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Research Article

The Diophantine Equation $8^x + p^y = z^2$

Lan Qi[1] and Xiaoxue Li[2]

[1] *College of Mathematics and Statistics, Yulin University, Yulin, Shaanxi 719000, China*
[2] *School of Mathematics, Northwest University, Xi'an, Shaanxi 710127, China*

Correspondence should be addressed to Xiaoxue Li; lxx20072012@163.com

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Let p be a fixed odd prime. Using certain results of exponential Diophantine equations, we prove that (i) if $p \equiv \pm 3 \pmod 8$, then the equation $8^x + p^y = z^2$ has no positive integer solutions (x, y, z); (ii) if $p \equiv 7 \pmod 8$, then the equation has only the solutions $(p, x, y, z) = (2^q - 1, (1/3)(q + 2), 2, 2^q + 1)$, where q is an odd prime with $q \equiv 1 \pmod 3$; (iii) if $p \equiv 1 \pmod 8$ and $p \neq 17$, then the equation has at most two positive integer solutions (x, y, z).

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. Let p be a fixed odd prime. Recently, the solutions (x, y, z) of the equation

$$8^x + p^y = z^2, \quad x, y, z \in \mathbb{N} \tag{1}$$

were determined in the following cases:

(1) (Sroysang [1]) if $p = 19$, then (1) has no solutions;
(2) (Sroysang [2]) if $p = 13$, then (1) has no solutions;
(3) (Rabago [3]) if $p = 17$, then (1) has only the solutions $(x, y, z) = (1, 1, 5), (2, 1, 9)$, and $(3, 1, 23)$.

In this paper, using certain results of exponential Diophantine equations, we prove a general result as follows.

Theorem 1. *If $p \equiv \pm 3 \pmod 8$, then (1) has no solutions (x, y, z). If $p \equiv 7 \pmod 8$, then (1) has only the solutions*

$$(p, x, y, z) = \left(2^q - 1, \frac{q+2}{3}, 2, 2^q + 1\right), \tag{2}$$

where q is an odd prime with $q \equiv 1 \pmod 3$.

If $p \equiv 1 \pmod 8$ and $p \neq 17$, then (1) has at most two solutions (x, y, z).

Obviously, the above theorem contains the results of [1, 2]. Finally, we propose the following conjecture.

Conjecture 2. *If $p \neq 17$, then (1) has at most one solution (x, y, z).*

2. Preliminaries

Lemma 3. *If $2^n - 1$ is a prime, where n is a positive integer, then n must be a prime.*

Proof. See Theorem 1.10.1 of [4]. □

Lemma 4. *If p is an odd prime with $p \equiv 1 \pmod 4$, then the equation*

$$u^2 - pv^2 = -1, \quad u, v \in \mathbb{N} \tag{3}$$

has solutions (u, v).

Proof. See Section 8.1 of [5]. □

Lemma 5. *The equation*

$$X^2 - 2^m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \ \gcd(X, Y) = 1, \ Y > 1, \quad m > 1, \quad n > 2 \tag{4}$$

has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

Proof. See Theorem 8.4 of [6]. □

Lemma 6. *Let D be a fixed odd positive integer. If the equation*

$$u^2 - Dv^2 = -1, \quad u, v \in \mathbb{N} \tag{5}$$

has solutions (u, v), then the equation

$$X^2 - D = 2^n, \quad X, n \in \mathbb{N}, \ n > 2 \tag{6}$$

has at most two solutions (X, n), except the following cases:

(i) *$D = 2^{2r} - 3 \cdot 2^{r+1} + 1$, $(X, n) = (2^r - 3, 3)$, $(2^r - 1, r + 2)$, $(2^r + 1, r + 3)$, and $(3 \cdot 2^r - 1, 2r + 3)$, where r is a positive integer with $r \geq 3$;*

(ii) *$D = ((1/3)(2^{2r+1} - 17))^2 - 32$, $(X, n) = ((1/3)(2^{2r+1} - 17), 5)$, $(1/3)(2^{2r+1} + 1, 2r + 3)$, and $((1/3)(17 \cdot 2^{2r+1} - 1), 4r + 7)$, where r is a positive integer with $r \geq 3$;*

(iii) *$D = 2^{2r_1} + 2^{2r_2} - 2^{r_1+r_2+1} - 2^{r_1+1} - 2^{r_2+1} + 1$, $(X, n) = (2^{r_2} - 2^{r_1} - 1, r_1 + 2)$, $(2^{r_2} - 2^{r_1} + 1, r_2 + 2)$, and $(2^{r_2} + 2^{r_1} - 1, r_1 + r_2 + 2)$, where r_1, r_2 are positive integers with $r_2 > r_1 + 1 > 2$.*

Proof. See [7]. □

Lemma 7. *If D is an odd prime and D belongs to the exceptional case (i) of Lemma 6, then $D = 17$.*

Proof. We now assume that D is an odd prime with $D = 2^{2r} - 3 \cdot 2^{r+1} + 1$. Then we have

$$(2^r - 1)^2 - 2^{r+2} = D, \tag{7}$$

$$(2^r + 1)^2 - 2^{r+3} = D. \tag{8}$$

If $2 \mid r$, since $r \geq 3$, then $r \geq 4$, and by (7), we have

$$(2^r - 1) + 2^{r/2+1} = D, \qquad (2^r - 1) - 2^{r/2+1} = 1. \tag{9}$$

But, by the second equality of (9), we get $1 \equiv (2^r - 1) - 2^{r/2+1} \equiv -1 (\text{mod } 8)$, a contradiction.

If $2 \nmid r$, then from (8) we get

$$(2^r + 1) + 2^{(r+3)/2} = D, \qquad (2^r + 1) - 2^{(r+3)/2} = 1. \tag{10}$$

Further, by the second equality of (10), we have $2^r = 2^{(r+3)/2}$, $r = 3$, and $D = 17$. Thus, the lemma is proved. □

Lemma 8. *If D is an odd prime and D belongs to the exceptional case (iii) of Lemma 6, then $D = 17$.*

Proof. Using the same method as in the proof of Lemma 7, we can obtain this lemma without any difficulty. □

Lemma 9. *If D belongs to the exceptional case (ii), then (6) has at most one solution (X, n) with $3 \mid n$.*

Proof. Notice that, for any positive integer r, there exists at most one number of 5, $2r + 3$, and $4r + 7$ which is a multiple of 3. Thus, by Lemma 6, the lemma is proved. □

Lemma 10. *The equation*

$$X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{N}, \ \min\{X, Y, m, n\} > 1 \tag{11}$$

has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

Proof. See [8]. □

3. Proof of Theorem

We now assume that (x, y, z) is a solution of (1). Then we have $\gcd(2p, z) = 1$.

If $2 \mid y$, since $\gcd(z + p^{y/2}, z - p^{y/2}) = 2$, then from (1) we get

$$z + p^{y/2} = 2^{3x-1}, \qquad z - p^{y/2} = 2, \tag{12}$$

where we obtain

$$z = 2^{3x-2} + 1, \tag{13}$$

$$p^{y/2} = 2^{3x-2} - 1. \tag{14}$$

Since $p > 1$, applying Lemma 10 to (14), we get

$$y = 2, \qquad p = 2^{3x-2} - 1. \tag{15}$$

Further, by Lemma 3, we see from the second equality of (15) that

$$p = 2^q - 1, \qquad q = 3x - 2 \tag{16}$$

is an odd prime with $q \equiv 1 (\text{mod} 3)$.

Therefore, by (13), (15), and (16), we obtain the solutions given in (2).

Obviously, if p satisfies (2), then $p \equiv 7 (\text{mod } 8)$. Otherwise, since $2 \nmid y$, we see from (1) that $p \equiv p^y \equiv z^2 - 8^x \equiv 1 (\text{mod } 8)$. It implies that if $p \equiv \pm 3 (\text{mod } 8)$, then (1) has no solutions (x, y, z). If $p \equiv 7 (\text{mod } 8)$, then (1) has only the solutions (2).

Here and below, we consider the remaining cases that $p \equiv 1 (\text{mod } 8)$. By the above analysis, we have $2 \nmid y$. If $y > 1$, then $y \geq 3$ and (4) has the solution $(X, Y, m, n) = (z, p, 3x, y)$ with $3 \mid m$. But, by Lemma 5, it is impossible. Therefore, we have

$$y = 1. \tag{17}$$

Substituting (17) into (1), the equation

$$X^2 - p = 2^n, \quad X, n \in \mathbb{N}, \ n > 2 \tag{18}$$

has the solution $(X, n) = (z, 3x)$ with $3 \mid n$. Since $p \equiv 1 (\text{mod } 8)$, by Lemma 4, (3) has solutions (u, v). Therefore, by Lemmas 6–9, (1) has at most two solutions (x, y, z). Thus, the theorem is proved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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