θ on A^L that

$$B_k(\omega, \epsilon) = C_{k+n-1}(\omega)$$

for all $k \in \mathbb{Z}_+$ and $\operatorname{diam}(C_j(\omega)) = \theta^{-\lambda_{j+1}}$. Thus comparing the two definitions, we obtain part (3). □

Remark 2.2. By Proposition 2.1(3) and our main theorem, which we will prove in Section 3, we can conclude that the Hausdorff dimension of G_μ coincides with $h_\mu(\mathcal{T})/\log\theta$ for any shift-invariant measure μ on A^L. This fact is related to a result of Grevich and Tempelman [**4**], who have estimated the Hausdorff dimension of G_μ for every Gibbs measure μ on more general shift spaces.

2.3. Almost product property

In this subsection, we define the almost product property, which is a weaker form of the specification property. The specification property for $\mathbb{Z}^d$-actions was introduced by Ruelle [**14**]. In [**3**], Eizenberg *et al* proved the large deviations results for the $\mathbb{Z}^d$-subshifts with the specification property. Lind and Schmidt [**8**] discussed this notion for continuous $\mathbb{Z}^d$-actions on a compact abelian group. On the other hand, the case of $\mathbb{Z}_+$-actions has also been considered by many authors (see for example [**1, 2, 15**]). Now we give the definition of the specification property for L-actions.

Definition 2.3. Let $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$ with $d \ge 1$. We say that a continuous L-action $\mathcal{T}$ has the *specification property* if, for any $\epsilon > 0$, there exists $M(\epsilon) \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$, any $x_1, \ldots, x_k \in X$ and any collection of subsets in L, $\{\Lambda(i)\}_{i=1}^k$, satisfying $\min_{i\ne j}\{\max_{1\le k\le d} |h_k - h'_k| : h = (h_1, \ldots, h_d) \in \Lambda(i), h' = (h'_1, \ldots, h'_d) \in \Lambda(j)\} \ge M(\epsilon)$, there exists $y \in X$ such that

$$y \in \bigcap_{i=1}^{k} B_{\Lambda(i)}(x_i, \epsilon),$$

where $B_\Lambda(x, \epsilon) := \{y \in X : \max_{h\in\Lambda} d(T^h x, T^h y) \le \epsilon\}$.

In [**13**], Pfister and Sullivan introduced the g-almost product property, which is a weaker form of the specification property defined as follows. Let $g\colon \mathbb{N} \to \mathbb{N}$ be a non-decreasing unbounded map satisfying $g(n) < n$ and $\lim_{n\to\infty} g(n)/n = 0$, which is called a *blow-up function*. A continuous transformation $f\colon X \to X$ has the *g-almost product property* with a blow-up function g if there exists a non-increasing function $m\colon \mathbb{R}^+ \to \mathbb{N}$, such that, for any $k \in \mathbb{N}$, any $x_1, \ldots, x_k \in X$, any $\epsilon_1 > 0, \ldots, \epsilon_k > 0$ and any integers $n_1 \ge m(\epsilon_1), \ldots, n_k \ge m(\epsilon_k)$, there exists a point $y \in X$ such that

$$y \in \bigcap_{j=1}^{k} f^{-M_{j-1}} B_{n_j}(g; x_j, \epsilon_j),$$

where $M_0 := 0$, $M_i := n_1 + \cdots + n_i$ and $B_n(g; x, \epsilon) := \{y \in X : \sharp\{0 \le j \le n-1 : d(f^j x, f^j y) > \epsilon\} \le g(n)\}$.

Let $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \ge 1$. Now we define a higher dimensional version of the g-almost product property.

Definition 2.4. Let $L = \mathbb{Z}^d$ or $L = \mathbb{Z}^d_+$ with $d \ge 1$. We say that a continuous L-action $\mathcal{T}$ has the *almost product property* if there exist a function $g\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} g(n)/n = 0$ and a function $m\colon \mathbb{R}^+ \to \mathbb{N}$ such that, for any $k \in \mathbb{N}$, any $x_1, \ldots, x_k \in X$, any $\epsilon_1 > 0, \ldots, \epsilon_k > 0$, any integers $n_1 \ge m(\epsilon_1), \ldots, n_k \ge m(\epsilon_k)$ and any disjoint $\Lambda(i) := \Lambda_{n_i} + h^{(i)} \subset L$,

$$\bigcap_{i=1}^{k} T^{-h^{(i)}} B_{n_i}(g; x_i, \epsilon_i) \neq \emptyset,$$

where $B_n(g; x, \epsilon) := \{y \in X : \sharp\{h \in \Lambda_n : d(T^h x, T^h y) > \epsilon\} \le g(\lambda_n)\}$.

In a similar way to the proof of Proposition 2.1 in [**13**] we can check that the specification property implies the almost product property. Furthermore, there is an example that does not have the specification property, but satisfies the almost product property. In the next example, we consider the F-shift, which is the generalization of the β-shift.

Example 2.2. Let $F\colon [0, \infty) \to [0, \infty)$ be a C^1 map with $F(0) = 0$ and

$$1 < \alpha \le F'(x) \le \beta \quad (x \in [0, 1])$$

for some constant $\beta \ge \alpha > 1$. Then $f\colon (0, 1] \to (0, 1]$ is defined by

$$f(x) = \begin{cases} 1 & \text{for } F(x) \in \mathbb{N}; \\ F(x) \bmod 1 & \text{otherwise.} \end{cases}$$

Let $a_1 < \cdots < a_p$ be discontinuous points of f and set

$$I_0 := (0, a_1], \quad I_1 := (a_1, a_2], \quad \ldots, \quad I_p := (a_p, 1].$$

Put $A := \{0, \ldots, p\}$ and consider the shift space $(A^{\mathbb{Z}^+}, \sigma)$. We define $(c_i) \in A^{\mathbb{Z}^+}$ as $c_i = j$ if and only if $f^i(1) \in I_j$ and set

$$X^F := \{\omega \in A^{\mathbb{Z}^+} : \sigma^k \omega \le (c_i),\ k \ge 0\}.$$

Here the above order is the lexicographical order, which is defined by $(a_i) < (b_i)$ if and only if $a_i < b_i$ for the least index i with $a_i \neq b_i$. Then X^F is a shift-invariant closed subset of the full shift. We call the restriction of σ into X^F the *F-shift*. When $F(x) = \beta x$ with $\beta > 1$, the F-shift coincides with the well-known β-shift. In the case of β-shifts it is known that the specification property holds only for a set of β of Lebesgue measure zero.

Then in a similar way to the case of the β-shifts [**13**], we can easily check that the F-shift has the almost product property.

Example 2.3. We can construct an example of a higher dimensional action, for which the specification property does not hold but the almost product property is satisfied.

Let $L=\mathbb{Z}_+^2$ and X^{F_i} be an F_i-shift defined as in Example 2.2 for $i=1,2$. We set $A:=A_1\times A_2$ where A_i is a finite set satisfying $X^{F_i}\subset A_i^{\mathbb{Z}_+}$ $(i=1,2)$, and

$$X^{F_1,F_2}:=\{(\omega_h,\nu_h)_{h=(h_1,h_2)\in\mathbb{Z}_+^2}\in A^{\mathbb{Z}_+^2}:(\omega_{(j,h_2)})_{j\in\mathbb{Z}_+}\in X^{F_1}\text{ and }$$
$$(\nu_{(h_1,j)})_{j\in\mathbb{Z}_+}\in X^{F_2}\text{ for any }h_1,h_2\in\mathbb{Z}_+\}.$$

We call the restriction of the shift action $\{\sigma^h\colon A^{\mathbb{Z}_+^2}\to A^{\mathbb{Z}_+^2}\}_{h\in\mathbb{Z}_+^2}$ into X^{F_1,F_2} the (F_1,F_2)-*shift action*. Clearly every (F_1,F_2)-shift action does not satisfy the specification property.

We show that every (F_1,F_2)-shift action has the almost product property. Let $(\omega_h^{(1)},\nu_h^{(1)}),\dots,(\omega_h^{(k)},\nu_h^{(k)})\in X^{F_1,F_2}$, $\epsilon_1>0,\dots,\epsilon_k>0$, $n_1,\dots,n_k\in\mathbb{N}$ and $\Lambda(i):=\Lambda_{n_i}+(h_1^{(i)},h_2^{(i)})\subset L$ $(1\le i\le k)$ be disjoint cubes. Then we put $\omega_h=\nu_h=0$ for $h\in\mathbb{Z}_+^2\setminus\bigcup_{i=1}^k\Lambda(i)$. Fix $1\le i\le k$ and an integer $h_2^{(i)}\le h_2<h_2^{(i)}+n_i$. We set

$$t_{h_2}^{(i)}:=\max\{h_1^{(i)}\le j<h_1^{(i)}+n_i:\omega_{(j,h_2)}^{(i)}\neq 0\}$$

and

$$\omega_{(j,h_2)}:=\begin{cases}\omega_{(j,h_2)}^{(i)}-1 & \text{when } j=t_{h_2}^{(i)},\\ \omega_{(j,h_2)}^{(i)} & \text{when } h_1^{(i)}\le j<h_1^{(i)}+n_i,\ j\neq t_{h_2}^{(i)},\end{cases}$$

for $h_2^{(i)}\le h_2<h_2^{(i)}+n_i$ and $1\le i\le k$. Then by the definition of the F_1-shift, we have $(\omega_{(j,h_2)})_{j\in\mathbb{Z}_+}\in X^{F_1}$ for any $h_2\in\mathbb{Z}_+$ (see [**13**]). Similarly, for fixed $h^{(i)}\le h_1<h^{(i)}+n_i$, we set

$$t_{h_1}^{(i)}:=\max\{h_2^{(i)}\le j<h_2^{(i)}+n_i:\nu_{(h_1,j)}^{(i)}\neq 0\}$$

and

$$\nu_{(h_1,j)}:=\begin{cases}\nu_{(h_1,j)}^{(i)}-1 & \text{when } j=t_{h_1}^{(i)},\\ \nu_{(h_1,j)}^{(i)} & \text{when } h_2^{(i)}\le j<h_2^{(i)}+n_i,\ j\neq t_{h_1}^{(i)},\end{cases}$$

for $h_1^{(i)}\le h_1<h_1^{(i)}+n_i$ and $1\le i\le k$. Then we have $(\nu_{(h_1,j)})_{j\in\mathbb{Z}_+}\in X^{F_2}$ for any $h_1\in\mathbb{Z}_+$. Thus we have $(\omega_h,\nu_h)_{h\in\mathbb{Z}_+^2}\in X^{F_1,F_2}$. Moreover, by the definition of $(\omega_h,\nu_h)_{h\in\mathbb{Z}_+^2}$, if we choose a blow-up function $g\colon\mathbb{N}\to\mathbb{N}$ and a function $m\colon\mathbb{R}^+\to\mathbb{N}$ suitably, then we get

$$(\omega_h,\nu_h)_{h\in\mathbb{Z}_+^2}\in\bigcap_{i=1}^k\sigma^{-h^{(i)}}B_{n_i}(g;\,(\omega_h^{(i)},\nu_h^{(i)})_{h\in\mathbb{Z}_+^2},\epsilon_i)$$

for $n_1\ge m(\epsilon_1),\dots,n_k\ge m(\epsilon_k)$, which implies that the (F_1,F_2)-shift action has the almost product property.

3. Main theorem

3.1. *Upper estimate of* $P_{G_\mu}(\mathcal{T},\varphi)$

The purpose of this subsection is to get an upper bound for $P_{G_\mu}(\mathcal{T},\varphi)$. For $F\subset\mathcal{M}(X)$, we set $X_{n,F}:=\{x\in X:\mathcal{E}_n(x)\in F\}$ and set

$$N_\varphi(F;n,\epsilon):=\sup_{E_n}\sum_{x\in E_n}e^{S_n\varphi(x)},$$

where the supremum is taken over all (n,ϵ)-separated sets E_n in $X_{n,F}$.

LEMMA 3.1. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \geq 1$. For any sequence of (n, ϵ)-separated subsets $\{E_n\}$ and any $\varphi \in C(X, \mathbb{R})$, we define*

$$\sigma_n := \frac{\sum_{y \in E_n} e^{S_n\varphi(y)}\delta_y}{\sum_{z \in E_n} e^{S_n\varphi(z)}} \quad \textit{and} \quad \mu_n := \frac{1}{\lambda_n}\sum_{h \in \Lambda_n} \sigma_n \circ T^{-h}.$$

Assume that $\lim_{n\to\infty} \mu_n = \mu$, *then*

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log \sum_{x \in E_n} e^{S_n\varphi(x)} \leq h_\mu(\mathcal{T}) + \int \varphi \, d\mu.$$

Proof. The following proof can be found in [**6**] and [**9**]. Let ξ be a finite measurable partition of X with diam $A < \epsilon$ and $\mu(\partial A) = 0$, where ∂A means the boundary of A, for all $A \in \xi$. Then we have

$$H_{\sigma_n}(\xi^n) := \sum_{A \in \xi^n} -\sigma_n(A) \log \sigma_n(A) = \sum_{y \in E_n} -\sigma_n(y) \log \sigma_n(y),$$

where $\xi^n := \bigvee_{h \in \Lambda_n} T^{-h}\xi$. It follows from the definition of σ_n that

$$H_{\sigma_n}(\xi^n) + \int S_n\varphi \, d\sigma_n = \log \sum_{z \in E_n} e^{S_n\varphi(z)}.$$

Let $m > 0$ and $n \geq 2m$. For $h \in \Lambda_n$, we set $S(h) := \{h + l_m k \in \Lambda_{n-m} : k \in L\}$ and $R(h) := \Lambda_n \setminus (S(h) + \Lambda_m)$, where l_m equals $2m - 1$ if $L = \mathbb{Z}^d$ and m if $L = \mathbb{Z}^d_+$. Then we can easily to see that

$$\sharp S(h) \geq \frac{\lambda_{n-2m}}{\lambda_m}, \quad \sharp R(h) \leq \lambda_n - \lambda_{n-2m} =: \gamma_{m,n}$$

and

$$\xi^n = \Big(\bigvee_{s \in S(h)} T^{-s}\xi^m\Big) \vee \Big(\bigvee_{k \in R(h)} T^{-k}\xi\Big).$$

Therefore, each $h \in \Lambda_m$ admits the estimate

$$\begin{aligned} \log \sum_{z \in E_n} e^{S_n\varphi(z)} &= H_{\sigma_n}(\xi^n) + \int S_n\varphi \, d\sigma_n \\ &\leq \sum_{s \in S(h)} H_{\sigma_n}(T^{-s}\xi^m) + \sum_{k \in R(h)} H_{\sigma_n}(T^{-k}\xi) + \int S_n\varphi \, d\sigma_n \\ &\leq \sum_{s \in S(h)} H_{\sigma_n}(T^{-s}\xi^m) + \int S_n\varphi \, d\sigma_n + \gamma_{m,n} \log \sharp\xi. \end{aligned}$$

Since $\bigcup_{h \in \Lambda_m} S(h) \subset \Lambda_n$, by summing up the above over all $h \in \Lambda_m$ we obtain

$$\begin{aligned} \frac{1}{\lambda_n} \log \sum_{z \in E_n} e^{S_n\varphi(z)} &\leq \frac{1}{\lambda_m} \sum_{k \in \Lambda_n} \frac{1}{\lambda_n} H_{\sigma_n}(T^{-k}\xi^m) + \frac{1}{\lambda_n} \int S_n\varphi \, d\sigma_n + \frac{\gamma_{m,n}}{\lambda_n} \log \sharp\xi \\ &= \frac{1}{\lambda_m} H_{\mu_n}(\xi^m) + \int \varphi \, d\mu_n + \frac{\gamma_{m,n}}{\lambda_n} \log \sharp\xi. \end{aligned}$$

Letting $n \to \infty$ we get

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log \sum_{z\in E_n} e^{S_n\varphi(z)} \le \frac{1}{\lambda_m} H_\mu(\xi^m) + \int \varphi \, d\mu,$$

for any $m > 0$, which proves the lemma. □

LEMMA 3.2. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \ge 1$. Then, for any $\mu \in \mathcal{M}_{\mathcal{T}}(X)$ and any $\varphi \in C(X, \mathbb{R})$,*

$$\overline{s}_\varphi(\mu) := \lim_{\epsilon\to 0} \inf_{F\ni\mu} \limsup_{n\to\infty} \frac{1}{\lambda_n} \log N_\varphi(F; n, \epsilon) \le h_\mu(\mathcal{T}) + \int \varphi \, d\mu,$$

where the infimum is taken over all neighborhoods F of μ.

Proof. Lemma 3.1 enables us to show this in a similar way to the proof of Proposition 3.1 in [**13**]. □

THEOREM 3.3. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \ge 1$. Then the following hold.*

(1) *Let $K \subset \mathcal{M}_{\mathcal{T}}(X)$ be a closed subset and let ${}^K G := \{x \in X : V(x) \cap K \ne \emptyset\}$, where $V(x)$ denotes the limit-point set of $\{\mathcal{E}_n(x)\}$. Then, for any $\varphi \in C(X, \mathbb{R})$,*

$$P_{{}^K G}(\mathcal{T}, \varphi) \le \sup_{\mu\in K} \bigg(h_\mu(\mathcal{T}) + \int \varphi \, d\mu \bigg).$$

(2) *For any $\mu \in \mathcal{M}_{\mathcal{T}}(X)$ and $\varphi \in C(X, \mathbb{R})$,*

$$P_{G_\mu}(\mathcal{T}, \varphi) \le h_\mu(\mathcal{T}) + \int \varphi \, d\mu.$$

Proof. Without loss of generality, we may assume $\sup_{\mu\in K}(h_\mu(\mathcal{T}) + \int \varphi \, d\mu) < \infty$. Given any $s > \sup_{\mu\in K}(h_\mu(\mathcal{T}) + \int \varphi \, d\mu)$, put

$$\delta := \frac{s - \sup_{\mu\in K}(h_\mu(\mathcal{T}) + \int \varphi \, d\mu)}{2}. \tag{3.1}$$

Since φ is uniformly continuous, there exists $\epsilon^* > 0$ such that

$$\sup\{|\varphi(x) - \varphi(y)| : d(x, y) < \epsilon^*\} < \frac{\delta}{2}. \tag{3.2}$$

Since $N_\varphi(F; n, \epsilon)$ does not decrease if ϵ decreases, by Lemma 3.2 we have

$$\inf_{F\ni\mu} \limsup_{n\to\infty} \frac{1}{\lambda_n} \log N_\varphi(F; n, \epsilon) \le h_\mu(\mathcal{T}) + \int \varphi \, d\mu \quad (\epsilon > 0).$$

We take an arbitrary $0 < \epsilon < \epsilon^*$. For any $\mu \in \mathcal{M}_{\mathcal{T}}(X)$, there exist a neighborhood $F(\mu, \epsilon)$ of μ and an integer $M(\mu, \epsilon) > 0$ such that

$$N_\varphi(F(\mu, \epsilon); n, \epsilon) \le e^{\lambda_n(h_\mu(\mathcal{T}) + \int \varphi \, d\mu + \delta/2)} \quad (n \ge M(\mu, \epsilon)). \tag{3.3}$$

Since K is compact, there exist $\mu_1, \ldots, \mu_{m_\epsilon} \in K$ such that $K \subset \bigcup_{j=1}^{m_\epsilon} F(\mu_j, \epsilon)$. Thus, if we set

$$A_M := \bigcup_{n\ge M} \bigcup_{j=1}^{m_\epsilon} X_{n, F(\mu_j, \epsilon)},$$

then ${}^K G \subset A_M$ $(M \ge 1)$.

Fix $M \geq \max_{1\leq j\leq m_\epsilon} M(\mu_j, \epsilon)$ and let $E_{n,j} \subset X_{n,F(\mu_j,\epsilon)}$ be a maximal (n, ϵ)-separated set for $n \geq M$ and $1 \leq j \leq m_\epsilon$. Since $E_{n,j}$ is also (n, ϵ)-spanning set of $X_{n,F(\mu_j,\epsilon)}$ (see [**6**, Definition 4.4.1]), we have

$$A_M \subset \bigcup_{n\geq M} \bigcup_{j=1}^{m_\epsilon} \bigcup_{x\in E_{n,j}} B_n(x, \epsilon).$$

Thus by (3.1), (3.2) and (3.3) we obtain

$$\begin{aligned} M_\varphi({}^K G, s, M, \epsilon) &\leq \sum_{n\geq M} \sum_{j=1}^{m_\epsilon} \sum_{x\in E_{n,j}} e^{-s\lambda_n + \sup_{y\in B_n(x,\epsilon)} S_n\varphi(y)} \\ &\leq \sum_{n\geq M} \sum_{j=1}^{m_\epsilon} \sum_{x\in E_{n,j}} e^{-s\lambda_n + (\delta/2)\lambda_n + S_n\varphi(x)} \\ &\leq \sum_{n\geq M} \sum_{j=1}^{m_\epsilon} e^{\lambda_n(-s+\delta+\sup_{\mu\in K}(h_\mu(\mathcal{T})+\int \varphi\, d\mu))} \\ &= m_\epsilon \sum_{n\geq M} e^{-\lambda_n\delta}. \end{aligned}$$

Letting $M \to \infty$, we get $M_\varphi({}^K G, s, \epsilon) = 0$, which proves part (1).

The assertion (2) follows from part (1) since $G_\mu \subset {}^{\{\mu\}}G$ for any $\mu \in \mathcal{M}_\mathcal{T}(X)$. □

3.2. *Lower estimate of $P_{G_\mu}(\mathcal{T}, \varphi)$*

The aim of this subsection is to obtain a lower estimate for $P_{G_\mu}(\mathcal{T}, \varphi)$. First, we review some notions found in [**13**]. Let $\epsilon > 0$, $\delta > 0$ and $n \in \mathbb{N}$. A subset $E \subset X$ is called (δ, n, ϵ)-*separated* if, for any $x, y \in E$ with $x \neq y$,

$$\sharp\{h \in \Lambda_n : d(T^h x, T^h y) > \epsilon\} \geq \delta n$$

holds. For a subset $F \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}_\mathcal{T}(X)$, we set

$$N(F; \delta, n, \epsilon) := \sharp E$$

where E is a (δ, n, ϵ)-separated set in $X_{n,F}$ with maximal cardinality and

$$\underline{s}(\mu; \delta, \epsilon) := \inf_{F\ni\mu} \liminf_{n\to\infty} \frac{1}{\lambda_n} \log N(F; \delta, n, \epsilon).$$

LEMMA 3.4. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}_+^d$, $d \geq 1$. Then, for any ergodic measure μ,*

$$h_\mu(\mathcal{T}) = \lim_{\epsilon\to 0} \lim_{\delta\to 0} \underline{s}(\mu; \delta, \epsilon).$$

Proof. Since a (δ, n, ϵ)-separated set is also (n, ϵ)-separated, we have

$$N(F; \delta, n, \epsilon) \leq N_{\mathbb{O}}(F; n, \epsilon),$$

where $\mathbb{O}$ denotes a constant function of X defined as $\mathbb{O}(x) \equiv 0$ $(x \in X)$. Thus it follows from Lemma 3.2 that

$$\lim_{\epsilon\to 0} \lim_{\delta\to 0} \underline{s}(\mu; \delta, \epsilon) \leq h_\mu(\mathcal{T}).$$

The other inequality follows from [**12**, Proposition 2.1] and so the lemma is proved. □

LEMMA 3.5. *Let $\mathcal{T}$ be a continuous L-action with $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \geq 1$, $\mu \in \mathcal{M}_{\mathcal{T}}(X)$, $h < h_\mu(\mathcal{T})$ and $t > 0$. Then there exist $\epsilon^* > 0$ and $\delta^* > 0$, which are independent of t, and $\alpha \in \mathcal{M}_{\mathcal{T}}(X)$ such that $\alpha = \sum_{i=1}^p a_i \mu_i$, where $a_i \in \mathbb{Q}$ and μ_i is ergodic, and*

$$h < \sum_{i=1}^{p} a_i \underline{s}(\mu_i; \delta^*, \epsilon^*), \qquad D(\mu, \alpha) < t.$$

Proof. Let $\{\mu_x\}$ be an ergodic decomposition of μ. Then, by the monotone convergence theorem and Lemma 3.4, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{\delta\to 0}\int \underline{s}(\mu_x; \delta, \epsilon)\, d\mu(x) &= \int \lim_{\epsilon\to 0}\lim_{\delta\to 0} \underline{s}(\mu_x; \delta, \epsilon)\, d\mu(x) \\ &= \int h_{\mu_x}(\mathcal{T})\, d\mu(x) \\ &= h_\mu(\mathcal{T}),\end{aligned}$$

where the last equality follows from the affinity of the entropy map (see [**5**]). Thus there exist $\epsilon^* > 0$ and $\delta^* > 0$ such that

$$\int \underline{s}(\mu_x; \delta^*, \epsilon^*)\, d\mu(x) > h_\mu(\mathcal{T}) - \eta/3,$$

where $\eta := h_\mu(\mathcal{T}) - h > 0$.

Let $\pi : X \to \mathcal{M}_{\mathcal{T}}(X)$ be a canonical projection defined by $\pi(x) := \mu_x$. Since $\mathcal{M}_{\mathcal{T}}(X)$ is compact, there exists a partition $\{A'_i\}_{i=1}^p$ of $\{\mu_x\}$ so that, for any $\mu_1 \in A'_1, \ldots, \mu_p \in A'_p$,

$$D\bigg(\mu, \sum_{i=1}^{p} a_i \mu_i\bigg) < \frac{t}{2}$$

holds, where $a_i := \mu(A_i)$ and $A_i := \pi^{-1}(A'_i)$. Thus, if we choose $x_i \in A_i$ so that

$$\sup_{x\in A_i} \underline{s}(\mu_x; \delta^*, \epsilon^*) < \underline{s}(\mu_{x_i}; \delta^*, \epsilon^*) + \eta/3,$$

then we have $\sum_{i=1}^p a_i \underline{s}(\mu_{x_i}; \delta^*, \epsilon^*) > h_\mu(\mathcal{T}) - \frac{2}{3}\eta$ and $D(\mu, \sum_{i=1}^p a_i \mu_{x_i}) < t/2$, which prove the lemma because $\sum_i a_i \underline{s}(\mu_{x_i}; \delta^*, \epsilon^*)$ and $D(\mu, \sum_{i=1}^p a_i \mu_{x_i})$ vary continuously on $\{(a_1, \ldots, a_p) : \sum_{i=1}^p a_i = 1\}$. □

THEOREM 3.6. *Let $L = \mathbb{Z}^d$ or $\mathbb{Z}^d_+$, $d \geq 1$. If a continuous L-action $\mathcal{T}$ has the almost product property, then, for any $\varphi \in C(X, \mathbb{R})$ and $\mu \in \mathcal{M}_{\mathcal{T}}(X)$,*

$$P_{G_\mu}(\mathcal{T}, \varphi) \geq h_\mu(\mathcal{T}) + \int \varphi\, d\mu.$$

Proof. We treat the case of $L = \mathbb{Z}^d$; the case of $L = \mathbb{Z}^d_+$ uses similar ideas. Let $h^* < h_\mu(\mathcal{T})$ and $s^* < \int \varphi\, d\mu$. Take $h^* < h' < h'' < h''' < h_\mu(\mathcal{T})$. By Lemma 3.5, there exist $\epsilon^* > 0$, $\delta^* > 0$ and, for any k, a finite convex combination of ergodic measures with rational coefficients

$$\alpha_k = \sum_{i=1}^{p_k} a_{i,k} \mu_{i,k}$$

such that $\{\alpha_k\}$ converges to μ and

$$h''' < \sum_{i=1}^{p_k} a_{i,k}\underline{s}(\mu_{i,k};\, \delta^*, \epsilon^*). \tag{3.4}$$

Let $\{\zeta_k\}$ and $\{\gamma_k\}$ be two strictly decreasing sequences such that $\lim_{k\to\infty} \zeta_k = 0$, $\lim_{k\to\infty} \gamma_k = 0$ and

$$\zeta_k \le \frac{h'' - h'}{h''}. \tag{3.5}$$

Since X is compact, we can find a strictly decreasing sequence $\{\epsilon_k\}$ with $\epsilon_k \le \epsilon^*/4$ and satisfying $D(\delta_x, \delta_y) \le \gamma_k$ whenever $d(x, y) \le \epsilon_k$. By the definition we have

$$\underline{s}(\mu_{i,k};\, \delta^*, \epsilon^*) \le \liminf_{n\to\infty} \frac{1}{\lambda_n} \log N(\mathbb{B}(\mu_{i,k}, \zeta_k);\, \delta^*, n, \epsilon^*)$$

and so we can choose a strictly increasing sequence $\{n_k\}$ such that n_k is divisible by n_{k-1},

$$\underline{s}(\mu_{i,k};\, \delta^*, \epsilon^*) - (h''' - h'') \le \frac{1}{\lambda_{n_k}} \log N(\mathbb{B}(\mu_{i,k}, \zeta_k);\, \delta^*, n_k, \epsilon^*), \tag{3.6}$$

$$\lambda_{n_k}\delta^* > 2g(\lambda_{n_k}) + 1, \qquad \frac{g(\lambda_{n_k})}{\lambda_{n_k}} \le \gamma_k \quad \text{and} \quad \frac{2d}{n_k} \le \zeta_k. \tag{3.7}$$

We also choose two sequences $\{J_k\}$ and $\{N_k\}$ such that $a_{i,k}J_k$ is an integer, $\sum_{j=1}^{k} N_j J_j n_j$ is divisible by $J_{k+1}n_{k+1}$ and

$$\max\left\{J_{k+1}n_{k+1},\, \sum_{j=1}^{k-1} N_j J_j n_j\right\} \le \zeta_k \sum_{j=1}^{k} N_j J_j n_j. \tag{3.8}$$

For $j \ge 1$ with $j = N_1 + \cdots + N_{k-1} + q$ $(1 \le q \le N_k)$, we set

$$M(j) := 2\sum_{l=1}^{k-1} N_l J_l n_l + 2q J_k n_k$$

with the convention $M(0) := 0$ and

$$\tilde{A}_j := \bigcup_{h\in A_j} (\Lambda_{n_k} + h),$$

where

$$\begin{aligned} A_j &:= \{h = (h_1, \ldots, h_d) \in \Lambda_{M(k)} \setminus \Lambda_{M(k-1)} : h_i = (2p-1)n_j,\ \ p \in \mathbb{Z}\} \\ &= (\Lambda_{M(k)} \setminus \Lambda_{M(k-1)}) \cap (2n_k\mathbb{Z}^d + (n_k, \ldots, n_k)). \end{aligned}$$

Clearly, $\tilde{A}_j$ consists of $\sharp A_j$ cubes with side length $2n_k - 1$ and $a_{i,k}\sharp A_j$ is an integer for each $1 \le i \le p_k$. Moreover, $\{\tilde{A}_1, \tilde{A}_2, \ldots\}$ is a partition of $\mathbb{Z}^d$ with very little gap. (The situation is sketched in Figures 1 and 2 for $L = \mathbb{Z}^2$.) It follows from (3.5) and (3.8) that

$$\sharp\left(\Lambda_{M(j)} \Big\backslash \bigcup_{l=1}^{j} \tilde{A}_l\right) \le \frac{h'' - h'}{h''}\lambda_{M(j)}. \tag{3.9}$$

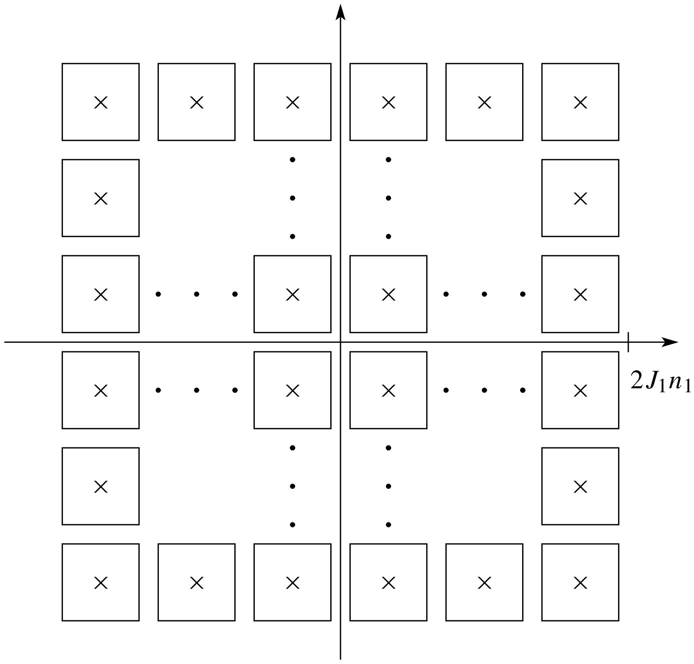

FIGURE 1. Sketch of the sets A_1 (which we denote by $\times$) and $\tilde{A}_1$ (solid box).

Let $\Gamma_{i,k}$ be a $(\delta^*, n_k, \epsilon^*)$-separated subset of $X_{n_k,\mathbb{B}(\mu_{i,k},\zeta_k)}$ with maximal cardinality, i.e.

$$\sharp\Gamma_{i,k} = N(\mathbb{B}(\mu_{i,k}, \zeta_k); \delta^*, n_k, \epsilon^*). \tag{3.10}$$

Now we construct a subset G such that, for each $x \in G$, the $\mathcal{T}$-orbit of x satisfies the following property. For $j \geq 1$ with $j = N_1 + \cdots + N_{k-1} + q$ $(1 \leq q \leq N_k)$, there are just $a_{i,k}\sharp A_j$ cubes in $\tilde{A}_j$ such that, on each cube, the $\mathcal{T}$-orbit of $x \in G$ partially ϵ_k-shadows the $\mathcal{T}$-orbit of some element in $\Gamma_{i,k}$. To construct such a subset G more precisely, we denote $A_1 = \{h^1, \ldots, h^{\sharp A_1}\}$, $A_2 = \{h^{\sharp A_1+1}, \ldots, h^{\sharp A_1+\sharp A_2}\}$ and so on. Then $\bigcup_k A_k$ can be expressed as $\{h^1, h^2, \ldots\}$. For $j \geq 1$ with

$$j = \sharp A_1 + \cdots + \sharp A_{(\sum_{l=1}^{k-1} N_l)+p} + (a_{1,k} + \cdots + a_{i-1,k})\sharp A_{(\sum_{l=1}^{k-1} N_l)+p+1} + q$$

for some $0 \leq p < N_k - 1$ and $0 < q \leq a_{i,k}\sharp A_{(\sum_{l=1}^{k-1} N_l)+p+1}$, we define three sequences $\{n'_j\}$, $\{\epsilon'_j\}$ and $\{\Gamma'_j\}$ as

$$n'_j := n_k, \quad \epsilon'_j := \epsilon_k, \quad \Gamma'_j := \Gamma_{i,k}$$

and let

$$G := \bigcap_{j=1}^{\infty} \bigcup_{x \in \Gamma'_j} T^{-h^j} B_{n'_j}(g; x, \epsilon'_j).$$

Then by (3.4), (3.10) and the choice of $\{n'_j\}$, $\{\epsilon'_j\}$ and $\{\Gamma'_j\}$, we have

$$\sharp\bigg(\prod_{h^j \in A_l} \Gamma'_j\bigg) \geq e^{\sharp A_l \lambda_{n'_l} h''} \tag{3.11}$$

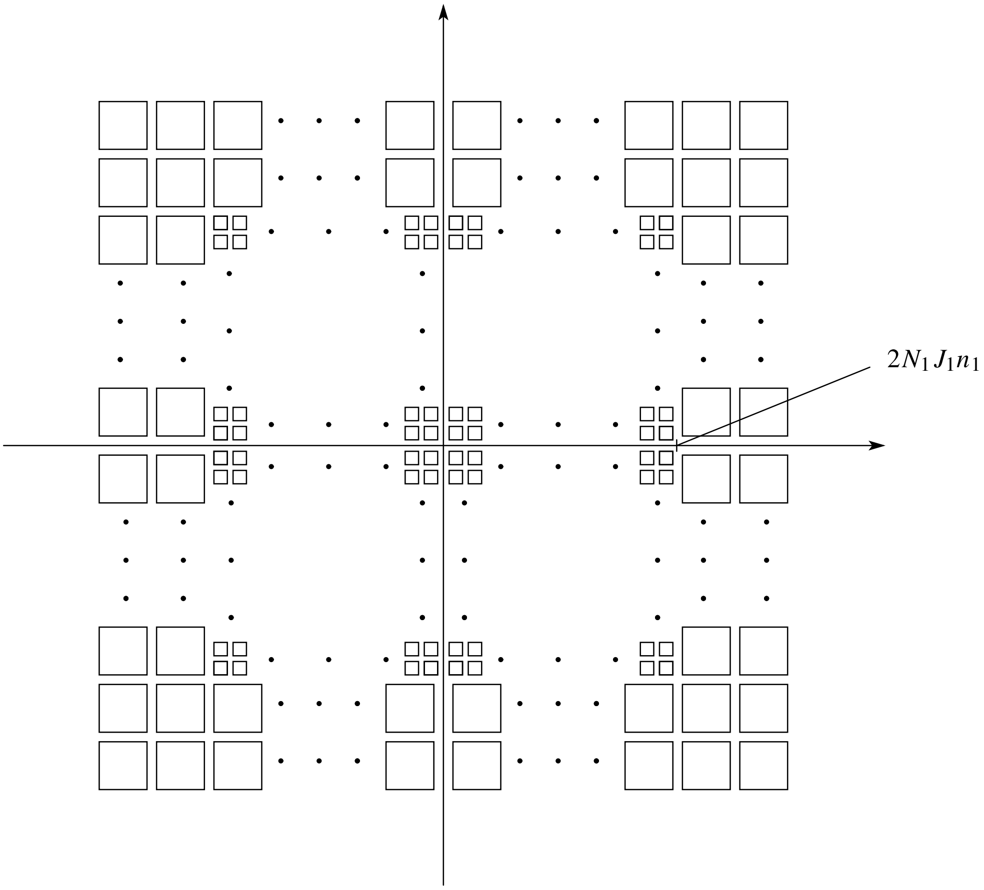

FIGURE 2. Sketch of the partition $\{\tilde{A}_1, \tilde{A}_2, \ldots\}$ of $\mathbb{Z}^2$. The side length of the small cube (respectively big cube) is $2n_1 - 1$ (respectively $2n_2 - 1$). So there is very little gap between any cubes.

for each $l \geq 1$. Since $\mathcal{T}$ has the almost product property, G is a non-empty compact set. Moreover, we can write

$$G = \bigcup \left\{ \bigcap_{j=1}^{\infty} T^{-h^j} B_{n'_j}(g; x_j, \epsilon'_j) : (x_1, x_2, \ldots) \in \prod_{j=1}^{\infty} \Gamma'_j \right\}. \tag{3.12}$$

LEMMA 3.7. *We have*

$$\lim_{m \to \infty} \sup_{y \in G} D(\mathcal{E}_m(y), \mu) = 0.$$

In particular, $G \subset G_\mu$.

Proof. Let $m > 0$ and $y \in G$. By (3.12), there exists $(x_1, x_2, \ldots) \in \prod_{j=1}^{\infty} \Gamma'_j$ such that $y \in \bigcap_{j=1}^{\infty} T^{-h^j} B_{n'_j}(g; x_j, \epsilon'_j)$. Take $k > 0$ as $M(\sum_{l=1}^{k} N_l) < m \leq M(\sum_{l=1}^{k+1} N_l)$. Now we decompose the cube Λ_m into four parts,

$$\Lambda_m = \Lambda^{(1)} \cup \Lambda^{(2)} \cup \Lambda^{(3)} \cup \Lambda^{(4)},$$

where

$$\Lambda^{(1)} := \Lambda_{M(\sum_{l=1}^{k-1} N_l)},$$

$$\Lambda^{(2)} := \bigcup \left\{ \Lambda_{n_k} + h^j : h^j \in \bigcup_{p=N_{k-1}+1}^{N_k} A_p \right\},$$

$$\Lambda^{(3)} := \bigcup \left\{ \Lambda_{n_{k+1}} + h^j \subset \Lambda_m : h^j \in \bigcup_{p=N_k+1}^{N_{k+1}} A_p \right\}$$

and

$$\Lambda^{(4)} := \Lambda_m \setminus (\Lambda^{(1)} \cup \Lambda^{(2)} \cup \Lambda^{(3)}).$$

Then, $D(\mathcal{E}_m(y), \mu) \le \sum_{t=1}^{4} (\sharp\Lambda^{(t)}/\lambda_m) D(\mathcal{E}_{\Lambda^{(t)}}(y), \mu)$. By the choice of $\{n_k\}$, $\{J_k\}$ and $\{N_k\}$, $\Lambda^{(1)}$ and $\Lambda^{(4)}$ are very small. Indeed we obtain the following estimates by (3.7) and (3.8): $\sharp\Lambda^{(1)}/\lambda_m \le \zeta_k^d$ and $\sharp\Lambda^{(4)}/\lambda_m \le (1+\zeta_k)^d - 1 + \zeta_k$.

To estimate the $D(\mathcal{E}_{\Lambda^{(2)}}(y), \mu)$, we decompose $\Lambda^{(2)}$ into many small cubes as

$$\Lambda^{(2)} = \bigcup_{i=1}^{p_k} \Lambda_i^{(2)},$$

$$\Lambda_i^{(2)} := \bigcup_j (\Lambda_{n_k} + h^j),$$

where the union is taken over all j such that $h^j \in \bigcup_{p=N_{k-1}}^{N_k} A_p$ and $\Gamma'_j = \Gamma_{i,k}$. Then, by the choice of $\{\Gamma'_j\}$, we can see that $\sharp\Lambda_i^{(2)}/\sharp\Lambda^{(2)} = a_{i,k}$. Thus we have

$$\begin{aligned}
D(\mathcal{E}_{\Lambda^{(2)}}(y), \mu) &\le D\left(\sum_{i=1}^{p_k} a_{i,k} \mathcal{E}_{\Lambda_i^{(2)}}(y), \ \sum_{i=1}^{p_k} a_{i,k} \mu_{i,k} \right) + D(\alpha_k, \mu) \\
&\le \sum_{i=1}^{p_k} a_{i,k} D(\mathcal{E}_{\Lambda_i^{(2)}}(y), \mu_{i,k}) + D(\alpha_k, \mu) \\
&\le \sum_{i=1}^{p_k} a_{i,k} \sum_{x_j \in \Gamma_{i,k}} \frac{\lambda_{n_k}}{\sharp\Lambda_i^{(2)}} (D(\mathcal{E}_{\Lambda_{n_k}+h^{(j)}}(y), \mu_{i,k})) + D(\alpha_k, \mu) \\
&\le \sum_{i=1}^{p_k} a_{i,k} \sum_{x_j \in \Gamma_{i,k}} \frac{\lambda_{n_k}}{\sharp\Lambda_i^{(2)}} \{D(\mathcal{E}_{\Lambda_{n_k}+h^{(j)}}(y), \mathcal{E}_{n_k}(x_j)) \\
&\quad + D(\mathcal{E}_{n_k}(x_j), \mu_{i,k})\} + D(\alpha_k, \mu).
\end{aligned}$$

Then by (3.7) and the choice of $\{\epsilon_k\}$ we have

$$D(\mathcal{E}_{\Lambda_{n_k}+h^{(j)}}(y), \mathcal{E}_{n_k}(x_j)) \le 2\gamma_k$$

and it follows from $x_j \in X_{n_k, \mathbb{B}(\mu_{i,k}, \zeta_k)}$ that

$$D(\mathcal{E}_{n_k}(x_j), \mu_{i,k}) \le \zeta_k.$$

Combining these estimations, we get

$$D(\mathcal{E}_{\Lambda^{(2)}}(y), \mu) \leq 2\gamma_k + \zeta_k + D(\alpha_k, \mu).$$

Similarly, we have

$$D(\mathcal{E}_{\Lambda^{(3)}}(y), \mu) \leq 2\gamma_{k+1} + \zeta_{k+1} + D(\alpha_{k+1}, \mu).$$

Finally we get

$$\begin{aligned} D(\mathcal{E}_m(y), \mu) &\leq \sum_{t=1}^{4} \frac{\sharp\Lambda^{(t)}}{\lambda_m} D(\mathcal{E}_{\Lambda^{(t)}}(y), \mu) \\ &\leq \zeta_k^d + 2\gamma_k + \zeta_k + D(\alpha_k, \mu) \\ &\quad + 2\gamma_{k+1} + \zeta_{k+1} + D(\alpha_{k+1}, \mu) + (1+\zeta_k)^d - 1 + \zeta_k, \end{aligned}$$

which implies $\lim_{m\to\infty} \sup_{y\in G} D(\mathcal{E}_m(y), \mu) = 0$. □

We continue the proof of Theorem 3.6. By Lemma 3.7 and the fact that $\lim_{n\to\infty} \lambda_{M(n+1)}/\lambda_{M(n)} = 1$, we can find an integer $Q > 0$ such that

$$h^*\lambda_{M(k+1)} \leq h'\lambda_{M(k)} \quad (M(k) \geq Q), \tag{3.13}$$

and

$$S_n\varphi(y) > \lambda_n s^* \quad (y \in G,\, n \geq Q). \tag{3.14}$$

For a sufficiently small $\epsilon > 0$, we take an arbitrary finite cover of G, $\mathcal{C} = \{B_{l_i}(x_i, \epsilon)\}_{i=1}^{q}$, with $\min_i l_i \geq Q$. For each $1 \leq i \leq q$, choose $k(i)$ as $M(k(i)) \leq l_i < M(k(i)+1)$. Then by (3.9), (3.11) and (3.12) we have

$$\begin{aligned} \sharp\Big(\prod\{\Gamma'_j : h^j \in \Lambda_{M(k(i))}\}\Big) &= \prod_{l=1}^{k(i)} \sharp\Big(\prod_{h^j\in A_l} \Gamma'_j\Big) \\ &\geq \prod_{l=1}^{k(i)} e^{\sharp A_l \lambda_{n'_l} h''} \\ &\geq \exp\Big\{\lambda_{M(k(i))} h'' \Big(1 - \frac{h''-h'}{h''}\Big)\Big\} \\ &= e^{h'\lambda_{M(k(i))}}. \end{aligned}$$

This inequality allows us to show that

$$1 \leq \sum_{i=1}^{q} e^{-h'\lambda_{M(k(i))}}$$

in a similar way to the proof of Lemma 5.1(4) in [**13**]. So it follows from (3.13) and (3.14) that

$$\begin{aligned}\sum_{i=1}^{q} e^{-(h^*+s^*)\lambda_{l_i} + \sup_{y\in B_{l_i}(x_i,\epsilon)} S_{l_i}\varphi(y)} &\geq \sum_{i=1}^{q} e^{-h^*\lambda_{l_i}} \\ &\geq \sum_{i=1}^{q} e^{-h^*\lambda_{M(k(i)+1)}} \\ &\geq \sum_{i=1}^{q} e^{-h'\lambda_{M(k(i))}} \\ &\geq 1.\end{aligned}$$

Since G is compact, we conclude that

$$1 \leq M_\varphi(G, h^* + s^*, Q, \epsilon),$$

which implies $P_G(\mathcal{T}, \varphi) \geq h^* + s^*$. □

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