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Optimum Step Stress Accelerated Life Testing for Lomax Distribution

Amal. S. Hassan, and Amani. S. Al-Ghamdi

Institute of Statistical Studies and Research (ISSR).Cairo University, Department of Mathematical Statistics. Faculty of Science, King Abdulaziz University

Abstract: The optimal times of changing stress level for simple stress plans under a cumulative exposure model using the Lomax distribution are determined for a wide range of values of the model parameters. The scale parameter of the Lomax failure time distribution at constant levels is assumed to be a log linear function of the stress level. The optimum test plan is obtained by minimizing with respect to the change time, the asymptotic variance (AV) of the maximum likelihood estimator (MLE) of a given 50th percentile of the distribution at the design stress. Interval estimations that generate narrow intervals to the unknown parameters of the distribution with high probability are obtained. Also, tests of hypotheses about model parameters using the likelihood ratio method are examined. Numerical study is discussed to illustrate the optimal time procedure using Mathcad (2001).

Key words: Accelerated life tests; Cumulative exposure model; Lomax distribution; Maximum likelihood estimation; Simple step stress.

INTRODUCTION

Accelerated life tests (ALTs) are used to estimate the lifetime of highly reliable products within a reasonable testing time. The test products are run at higher than usual levels of stress(which includes temperature, voltage, pressure, etc) to induce early failures. Data are obtained at accelerated conditions, and based on regression type model, results are extrapolated to the design stress to estimate the life time distribution, such overstress testing reduces time and cost. The stress can be applied in different ways: commonly used methods are constant stress, step stress and progressive stress Nelson[13].

The former put each experimental unit to only one of the stress levels. Some of the important early works in constant stress test can be found in Kelpinski and Nelson[7], Nelson and Meeker [14] and Meeker [10]. In progerssive-stress ALT, the stress applied to a test product is continuously increasing in time see, for example, Bai *et al.,* [4], Wang and Fei [15], Abdel-Hamid and Al-Hussaini [1].

The problem of designing optimum step stress ALT, and making inference, have been studied by several authors. Nelson [12] introduced step-stress ALT that allows test condition to change during testing. In this type of testing, a unit is placed on test at an initial low stress; and if does not fail in a predetermined time (τ), stress is increased. If there is a single change of stress, the ALT is called simple step-stress test. Most of the available literature on step stress ALT deals with exponential exposure model. Miller and Nelson [11] obtained simple step stress ALT plans for the case where test units have exponentially distributed life times. Khamis and Higgins [5,6] proposed a new model for step stress ALT as an alternative to the Weibull cumulative exposure model. Lu and Rudy [9] studied the Weibull step stress model in step stress ALT. Alhadeed and Yang [2] obtained the optimum design for the lognormal simple step stress model. Al-Haj Ebrahem and Al-Masri [3] studied the optimum times of changing stress levels under a log-logistic cumulative exposure model.

The Lomax distribution was originally proposed as a second kind of the Pareto distribution by Lomax [8]. It is used to provide a good model in biomedical problems. It is considered as an important model of lifetime models. Also, it has been used in relation with studies of income, size of cities and reliability modeling. It is being widely used for stochastic modeling of decreasing failure rate life components. It also serves as a useful model in the study of queuing theory and biological analysis.

The probability density function (p.d.f) of a random variable that has the Lomax distribution is given by

$$f(t,\theta,\lambda)=\frac{\lambda}{\theta}(1+\frac{t}{\theta})^{-(\lambda+1)},t>0. \qquad (1)$$

and the cumulative distribution function (c.d.f) is given by

Corresponding Author: Amal. S. Hassan, Institute of Statistical Studies and Research (ISSR).Cairo University, Department of Mathematical Statistics. Faculty of Science, King Abdulaziz University
E-mail: amanisg @hot mail. com Amal52-Soliman@yahoo.com

$$F(t,\theta,\lambda) = 1-(1+\frac{t}{\theta})^{-\lambda}, t > 0. \quad (2)$$

The purpose of this article is to choose times to change stress levels to minimize the asymptotic variance (AV) of the estimates of a given percentile at the design stress. The life time of test units is assumed to be Lomax with scale parameter is log linear function of stresses and cumulative exposure model is considered. The estimates of parameters, confidence intervals and testing hypotheses are discussed. A numerical illustration carried out to explain the theoretical results through Mathcad program (2001).

In addition to this introductory Section, this article includes seven Sections. Model and assumptions are described in Section 2. The maximum likelihood estimators(MLEs) of model parameters, in addition, asymptotic variance-covariance matrix of the estimators is given in Section 3. The confidence intervals of the model parameters that based on the asymptotic normality of the MLE are obtained in Section 4. In the same Section, test of hypotheses for parameters are obtained using likelihood ratio method. Section 5 provides the optimal life testing plan which will enable us to accurately estimate 50th percentile of life time of a product being testing without having to wait long time for the product to fail. For illustration, simulation studies are given in Section 6. Finally conclusions are reported in Section 7.

Model and Assumptions: Under any constant stress, the failure time of a test unit follows a Lomax distribution with c.d.f given by

$$F_i(t) = 1-(1+\frac{t}{\theta_i})^{-\lambda}, t > 0. \quad (2.1)$$

The following assumptions are made:

1) The scale parameter θ_i at stress level i, i = 0, 1, 2 is a log-linear function of stress, that is, $\log(\theta_i) = \beta_0+\beta_1 S_i$ where β_0 and β_1 are unknown parameters depending on the nature of the product and the method of the test.

2) The test is conducted as follows. All n units are initially put on the lower stress S_1 and run until time τ. Then, the stress is changed to high level S_2, and the test continues until all units fail.

3) Total of n_i failure are observed at time $t_{ij}, j = 1,2,..,n_i$ level $S_i, i = 1,2$

4) The life times of test units are i.i.d.

5) The cumulative exposure model of time to failure step-stress ALT is given by

$$F(t) = \begin{cases} F_1(t) & 0 \le t < \tau \\ F_2(t-\tau+\tau') & \tau \le t < \infty \end{cases} \quad (2.2)$$

where the equivalent starting time, τ', is a solution of $F_1(\tau) = F_2(\tau')$ solving for τ', then $\tau' = \frac{\theta_2}{\theta_1}\tau$

Thus, the Lomax cumulative exposure model for simple step-stress is given by

$$F(t) = \begin{cases} 1-(1+\frac{t}{\theta_1})^{-\lambda} & 0 \le t < \tau \\ 1-\left[1+\frac{1}{\theta_2}(t-\tau(1-\frac{\theta_2}{\theta_1}))\right]^{-\lambda} & \tau \le t < \infty \end{cases} \quad (2.3)$$

6) The corresponding p.d.f of the failure time is obtained as follows:

$$f(t) = \begin{cases} \frac{\lambda}{\theta_1}(1+\frac{t}{\theta_1})^{-\lambda-1} & 0 \le t < \tau \\ \frac{\lambda}{\theta_2}\left[1+\frac{1}{\theta_2}(t-\tau(1-\frac{\theta_2}{\theta_1}))\right]^{-\lambda-1} & \tau \le t < \infty \end{cases} \quad (2.4)$$

Likelihood Function and Fisher Information Matrix: In order to obtain the maximum likelihood estimate (MLE) of the model parameters, let $t_{ij}, j = 1,2,\ldots,n_i, i = 1,2$ be the observed failure time of a test unit j under stress level i, where n_1 denotes the number of units failed at the low stress S_1 and n_2 denotes the number of units failed at the high stress S_2.

The likelihood function is given by

$$L(\theta_1,\theta_2,\lambda)=\prod_{j=1}^{n_1}\frac{\lambda}{\theta_1}(1+\frac{t_{1j}}{\theta_1})^{-(\lambda+1)}\prod_{j=1}^{n_2}\frac{\lambda}{\theta_2}\{1+\frac{1}{\theta_2}[t_{2j}-\tau(1-\frac{\theta_2}{\theta_1})]^{-(\lambda+1)}\} \tag{3.1}$$

Taking the logarithm of the likelihood function (3.1),then

$$\log L(\theta_1,\theta_2,\lambda)=n\log\lambda-n_1\log\theta_1-n_2\log\theta_2-(\lambda+1)\sum_{j=1}^{n_1}\log(1+\frac{t_{1j}}{\theta_1})$$
$$-(\lambda+1)\sum_{j=1}^{n_2}\log\{1+\frac{1}{\theta_2}[t_{2j}-\tau(1-\frac{\theta_2}{\theta_1})]\}, \tag{3.2}$$

where, $n=n_1+n_2$

by using the relation $\log\theta_i=\beta_0+\beta_1 S_i,\ i=0,1,2,$ The log likelihood function (3.2) becomes,

$$\log L(\beta_0,\beta_1,\lambda)=n\log\lambda-n_1(\beta_0+\beta_1S_1)-(\lambda+1)\sum_{j=1}^{n_1}\log(1+t_{1j}e^{-(\beta_0+\beta_1S_1)})$$
$$-(\lambda+1)\sum_{j=1}^{n_2}\log\{1+e^{-(\beta_0+\beta_1S_2)}(t_{2j}-\tau)+\tau e^{-(\beta_0+\beta_1S_1)}\} \tag{3.3}$$
$$-n_2(\beta_0+\beta_1S_2).$$

For simplicity, we write log L instead of log $L(\beta_0,\beta_1,\lambda)$: The MLEs $\hat{\beta}_0,\hat{\beta}_1\ and\ \hat{\lambda}$ for the model parameters β_0, β_1 and λ are the values which maximize the likelihood function (3.1). The first derivatives of the logarithm of the likelihood function with respect to each one of β_0 , β_1 and λ are btained as follows:

$$\frac{\partial\log L}{\partial\beta_0}=-n+(\hat{\lambda}+1)\{\sum_{j=1}^{n_1}\frac{t_{1j}\theta_1^{-1}}{1+t_{1j}\theta_1^{-1}}+\sum_{j=1}^{n_2}\frac{t_{2j}\theta_2^{-1}+\tau[\theta_1^{-1}-\theta_2^{-1}]}{1+t_{2j}\theta_2^{-1}-\tau[\theta_2^{-1}-\theta_1^{-1}]}\}=0, \tag{3.4}$$

where, $\theta_1^{-1}=e^{-(\hat{\beta}_0+\hat{\beta}_1S_1)},$ and $\theta_2^{-1}=e^{-(\hat{\beta}_0+\hat{\beta}_1S_2)},$

$$\frac{\partial\log L}{\partial\beta_1}=-n_1S_1-n_2S_2+(\hat{\lambda}+1)\{\sum_{j=1}^{n_1}\frac{S_1t_{1j}\theta_1^{-1}}{(1+t_{1j}\theta_1^{-1})}+\sum_{j=1}^{n_2}\frac{t_{2j}S_2\theta_2^{-1}+\tau[S_1\theta_1^{-1}-S_2\theta_2^{-1}]}{1+t_{2j}\theta_2^{-1}-\tau[\theta_2^{-1}-\theta_1^{-1}]}\}=0, \tag{3.5}$$

and

$$\frac{\partial\log L}{\partial\lambda}=\frac{n}{\hat{\lambda}}-\sum_{j=1}^{n_1}\log(1+\theta_1^{-1}t_{1j})-\sum_{j=1}^{n_2}\log[1+\theta_2^{-1}t_{2j}-\tau(\theta_2^{-1}-\theta_1^{-1})]=0, \tag{3.6}$$

$$\hat{\lambda}=\frac{n}{\sum_{j=1}^{n_1}\log(1+t_{1j}\theta_1^{-1})+\sum_{j=1}^{n_2}\log[1+\theta_2^{-1}t_{2j}-\tau(\theta_2^{-1}-\theta_1^{-1})]} \tag{3.7}$$

Substitute equation (3.7) in equations (3.4) and (3.5), then we obtain the following two nonlinear equations

$$-n+\left(\frac{n}{\sum_{j=1}^{n_1}\log(1+t_{1j}\theta_1^{-1})+\sum_{j=1}^{n_2}\log[1+\theta_2^{-1}t_{2j}-\tau(\theta_2^{-1}-\theta_1^{-1})]}+1\right)\times \left\{\sum_{j=1}^{n_2}\frac{t_{2j}\theta_2^{-1}+\tau[\theta_1^{-1}-\theta_2^{-1}]}{1+t_{2j}\theta_2^{-1}-\tau(\theta_2^{-1}-\theta_1^{-1})}+\sum_{j=1}^{n_1}\frac{t_{1j}\theta_1^{-1}}{1+t_{1j}\theta_1^{-1}}\right\}=0, \tag{3.8}$$

and

$$-n_1S_1-n_2S_2+\left(\frac{n}{\sum_{j=1}^{n_1}\log(1+t_{1j}\theta_1^{-1})+\sum_{j=1}^{n_2}\log[1+\theta_2^{-1}t_{2j}-\tau(\theta_2^{-1}-\theta_1^{-1})]}+1\right)\times \left\{\sum_{j=1}^{n_2}\frac{t_{2j}S_2\theta_2^{-1}+\tau[S_1\theta_1^{-1}-S_2\theta_2^{-1}]}{1+t_{2j}\theta_2^{-1}-\tau(\theta_2^{-1}-\theta_1^{-1})}+\sum_{j=1}^{n_1}\frac{S_1t_{1j}\theta_1^{-1}}{(1+t_{1j}\theta_1^{-1})}\right\}=0, \tag{3.9}$$

Obviously, it is not easy to obtain a closed form solution for the two non-linear equations (3.8) and (3.9). So, iterative procedure must be applied to solve these equations numerically. Newton Raphson method is used to obtain the MLE of $\beta_0$and β_1. Thus, once the values of β_0 and $\beta_1$are determined, an estimate of λ can be obtained from equation (3.7).

The Fisher information matrix is obtained by taking the expected value of the second and mixed partial derivatives of log L with respect to β_0, β_1 and λ. Unfortunately, the exact mathematical expression for the expectation is very difficult to find. Therefore, it can be approximated by numerically inverting the asymptotic Fisher information matrix. It is composed of the negative second and mixed derivatives of the natural logarithm of the likelihood function evaluated at the MLE. So, the asymptotic Fisher information matrix can be written as follows:

$$F=\begin{bmatrix}-\frac{\partial^2\log L}{\partial\beta_0^2} & -\frac{\partial^2\log L}{\partial\beta_0\,\partial\beta_1} & -\frac{\partial^2\log L}{\partial\beta_0\,\partial\lambda}\\ -\frac{\partial^2\log L}{\partial\beta_0\,\partial\beta_1} & -\frac{\partial^2\log L}{\partial\beta_1^2} & -\frac{\partial^2\log L}{\partial\beta_1\,\partial\lambda}\\ -\frac{\partial^2\log L}{\partial\beta_0\,\partial\lambda} & -\frac{\partial^2\log L}{\partial\beta_1\partial\lambda} & -\frac{\partial^2\log L}{\partial\lambda^2}\end{bmatrix}\downarrow(\hat\beta_0,\hat\beta_1,\hat\lambda)$$

The second and mixed partial derivatives of the logarithm of the likelihood function are obtained as follows:

$$\frac{\partial^2\log L}{\partial\beta_0^2}=-(\hat\lambda+1)\left\{\sum_{j=1}^{n_1}\frac{t_{1j}\theta_1^{-1}}{(1+t_{1j}\theta_1^{-1})^2}+\sum_{j=1}^{n_2}\frac{t_{2j}\theta_2^{-1}+\tau[\theta_1^{-1}-\theta_2^{-1}]}{[1+t_{2j}\theta_2^{-1}-\tau(\theta_2^{-1}-\theta_1^{-1})]^2}\right\},$$

$$\frac{\partial^2 \log L}{\partial \beta_1^2} = (\hat{\lambda}+1)\sum_{j=1}^{n} \frac{\tau(S_2^2\theta_2^{-1} - S_1^2\theta_1^{-1})[1+t_{2j}\theta_2^{-1} - \tau(\theta_2^{-1} - \theta_1^{-1})] + S_2^2 t_{2j}\theta_2^{-1}(\tau\theta_2^{-1} - \tau\theta_1^{-1} - 1)}{[1+t_{2j}\theta_2^{-1} - \tau(\theta_2^{-1} - \theta_1^{-1})]^2}$$

$$+ (\hat{\lambda}+1)\sum_{j=1}^{n} \frac{\tau(S_1\theta_1^{-1} - S_2\theta_2^{-1})[2S_2 t_{2j}\theta_2^{-1} + \tau(\theta_1^{-1}S_1 - S_2\theta_2^{-1})}{[1+t_{2j}\theta_2^{-1} - \tau(\theta_2^{-1} - \theta_1^{-1})]^2}$$

$$- (\hat{\lambda}+1)\sum_{j=1}^{n} \frac{S_1^2 t_{1j}\theta_1^{-1}}{(1+t_{1j}\theta_1^{-1})^2},$$

$$\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = -(\hat{\lambda}+1)\{\sum_{j=1}^{n} \frac{S_1 t_{1j}\theta_1^{-1}}{(1+t_{1j}\theta_1^{-1})^2} + \sum_{j=1}^{n} \frac{S_2 t_{2j}\theta_2^{-1} + \tau[\theta_1^{-1}S_1 - \theta_2^{-1}S_2}{[1+t_{2j}\theta_2^{-1} - \tau(\theta_2^{-1} - \theta_2^{-1})]^2}\},$$

$$\frac{\partial^2 \log L}{\partial \beta_0 \partial \lambda} = \sum_{j=1}^{n} \frac{S_1 t_{1j}\theta_1^{-1}}{1+t_{1j}\theta_1^{-1}} + \sum_{j=1}^{n} \frac{S_2 t_{2j}\theta_2^{-1} + \tau[S_1\theta_1^{-1} - S_2\theta_2^{-1}]}{[1+t_{2j}\theta_2^{-1} - \tau(\theta_2^{-1} - \theta_1^{-1})},$$

and, $\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{n}{\lambda^2}$

Confidence Intervals and Testing Hypotheses: Based on the asymptotic normality, the confidence intervals of the model parameters of the MLE are obtained. Test of hypotheses for parameters are obtained using likelihood ratio test method.

For large sample size, the MLEs $\hat{\varphi}$ under appropriate regularity conditions, are consistent and asymptotically normally distributed with mean φ and variance $\sigma_n^2(\hat{\varphi})$ where $\sigma_n^2(\hat{\varphi})$ indicates that the variance is a function of $\hat{\varphi}$ and the sample size n. Consequently, a confidence interval with an approximate confidence coefficient 1 -α for the MLE $\hat{\varphi}$ of a population parameter φ can be constructed, such that

$$P[\hat{\varphi} - z\sigma(\hat{\varphi}) \leq \varphi \leq \hat{\varphi} + z\sigma(\hat{\varphi})] \cong 1-\alpha \tag{4.1}$$

The two sided approximate confidence limits for the model parameters β_0, β_1 and λ will be obtained respectively as follows

$$\begin{aligned} LL(\beta_0) &= \hat{\beta}_0 - z\sigma(\hat{\beta}_0) \quad & UL(\beta_0) &= \hat{\beta}_0 + z\sigma(\hat{\beta}_0), \\ LL(\beta_1) &= \hat{\beta}_1 - z\sigma(\hat{\beta}_1) \quad & UL(\beta_1) &= \hat{\beta}_1 + z\sigma(\hat{\beta}_1), \\ LL(\lambda) &= \hat{\lambda} - z\sigma(\hat{\lambda}) \quad & UL(\lambda) &= \hat{\lambda} + z\sigma(\hat{\lambda}), \end{aligned} \tag{4.2}$$

Therefore, the two sided approximate confidence limits for β_0, β_1 and λ under different stresses and different sample sizes will be constructed using equation (4.2) with confidence levels 95% and 99% (see Tables (3-6))
Tests of hypotheses for parameters can be obtained either by using the likelihood ratio method, or the approximate s-normality of the MLE in large samples. An important inference problem concerning the regression coefficients is the test of hypothesis

$$H_0: \beta_1 = 0 \ against \quad H_1: \beta_1 \neq 0, \tag{4.3}$$

where β_1 is the parameter of the log linear relationship between stresses S_i, i = 1, 2 and the scale parameter. For large sample size, the generalized likelihood ratio is defined to be

$$-2\log\Lambda = -2\log[\frac{L(\tilde{\beta}_0, 0)}{L(\hat{\beta}_0, \hat{\beta}_1)}], \tag{4.4}$$

where $\tilde{\beta}_0$ and $\hat{\beta}_1$ are the unrestricted MLE of the parameters β_0 and β_1 which obtained by solving the two non-linear equations (3.8) and (3.9) numerically using the Newton Raphson method. While $\tilde{\tilde{\beta}}_0$ is the restricted MLE of the parameter β_0 under H_0, it is obtained by setting the value β_1 to be equal zero and then find the MLE of β_0. The test with approximate size α is given by the following; reject H_0 if and only if, $-2\log\Lambda > \chi^2_{1-\alpha,1}$ where $\chi^2_{1-\alpha,1}$ is the (1-α)th quantile of the chi-square distribution with one degrees of freedom.

Therefore, the test with approximate size for the parameter β_1 will be constructed using the test of hypothesis (4.3) and the generalized likelihood ratio defined in equation (4.4).

Optimization Criterion: The optimal test plan is to determine the duration of the lower stress level. According to Al-Haj Ebrahem and Al-Masri [3], an optimum test plan can be determined with respect to the change time, the asymptotic variance (AV) of the MLE of a given 100 p-th percentile at the design stress S_0. The log of the 100 p-th percentile of the lifetime $t_p(S)$ at the design stress S_0 is given by

$$\eta_p(S_0) = \log(t_p(S_0)) = \beta_0 + \beta_1 S_0 + \log[\theta_i[(1-p)^{-\frac{1}{\lambda}} - 1]] \tag{5.1}$$

Generally, the optimization criterion is defined to minimize the asymptotic variance of the percentile estimate at the design stress. The MLE is used for the percentile estimate. Then, the AV of the percentile estimate at the design stress can be obtained as follows:

$$\begin{aligned} AV(\hat{\eta}_p(S_0) &= AV\{\hat{\beta}_0 + \hat{\beta}_1 S_0 + \log[\hat{\theta}_i[(1-p)^{-\frac{1}{\lambda}} - 1]]\}, \\ &= H \;\; F^{-1} H' \end{aligned} \tag{5.2}$$

where F is the asymptotic Fisher information matrix given in Section (3), and

$$H = [\frac{\partial\hat{\eta}_p(S_0)}{\partial\hat{\beta}_0} \frac{\partial\hat{\eta}_p(S_0)}{\partial\hat{\beta}_1} \frac{\partial\hat{\eta}_p(S_0)}{\partial\hat{\lambda}}].$$

Then

$$AV(\hat{\eta}_p(S_0) = \left[1 \; S_0 \; \frac{(1-p)^{-\frac{1}{\lambda}}\log(1-p)}{\hat{\lambda}^2[(1-p)^{\frac{1}{\lambda}} - 1]}\right] F^{-1} \left[1 \; S_0 \; \frac{(1-p)^{-\frac{1}{\lambda}}\log(1-p)}{\hat{\lambda}^2[(1-p)^{\frac{1}{\lambda}} - 1]}\right] \tag{5.3}$$

Therefore, the optimal times τ^* to change stress levels under different values of stresses and sample sizes will be obtained numerically using equation (5.3) and a similar method discussed by Alhadeed and Yang [2] with percentile (p) = 0.5 (see Table (8)).

Numerical Illustration: The concept of the optimum test plan is to determine the duration of the lower stress level by minimizing the AV of the MLE of the 100 p-th percentile at the design stress. To find the formula for the optimum time of changing stress level, some properties of the derived estimators by the MLE method will be obtained. Unfortunately, it is too complicated to study the derived expression analytically. So, a simulation study will be carried out using Mathcad (2001) for illustrating the theoretical results of the estimation problem. The performance of the resulting estimators of the shape parameter and the parameters of the log linear relationship between stresses and the scale parameter in the Lomax lifetime distribution has been considered in terms of their absolute relative bias (RBias) of estimators; which is the absolute difference between the estimator and its true value divided by the true value of the parameter, mean square error (MSE) of estimators; which is a reasonable criterion that takes into account both the variance and the bias of an estimator and relative error (RE); which is the square root of mean square error divided by the value of the parameter. In addition, the asymptotic variance and covariance matrix, two-sided confidence interval and tests of hypotheses for the parameters are obtained. The simulation procedures are described through the following steps:

Step (1): For given values of the parameters β_0, β_1 and λ and given values of stresses S_1 = 0.2, 0.4, 0.8 and S_2 = 0.6; 1 calculate $\theta_j = e^{\beta_0 + \beta_1 S_j}, j = 1,2$

Step (2) A random samples T_1; T_2,...T_n of sizes $n = n_1 + n_2$ = 50(10)100, were generated from cumulative exposure model (2.3) defined in Section 2.

Step (3): For each sample size and for the selected values of the parameters β_0, β_1 and λ, the MLEs $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\lambda}$ were obtained by solving the two non-linear equations (3.8) and (3.9). The exact solution of these equations dose not exists, so Newton Raphson method will be used for solving the non-linear equations. Once the estimates of β_0 and β_1 are computed, the estimates of λ is easily computed using equation (3.7).

Step (4): The performance of the estimators can be evaluated through some measures of accuracy which are RBias, MSE and RE for all sample sizes and under different stress levels (see Tables (1-2)).

Step (5): The approximate two sided confidence limit with confidence level 95% and 99% of parameters β_0; β_1 and λ are constructed. The evaluated upper and lower confidence intervals are reported in Tables (3-6).

Step (6): The results of the likelihood ratio test under the hypothesis for the parameter β_1 were obtained using equation (4.4).

Step (7): The resulting estimates of the parameters β_0, β_1 and λ are used to obtain the asymptotic variance-covariance matrix.

Step (8): Based on n; S_1, S_2, and for given value of p = 0.5 the optimal value of τ^* tha minimize AV of MLE was obtained from equation (5.3).

Simulation results are summarized as follows:

Tables (1-2) give the RBias, MSE and RE of the estimators. The approximate confidence limits at 95% and 99% for the shape parameter and the parameters of the log linear relationship between stresses and the scale parameter presented in Tables (3-6). The results of the likelihood ratio test for the parameters are presented in Table (7). Table (8) gives the optimum time of changing stress levels.

The following conclusions can be observed on the properties of estimated parameters from the Lomax lifetime distribution step stress ALT:

1.As the sample size increases with the same stresses S_1 and S_2 RBias, MSE and RE of the parameters are decreasing (see Tables (1-2)).

2.The interval of the estimators decreases when the sample size is increasing. Also, the interval of the estimator at α = 0.05 is smaller than the interval of estimator at α = 0.01(see Tables (3-6))

3.The optimal value of τ^* is decreasing as the sample size increasing with the same stresses S_1 and S_2 (see Table (8))

4.At the same sample size, for the same high stress S_2 and the low stress S_1 is increasing, the optimal value of τ^* is decreasing (see Table (8))

5. At the same sample size, for the same low stress S_1 and the high stress S_2 is increasing, the optimal value of τ^* is decreasing (see Table (8))

6. The decision rule for hypothesis testing of β_1 is rejected H_0 (i.e. $\beta_1 \neq 0$), for all sample sizes and different values of stresses (see Table (7))

Discussion and Conclusions: ALT are used to estimate the lifetime of highly realizable products within a reasonable testing time. The test units are run at higher than usual levels of stress to induce early failures. The test data obtained at the accelerated conditions are analyzed in terms of a model, and then extrapolated to design stress to estimate the life time distribution.

This article considered the inference on the parameters and optimally designing simple step stress for the Lomax distribution. The MLEs of the model parameters were obtained. Performance of step stress testing plans and model assumptions are usually evaluated by the properties of the MLEs of model parameters. The asymptotic variance and covariance of estimators are obtained. Based on the asymptotic normality, the two-sided confidence limits of the model parameters are constructed, also test of hypotheses are discussed. The optimality criterion adopted was minmization with respect to the change time the AV of the MLE of a given percentile of the distribution at design stress.

Table 1: The RBias, MSE and RE of the parameters (β_0, β_1, λ) for different stress levels and di§erent sample sizes

Sample size	Parameters	Stresses $S_1 = 0.2, S_2 = 0.6$			$S_1 = 0.2, S_2 = 1$			$S_1 = 0.4, S_2 = 0.6$		
n	β_0 ,β_1,λ	RBias	MSE	RE	RBias	MSE	RE	RBias	MSE	RE
50	β_0	0.084	0.212	0.092	0.061	0.148	0.077	0.113	0.361	0.182
	β_1	0.769	0.441	0.664	0.961	0.715	0.846	0.974	0.875	0.935
	λ	0.256	0.213	0.257	0.313	0.394	0.314	0.314	0.396	0.281
60	β_0	0.036	0.073	0.054	0.051	0.114	0.067	0.093	0.275	0.149
	β_1	0.605	0.332	0.576	0.355	0.693	0.832	0.553	0.715	0.648
	λ	0.229	0.183	0.231	0.311	0.393	0.312	0.279	0.315	0.271
70	β_0	0.028	0.044	0.042	0.038	0.056	0.047	0.074	0.158	0.120
	β_1	0.604	0.295	0.543	0.300	0.546	0.740	0.365	0.420	0.492
	λ	0.213	0.146	0.214	0.303	0.369	0.304	0.278	0.312	0.257
80	β_0	0.022	0.043	0.034	0.025	0.052	0.045	0.065	0.138	0.105
	β_1	0.559	0.256	0.506	0.278	0.515	0.718	0.217	0.242	0.457
	λ	0.188	0.155	0.191	0.290	0.338	0.291	0.274	0.311	0.248
90	β_0	0.016	0.028	0.033	0.015	0.032	0.036	0.061	0.112	0.074
	β_1	0.432	0.246	0.496	0.169	0.442	0.665	0.158	0.104	0.322
	λ	0.170	0.126	0.177	0.259	0.263	0.260	0.270	0.293	0.058
100	β_0	0.005	0.028	0.050	0.012	0.026	0.032	0.060	0.102	0.073
	β_1	0.343	0.161	0.401	0.155	0.109	0.331	0.086	0.051	0.226
	λ	0.162	0.109	0.165	0.256	0.259	0.257	0.247	0.247	0.055

Table 2: The RBias, MSE and RE of the parameters (β_0, β_1, λ) for different stress levels and different sample sizes

Sample size	Parameters	Stresses $S_1 = 0.4; S_2 = 1$			$S_1 = 0.8, S_2 = 1$		
n	β_0, β_1, λ	RBias	MSE	RE	RBias	MSE	RE
50	β_0	0.095	0.256	0.101	0.128	0.808	0.177
	β_1	0.496	0.851	0.922	0.662	0.660	0.812
	λ	0.357	0.512	0.358	0.438	0.770	0.439
60	β_0	0.065	0.139	0.075	0.118	0.679	0.165
	β_1	0.417	0.683	0.826	0.519	0.546	0.739
	λ	0.348	0.488	0.349	0.426	0.726	0.426
70	β_0	0.055	0.107	0.065	0.130	0.469	0.127
	β_1	0.376	0.678	0.823	0.425	0.428	0.654
	λ	0.337	0.456	0.338	0.420	0.706	0.420
80	β_0	0.040	0.062	0.046	0.075	0.400	0.083
	β_1	0.179	0.438	0.662	0.369	0.295	0.544

Table 2: Continue

	λ	0.275	0.380	0.308	0.415	0.650	0.403
90	β_0	0.039	0.054	0.037	0.066	0.179	0.077
	β_1	0.176	0.387	0.622	0.284	0.257	0.507
	λ	0.306	0.377	0.307	0.403	0.647	0.402
100	β_0	0.026	0.035	0.050	0.047	0.149	0.078
	β_1	0.082	0.313	0.559	0.174	0.218	0.215
	λ	0.303	0.304	0.276	0.395	0.629	0.397

Table 3: Confidence bounds of the parameters at confidence level 0.95

		Stresses								
Sample size	Parameters	S_1 = 0.2, S_2 = 0.6			S_1 = 0.2, S_2 = 1			S_1 = 0.4, S_2 = 0.6		
n	β_0,β_1,λ	LL	UL	Length	LL	UL	Length	LL	UL	Length
	β_0	4.621	5.436	0.815	4.232	5.519	1.287	4.420	5.721	1.301
50	β_1	0.417	1.015	0.598	0.312	1.133	0.821	0.549	1.123	0.574
	λ	1.465	2.353	0.888	1.046	2.220	1.174	1.340	2.066	0.726
	β_0	4.788	5.575	0.787	4.817	5.690	0.873	4.987	5.943	0.956
60	β_0	0.719	1.390	0.671	0.529	1.161	0.632	0.505	1.060	0.555
	λ	1.551	2.300	0.749	1.171	2.295	1.124	1.336	2.052	0.716
	β_0	5.063	5.781	0.718	4.732	5.390	0.658	5.126	5.839	0.713
70	β_1	0.716	1.116	0.400	0.390	1.010	0.620	0.668	1.183	0.515
	λ	1.507	2.239	0.732	1.087	2.161	1.074	1.338	2.045	0.707
	β_0	4.838	5.443	0.605	4.797	5.353	0.556	4.977	5.676	0.699
80	β_1	0.700	1.077	0.377	0.487	1.051	0.564	0.889	1.195	0.306
	λ	1.429	2.156	0.727	1.091	2.149	1.058	1.386	2.034	0.648
	β_0	4.792	5.369	0.577	4.942	5.436	0.494	5.076	5.661	0.585
90	β_1	0.682	1.054	0.372	0.877	1.091	0.214	0.919	1.139	0.220
	λ	1.491	2.156	0.665	1.194	2.251	1.057	1.527	2.117	0.590
	β_0	4.859	5.356	0.497	4.991	5.368	0.377	5.073	5.524	0.451
100	β_1	0.778	1.092	0.314	1.287	1.461	0.174	1.338	1.545	0.207
	λ	1.561	2.205	0.644	1.146	2.194	1.048	1.516	2.026	0.510

Table 4: Confidence bounds of the parameters at confidence level 0.95

		Stresses					
Sample size	Parameters	S_1 = 0.4, S_2 = 1			S_1 = 0.8, S_2 = 1		
n	β_0,β_1, λ	LL	UL	Length	LL	UL	Length
50	β_0	4.967	5.684	0.717	4.122	5.598	1.476
	β_1	0.212	1.098	0.886	0.110	1.197	1.087
	λ	1.535	2.242	0.707	0.846	2.050	1.204

Table 4: Continue

60	β_0	5.123	5.829	0.706	5.649	6.646	0.997
	β_1	0.190	1.057	0.867	0.354	1.077	0.723
	λ	1.412	2.093	0.681	0.972	2.025	1.053
70	β_0	4.930	5.621	0.691	5.148	6.034	0.886
	β_1	0.157	1.009	0.852	0.488	1.173	0.685
	λ	1.388	2.059	0.671	1.019	2.070	1.051
80	β_0	4.904	5.492	0.588	5.242	6.062	0.820
	β_1	0.902	1.140	0.238	0.391	1.044	0.653
	λ	1.407	2.064	0.657	1.138	2.055	0.917
90	β_0	4.952	5.440	0.488	4.932	5.726	0.794
	β_1	0.908	1.143	0.235	0.463	1.098	0.635
	λ	1.559	2.093	0.534	1.310	2.0ë11	0.701
100	β_0	4.800	5.381	0.581	5.021	5.730	0.709
	β_1	0.968	1.180	0.212	0.532	1.124	0.592
	λ	1.574	2.104	0.530	1.374	2.072	0.698

Table 5: Confidence bounds of the parameters at confidence level 0.99

		Stresses								
Sample size	Parameters	S_1 = 0.2, S2 = 0.6			S_1 = 0.2, S_2 = 1			S_1 = 0.4, S_2 = 0.6		
n	β_0, β_1, λ	LL	UL	Length	LL	UL	Length	LL	UL	Length
50	β_0	4.492	5.565	1.073	4.088	5.305	1.217	4.897	6.243	1.346
	β_1	0.310	1.202	0.892	0.440	1.304	0.864	0.418	1.147	0.729
	λ	1.404	2.415	1.011	0.943	2.130	1.187	1.289	2.079	0.790
60	β_0	4.663	5.700	1.073	4.679	5.825	1.146	4.835	6.094	1.259
	β_1	0.123	1.012	0.889	0.229	1.061	0.832	0.497	1.221	0.724
	λ	1.512	2.339	0.827	0.899	2.031	1.132	1.302	2.087	0.785
70	β_0	4.949	5.895	0.946	4.627	5.494	0.867	4.866	5.287	0.421
	β_1	0.199	1.037	0.838	0.292	1.108	0.816	0.610	1.261	0.651
	λ	1.471	2.275	0.804	0.960	2.088	1.128	1.305	2.078	0.773
80	β_0	4.743	5.538	0.795	4.709	5.441	0.732	4.984	5.753	0.769
	β_1	0.468	1.089	0.621	0.298	1.040	0.742	0.728	1.149	0.421
	λ	1.393	2.192	0.799	0.969	2.076	1.107	1.302	2.011	0.709
90	β_0	4.701	5.460	0.759	4.864	5.514	0.650	5.013	5.951	0.938
	β_1	0.512	1.057	0.545	0.743	1.025	0.282	0.640	1.044	0.404
	λ	1.465	2.182	0.717	0.970	2.071	1.101	1.362	2.057	0.695

Table 7: Continue

100	β_0	4.780	5.435	0.655	4.931	5.428	0.497	5.002	5.595	0.593
	β_1	0.653	1.179	0.526	1.260	1.488	0.228	1.305	1.578	0.273
	λ	1.538	2.232	0.694	0.933	2.017	1.084	1.496	2.147	0.651

Table 6: Confidence bounds of the parameters at confidence level 0.99

Sample Size	Parameters	Stresses					
n	β_0, β_1, λ	S_1=0.4; S_2=1			S_1=0.8; S_2=1		
		LL	UL	Length	LL	UL	Length
50	β_0	4.854	5.747	0.893	4.333	6.962	2.629
	β_1	0.222	1.144	0.922	0.021	1.394	1.373
	λ	1.162	2.085	0.923	0.798	2.098	1.300
60	β_0	5.012	5.935	0.923	3.731	5.990	2.259
	β_1	0.311	1.230	0.919	0.218	1.320	1.102
	λ	1.186	2.091	0.905	0.848	2.049	1.201
70	β_0	4.820	5.730	0.910	5.008	6.174	1.166
	β_1	0.230	1.123	0.893	0.240	1.192	0.952
	λ	1.183	2.023	0.840	0.895	2.094	1.199
80	β_0	4.811	5.585	0.774	5.113	6.192	1.079
	β_1	0.682	1.153	0.471	0.180	1.081	0.901
	λ	1.382	2.189	0.807	0.919	2.073	1.154
90	β_0	1.875	5.517	3.642	4.806	5.852	1.046
	β_1	8.640	1.178	0.314	0.312	1.047	0.735
	λ	1.138	2.014	0.876	0.993	2.027	1.034
100	β_0	4.821	5.457	0.636	4.909	5.842	0.933
	β_1	0.839	1.097	0.258	0.563	1.198	0.635
	λ	1.453	2.024	0.571	1.195	2.088	0.893

Table 7: Testing of hypothesis for the parameter λ_1 with χ_1^2 at significance level α = 0.05

Stresses	Likelihood Ratio Test						Decision Rule
	n=50	n=60	n=70	n=80	n=90	n=100	
S_1 = 0.2; S_2 = 0.6	49.53	64.68	69.95	146.33	112.25	42.33	
S_1 = 0.2; S_2 = 1	138.40	100.57	172.62	118.59	283.53	173.20	Reject H_0
S_1 = 0.4; S_2 = 0.6	174.82	35.72	41.47	103.43	43.64	273.96	
S_1 = 0.4; S_2 = 1	163.43	148.04	143.00	217.96	296.33	275.86	
S_1=0.8; S_2 =1	205.45	281.92	241.82	351.84	190.56	516.80	

Table 8: Optimum time changing stress τ^*

Stresses	Optimum time τ^*					
	n=50	n=60	n=70	n=80	n=90	n=100
$S_1 = 0.2; S_2 = 0.6$	216.45	207.28	157.13	152.33	93.69	82.91
$S_1 = 0.2; S_2 = 1$	104.13	83.66	59.59	53.39	49.34	48.84
$S_1 = 0.4; S_2 = 0.6$	60.01	43.08	41.17	36.14	32.23	20.67
$S_1 = 0.4; S_2 = 1$	39.54	25.88	21.30	20.21	16.35	13.08
$S_1 = 0.8; S_2 = 1$	15.84	13.09	12.53	4.49	2.95	2.74

As shown from the numerical results, as the sample size increases with the same stresses, we get, RBias, MSE and RE of the parameters are decreasing. The standard deviation of the parameters under the Lomax distribution is decreasing. As the stresses and sample size increase the optimal time approach the changing time is decreasing.

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