

A Complete Kinematic Analysis of the SNU 3-UPU Parallel Robot

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ABSTRACT. Several articles have been published about the SNU 3-UPU parallel robot, since the prototype built at the Seoul National University (SNU) showed a rather unexpected behavior, being completely mobile although none of the prismatic joints was actuated.

The main goal of this work is to describe all possible poses of the robot by a system of algebraic equations using Study parameters, such that theoretical questions concerning assembly modes and mobility can be answered on the basis of the solutions of this system. We study the number of possible assembly modes for fixed limb lengths, also including the case where all lengths are equal. For the first time a complete analysis of the forward kinematics is given showing that the manipulator has theoretically up to 78 assembly modes, most of them being always complex. Investigating the Jacobian of the system of equations we show that for equal limb lengths the manipulator has some highly singular poses. Furthermore we discuss possible operation modes of the manipulator when the prismatic joints are actuated. To obtain these modes methods from algebraic geometry prove to be very useful. Moreover it is examined for which fixed design and joint parameters the mechanism allows self-motion. It is shown that there are only two such mobile robots. Their operation mode has no similarity with the pathologically mobile prototype.

1. Introduction

In 2001 during the Computational Kinematics workshop in Seoul F. Park showed a highly accurate machined model of a 3-UPU parallel manipulator which had an unexpected mobility although the prismatic joints were locked. From theoretical point of view this manipulator should have been rigid in this circumstance. After the workshop there were many attempts to elucidate this unexpected behavior. In all subsequent papers the authors tried to explain the mobility using different approaches, see e.g. [BZ01], [HKKP02], [WSP02] and [LLL03]. More general discussions of this and other related mechanisms regarding DOFs and possible translational motion were published by e.g. [Tsa96] and [GPC98]. Furthermore the publication by [ZBG02] should be mentioned where the DYMO 3-URU parallel robot is discussed which is very similar to the SNU 3-UPU robot.

2000 *Mathematics Subject Classification*. Primary 53A17, 70B15; Secondary 13P10.

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In the following we present a complete discussion of the SNU 3-UPU parallel robot regarding assembly modes and possible self-motion when the limb lengths are considered to be design parameters i.e. fixed. In addition to that we give an algebraic description of the manipulator's operation modes. Such modes are different types of motion which can occur when the prismatic joints are actuated. The main goal was to explore the robot's theoretical properties using Study parameters and a set of equations, where each solution of the system corresponds to an allowed pose of the platform. As we expected the set of solutions is finite for arbitrary design parameters, and even if the limbs have equal length there is no self-motion possible. The number of solutions is 78 resp. 72 in the special case. It is remarkable that the position where the prototype was extremely mobile corresponds to a solution of multiplicity four. The Jacobian of the system has a rank defect of two at this point. To solve the system of eight equations techniques from algebraic geometry were used, especially the method of decomposing ideals into primary components was extremely useful to split the system into ten smaller systems.

On the basis of this decomposition it could be shown that each of these smaller systems corresponds to a special operation mode of the manipulator, as described in [ZBG02], e.g. a translational or a rotational mode.

Another question we tried to answer was for which design parameters the mechanism allows self-motion. Special importance was given to non-degenerate mobile mechanisms where the solutions are real. By adding additional equations to the system, which were generated from the univariate polynomials of the corresponding ideal, we could deduce conditions for the design parameters of the manipulator that lead to real self-motion. Further inspection of these conditions showed that there are only two essentially different sets of conditions. One of these self-mobile mechanisms allows 1-DOF motion, with the other one even 2-DOF motion is possible.

The paper is organized as follows. In Section 2 the design of the SNU 3-UPU parallel robot is described. Section 3 shows how the constraint equations are deduced. When the system is solved in Section 4 to get all possible assembly modes two cases are discussed, in Subsection 4.1 arbitrary design parameters are used, in Subsection 4.2 however the limbs are considered to be of equal length. In Subsection 4.3 we describe the manipulator's operation modes and finally the two mechanisms with self-motion are presented in Section 5.

All computations were done using the computer algebra system *Maple 12*, except the primary decomposition in Section 4, which was computed with *Singular*.

2. Design of the robot

Referring to Figure 1 the design of the mechanism is described as follows: In the base we have three points \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 which form an equilateral triangle with circumradius h_1 . The frame Σ_0 is fixed in the base such that its origin lies in the circumcenter of the triangle, the yz -plane coincides with the plane formed by the triangle and the z -axis passes through \mathbf{A}_3 . The same situation is established in the platform. There we have an equilateral triangle with vertices \mathbf{B}_1 , \mathbf{B}_2 , \mathbf{B}_3 and circum-radius h_2 . The parameters h_1 and h_2 are the two first design parameters.

Now each pair of points \mathbf{A}_i , \mathbf{B}_i is connected by a limb of length d_i with U-joints at each end. The second and the third axis of this link-combination are parallel to each other and perpendicular to the axis of the limb. The first and the fourth axis are embedded in the base resp. platform such that each of them points to the

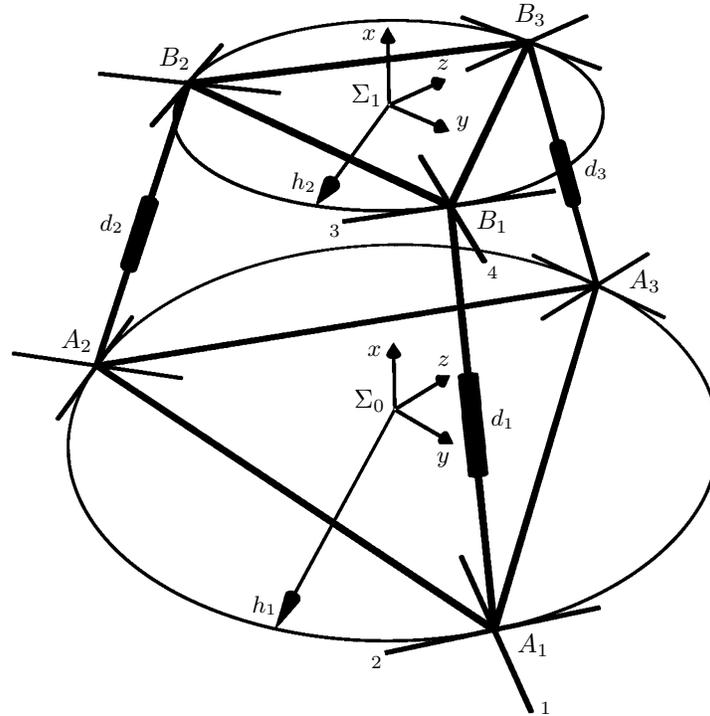


FIGURE 1. The numbers at the first limb describe the order of the rotational axes of the U-joints.

corresponding circumcenter (see Figure 1). This is the main difference to the so called translational 3-UPU parallel robot which was discussed by Tsai in [Tsa96]. That robot has almost the same design except that the roles of the first and the second axis resp. the third and fourth axis are swapped. Tsai showed that if the prismatic joints are actuated the platform performs a pure translational motion. This is a property the SNU 3-UPU robot also has as we will see in Section 4 when we solve the system of equations. A practical application of that translational motion is rather doubtful.

All in all we need five parameters to describe the 3-UPU mechanism: d_1 , d_2 , d_3 , h_1 and h_2 . Two of them (h_1 and h_2) determine the design of the robot. d_1 , d_2 , d_3 are the joint parameters, that determine the motion of the manipulator. When the motion parameters are fixed, they also can be considered as design parameters. We will take this point of view when we discuss the direct kinematics of the robot. We assume that all parameters are always strictly positive. This assumption is important in Section 5 where we want to exclude mobile mechanisms with e.g. a platform where \mathbf{B}_1 , \mathbf{B}_2 and \mathbf{B}_3 coincide or mechanisms with limbs of length zero.

3. Constraint equations

To derive equations which describe the possible poses of Σ_1 i.e. the platform, we use an ansatz with Study parameters. First of all we need the coordinates of all vertices with respect to the corresponding frame. In the following we use projective

coordinates to describe the vertices in which the homogenizing coordinate is the first one. Furthermore we write coordinates with respect to Σ_0 with capital letters and coordinates with respect to Σ_1 with lower case letters.

$$\mathbf{A}_1 = (1, 0, \sqrt{3}h_1/2, -h_1/2) \quad \mathbf{A}_2 = (1, 0, -\sqrt{3}h_1/2, -h_1/2) \quad \mathbf{A}_3 = (1, 0, 0, h_1)$$

$$\mathbf{b}_1 = (1, 0, \sqrt{3}h_2/2, -h_2/2) \quad \mathbf{b}_2 = (1, 0, -\sqrt{3}h_2/2, -h_2/2) \quad \mathbf{b}_3 = (1, 0, 0, h_2)$$

To obtain the coordinates of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ with respect to Σ_0 a coordinate transformation has to be applied. Here we use Study's well known transformation matrix $\mathbf{M} \in \mathbf{SE}(\mathbf{3})$ representing a Euclidean spatial transformation (see [Pfu06] for further information about this parametrization).

$$\mathbf{M} = \begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 \\ \mathbf{M}_T & \mathbf{M}_R \end{pmatrix}$$

The translational part \mathbf{M}_T and the rotational part \mathbf{M}_R of \mathbf{M} are as follows:

$$\mathbf{M}_T = \begin{pmatrix} 2(-x_0 y_1 + x_1 y_0 - x_2 y_3 + x_3 y_2) \\ 2(-x_0 y_2 + x_1 y_3 + x_2 y_0 - x_3 y_1) \\ 2(-x_0 y_3 - x_1 y_2 + x_2 y_1 + x_3 y_0) \end{pmatrix}$$

$$\mathbf{M}_R = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1 x_2 - x_0 x_3) & 2(x_1 x_3 + x_0 x_2) \\ 2(x_1 x_2 + x_0 x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2 x_3 - x_0 x_1) \\ 2(x_1 x_3 - x_0 x_2) & 2(x_2 x_3 + x_0 x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

The parameters $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ appearing in the matrix \mathbf{M} are the Study parameters and each projective point $[x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]$ on the 6-dimensional Study-quadric $S \in \mathbb{P}^7$ corresponds to exactly one spatial transformation and vice versa. The Study-quadric S is a semi-algebraic set described by

$$(3.1) \quad \begin{aligned} x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 &= 0 \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 &\neq 0. \end{aligned}$$

These two conditions will be used in the following computations to simplify expressions. Using \mathbf{M} we can compute the coordinates of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ with respect to Σ_0 by

$$\mathbf{B}_i = \mathbf{M} \cdot \mathbf{b}_i, \quad i = 1, \dots, 3.$$

Now we have the vertices of the platform given in terms of the transformation parameters. To deduce the constraint equations corresponding to constant parameters, we will firstly exploit that the distance between \mathbf{A}_i and \mathbf{B}_i is constant $d_i = \text{const}$. Note that with this assumption the manipulator is immobilized in general. The computation of the squared Euclidean distance can easily be done. After removal of the denominator $(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$ which comes from the normalization of \mathbf{B}_i we have an equation in the Study parameters of degree four. To get a lower degree we apply the method used by M. Husty in [Hus96] to simplify the equations for the Stewart-Gough-platform. After adding $4(x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3)^2$ the result can be factorized and the smaller factor $(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ can be removed. So we get for each limb a quadratic equation which guarantees that the distance of \mathbf{A}_i and \mathbf{B}_i remains constant. Applying this procedure to all the limbs we obtain

the first three equations which are denoted by g_1 , g_2 and g_3 .

$$(3.2) \quad g_1 : (h_1 - h_2)^2 x_0^2 + (h_1 + h_2)^2 x_1^2 + (h_1^2 + h_2^2 - h_1 h_2) x_2^2 + \\ + (h_1^2 + h_2^2 + h_1 h_2) x_3^2 - 2(h_1 - h_2) x_0 y_3 - 2(h_1 + h_2) x_1 y_2 + \\ + 2(h_1 + h_2) x_2 y_1 + 2(h_1 - h_2) x_3 y_0 + 2\sqrt{3}(h_1 - h_2) x_0 y_2 - \\ - 2\sqrt{3}(h_1 + h_2) x_1 y_3 - 2\sqrt{3}(h_1 - h_2) x_2 y_0 + 2\sqrt{3}(h_1 + h_2) x_3 y_1 + \\ + 2\sqrt{3} h_1 h_2 x_2 x_3 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2) d_1^2 = 0$$

$$(3.3) \quad g_2 : (h_1 - h_2)^2 x_0^2 + (h_1 + h_2)^2 x_1^2 + (h_1^2 + h_2^2 - h_1 h_2) x_2^2 + \\ + (h_1^2 + h_2^2 + h_1 h_2) x_3^2 - 2(h_1 - h_2) x_0 y_3 - 2(h_1 + h_2) x_1 y_2 + \\ + 2(h_1 + h_2) x_2 y_1 + 2(h_1 - h_2) x_3 y_0 - 2\sqrt{3}(h_1 - h_2) x_0 y_2 + \\ + 2\sqrt{3}(h_1 + h_2) x_1 y_3 + 2\sqrt{3}(h_1 - h_2) x_2 y_0 - 2\sqrt{3}(h_1 + h_2) x_3 y_1 - \\ - 2\sqrt{3} h_1 h_2 x_2 x_3 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2) d_2^2 = 0$$

$$(3.4) \quad g_3 : (h_1 - h_2)^2 x_0^2 + (h_1 + h_2)^2 x_1^2 + (h_1 + h_2)^2 x_2^2 + (h_1 - h_2)^2 x_3^2 + \\ + 4(h_1 - h_2) x_0 y_3 + 4(h_1 + h_2) x_1 y_2 - 4(h_1 + h_2) x_2 y_1 - \\ - 4(h_1 - h_2) x_3 y_0 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2) d_3^2 = 0$$

It can easily be seen that each of our link-combinations reduces the degrees of freedom of the platform by two. The first restriction was the condition of constant distance, we already have handled. The second restriction is induced by the two U-joints: the platform cannot be rotated about the axis of the limb. From this fact it follows that the vertices \mathbf{A}_i , \mathbf{B}_i and the circumcentres of base and platform have to form a planar quadrangle. But such a condition can easily be translated to an equation, because it is fulfilled if and only if the determinant of the 4x4-matrix built by these four points vanishes. This argumentation was also used in [BZ01].

So for each limb we take the coordinates of \mathbf{A}_i , \mathbf{B}_i , the origin of Σ_0 and the origin of Σ_1 with respect to Σ_0 , build the 4x4-matrix and compute its determinant. Here it is not necessary to normalize the coordinates. After removal of non-vanishing factors these determinants are reduced with the polynomial $x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$ from (3.1) with respect to the total degree order $x_0 \succ x_1 \succ x_2 \succ x_3 \succ y_0 \succ y_1 \succ y_2 \succ y_3$ which has the effect that the result can be again factorized and the smaller factor $-(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ can be removed. Readers who are not familiar with this type of simplification might consult e.g. [CLO05] for information about this very useful technique. We obtain the following equations which are again quadratic and completely independent from all design parameters.

$$(3.5) \quad g_4 : 4 x_1 y_1 + x_2 y_2 + \sqrt{3} x_2 y_3 + \sqrt{3} x_3 y_2 + 3 x_3 y_3 = 0$$

$$(3.6) \quad g_5 : 4 x_1 y_1 + x_2 y_2 - \sqrt{3} x_2 y_3 - \sqrt{3} x_3 y_2 + 3 x_3 y_3 = 0$$

$$(3.7) \quad g_6 : x_1 y_1 + x_2 y_2 = 0$$

The Study-quadratic equation

$$(3.8) \quad g_7 : x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0,$$

which was already used to simplify the first six equations completes the system. This system of algebraic equations describes the mechanism and we could ask now for all projective points in \mathbb{P}^7 which fulfill all these seven equations, under the condition that $x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0$, to get all possible poses of the platform. This would be the solution of the direct kinematics of this manipulator. But because it is more convenient to do all computations in affine space we add the following equation for normalization:

$$(3.9) \quad g_8 : x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

Now it is guaranteed that no solution of this final system lies in the forbidden variety described by $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. The downside of the normalization is that for each projective solution point we get two affine representatives as solutions for (3.2)-(3.9). This has to be taken in account when we count different solutions.

4. Solving the system

Now we have to study the system of equations (3.2)-(3.9). In the following this system of equations is always written as a polynomial ideal. Therefore, the ideal we have to deal with is

$$\mathcal{I} = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle$$

where each g_i here stands for the polynomial on the left hand side of the corresponding equation. First of all we will inspect the following ideal which is independent of the design parameters

$$\begin{aligned} \mathcal{J} &= \langle g_4, g_5, g_6, g_7 \rangle \\ &= \langle 4x_1y_1 + x_2y_2 + \sqrt{3}x_2y_3 + \sqrt{3}x_3y_2 + 3x_3y_3, \\ &\quad 4x_1y_1 + x_2y_2 - \sqrt{3}x_2y_3 - \sqrt{3}x_3y_2 + 3x_3y_3, x_1y_1 + x_2y_2, \\ &\quad x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 \rangle \end{aligned}$$

Due to the fact that we want to use methods from algebraic geometry like reductions, Groebner bases etc., we can be lucky to have equations independent from design parameters, because otherwise we would have to be careful concerning specialisations of the design parameters where e.g. a basis computed in general isn't a basis anymore. For treating such parametric systems the theory of comprehensive Groebner bases was developed, see e.g. [Wei92].

Although our ideal is not that complicated, we try to make it simpler and computation of the primary decomposition of \mathcal{J} shows that it can be written in a very simple way:

$$\mathcal{J} = \bigcap_{i=1}^{10} \mathcal{J}_i$$

with

$$\begin{aligned} \mathcal{J}_1 &= \langle y_0, y_1, y_2, y_3 \rangle, \quad \mathcal{J}_2 = \langle x_0, y_1, y_2, y_3 \rangle, \quad \mathcal{J}_3 = \langle y_0, x_1, y_2, y_3 \rangle, \quad \mathcal{J}_4 = \langle x_0, x_1, y_2, y_3 \rangle, \\ \mathcal{J}_5 &= \langle y_0, y_1, x_2, x_3 \rangle, \quad \mathcal{J}_6 = \langle x_0, y_1, x_2, x_3 \rangle, \quad \mathcal{J}_7 = \langle y_0, x_1, x_2, x_3 \rangle, \\ \mathcal{J}_8 &= \langle x_2 - ix_3, y_2 + iy_3, x_0y_0 + x_3y_3, x_1y_1 + x_3y_3 \rangle, \\ \mathcal{J}_9 &= \langle x_2 + ix_3, y_2 - iy_3, x_0y_0 + x_3y_3, x_1y_1 + x_3y_3 \rangle, \\ \mathcal{J}_{10} &= \langle x_0, x_1, x_2, x_3 \rangle. \end{aligned}$$

To compute this decomposition the software *Singular* was used, a very powerful tool to do computations with polynomials. Actually all these ideals are prime ideals and there are no embedded components. It has to be noted that an ideal has to be very special to allow such a decomposition in so many small components. For the zero set or vanishing set $\mathcal{V}(\mathcal{J})$ of \mathcal{J} it follows that

$$\mathcal{V}(\mathcal{J}) = \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i).$$

Now we add the remaining equations and by writing $\mathcal{K}_i := \mathcal{J}_i \cup \langle g_1, g_2, g_3, g_8 \rangle$ the vanishing set of the essential system \mathcal{I} can be written as

$$\begin{aligned} \mathcal{V}(\mathcal{I}) &= \mathcal{V}(\mathcal{J} \cup \langle g_1, g_2, g_3, g_8 \rangle) = \mathcal{V}(\mathcal{J}) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle) \\ &= \left(\bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i) \right) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle) = \bigcup_{i=1}^{10} (\mathcal{V}(\mathcal{J}_i) \cap \mathcal{V}(\langle g_1, g_2, g_3, g_8 \rangle)) \\ &= \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{J}_i \cup \langle g_1, g_2, g_3, g_8 \rangle) = \bigcup_{i=1}^{10} \mathcal{V}(\mathcal{K}_i). \end{aligned}$$

So, instead of studying the system as a whole, we can look for solutions of the smaller systems \mathcal{K}_i . Then the solution of system \mathcal{I} is the union of the solutions of the sub-systems.

It can easily be seen that the last set $\mathcal{V}(\mathcal{K}_{10})$ is empty because \mathcal{K}_{10} contains equations $\{x_0, x_1, x_2, x_3, x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1\}$ which cannot vanish simultaneously. Therefore it is only necessary to study systems $\mathcal{K}_1, \dots, \mathcal{K}_9$.

4.1. Solutions for arbitrary design parameters. Here all computations are made under the assumption that the five design parameters are arbitrary i.e. generic. To find out the Hilbert dimension of each ideal \mathcal{K}_i the necessary Groebner bases are not computed for general parameters. Instead of that randomly chosen parameters are substituted before the computation of the basis. This approach is quite reasonable because computations are much faster and the probability to choose a parameter set where the dimension is not the one from the generic case, is extremely small. So, for arbitrary design parameters we get the result that

$$\dim(\mathcal{K}_i) = 0, \quad i = 1, \dots, 9$$

which means that all sub-systems have finitely many solutions. Reusing the computed bases from above the number of solutions can be determined for each system \mathcal{K}_i . Due to the fact that always two solutions of a system describe the same position of the platform, each number has to be halved (see paragraph below (3.9)). In the following we will always only talk about these essentially different solutions. The number of solutions for each system \mathcal{K}_i in the generic case is

$$\begin{aligned} |\mathcal{V}(\mathcal{K}_1)| &= |\mathcal{V}(\mathcal{K}_2)| = |\mathcal{V}(\mathcal{K}_3)| = 8, \quad |\mathcal{V}(\mathcal{K}_4)| = 6, \quad |\mathcal{V}(\mathcal{K}_5)| = 4, \\ |\mathcal{V}(\mathcal{K}_6)| &= |\mathcal{V}(\mathcal{K}_7)| = 2, \quad |\mathcal{V}(\mathcal{K}_8)| = |\mathcal{V}(\mathcal{K}_9)| = 20. \end{aligned}$$

So all together we have 78 essentially different solutions, i.e. 78 possible poses of the platform, theoretically. It is clear that for most choices of the real parameters h_1, h_2, d_1, d_2, d_3 all these solutions are complex. Mechanically this means that the manipulator cannot be assembled because of e.g. too different limb lengths. But, on the other hand it can be shown that systems \mathcal{K}_8 and \mathcal{K}_9 always lead to complex

solutions, unless all limb lengths are equal. This can be seen easily because to get real solutions it is necessary that

$$x_3 = y_3 = 0.$$

Adding these two conditions to the corresponding ideal and eliminating all unknowns by intersecting it with $\mathbb{C}[h_1, h_2, d_1, d_2, d_3]$ leads to the ideal

$$\langle d_1^2 - d_3^2, d_2^2 - d_3^2 \rangle$$

from which we can conclude equality of the limb lengths as a necessary condition. For the remaining 38 solutions it is not clear how many of them can be real although at least some of the systems can be solved in closed form. The resulting expressions are simply too large. We tried some examples with reasonably chosen parameters and the number of real solutions never exceeded 16. A strict proof for this number to be an upper bound for real solutions is missing. Furthermore the system became numerically instable for nearly equal limb lengths. This was also the case for only slightly different circumradii. These two facts already indicate a special mechanical behavior.

Concerning singular solutions it can be shown that the Jacobian of system \mathcal{I} does not vanish at the solutions if the parameters are arbitrary, even when two limbs have equal length.

4.2. Solutions for equal limb lengths. Here we assume that all limbs are of equal length.

$$d_1 := d \quad d_2 := d \quad d_3 := d$$

Now we can perform the same computations we have done in the previous subsection to get the Hilbert dimension of each ideal. Due to the fact that we have less parameters all the Groebner bases can be computed without specifying parameters. We get the same dimensions as in the previous case.

$$\dim(\mathcal{K}_i) = 0, \quad i = 1, \dots, 9$$

When the number of solutions is computed for each system and halved afterwards the following results are obtained.

$$|\mathcal{V}(\mathcal{K}_1)| = |\mathcal{V}(\mathcal{K}_2)| = |\mathcal{V}(\mathcal{K}_3)| = 8, \quad |\mathcal{V}(\mathcal{K}_4)| = 6,$$

$$|\mathcal{V}(\mathcal{K}_5)| = |\mathcal{V}(\mathcal{K}_6)| = |\mathcal{V}(\mathcal{K}_7)| = 2, \quad |\mathcal{V}(\mathcal{K}_8)| = |\mathcal{V}(\mathcal{K}_9)| = 18.$$

Here we have theoretically 72 solutions for the platform's position, six less than before. Concerning the question where they could have gone, one should not forget that we have that forbidden subvariety on the Study-quadric. It is possible that for special design parameters a solution lies on this subvariety.

For reasonably chosen parameters the number of real solutions never exceeded 16 in our examples as in the previous subsection.

All together this means that the mechanism, at least theoretically, should be rigid. We have seen in machined models that this is not the case. An important difference to the previous case is that lots of singular solutions appear. When all these 72 solutions are pooled and then each of them is counted with multiplicity we get 30 solutions with multiplicity 1, exactly four solutions with multiplicity 4 and even two solutions with multiplicity 13. These two highly singular solutions correspond to planar mechanisms where both circumcentres coincide.

The solution which is most interesting regarding the unexpected mobility is the so called “home position” which can be seen in Figure 1, described by

$$\begin{aligned} x_0 &= 1, \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \\ y_0 &= 0, \quad y_1 = \sqrt{d^2 - (h_1 - h_2)^2}/2, \quad y_2 = 0, \quad y_3 = 0. \end{aligned}$$

It has multiplicity 4 and the Jacobian has rank six at this point, instead of eight. Therefore it is quite natural that the robot is at least shaky in this position. There is no higher-dimensional solution set at all which could be responsible for the unexpected motion of the physical model. To simulate that shakiness we made just a few experiments where we added in each limb a very small rotation around the limb’s axis so that equations (3.5), (3.6) and (3.7) were slightly perturbed i.e. the points \mathbf{A}_i , \mathbf{B}_i and the circumcentres were only almost coplanar. The angle of rotation ranged between -1 and 1 degrees. Then the solutions of this modified system \mathcal{T}' were computed numerically, which was not easy due to the numerical instability at this region, and we obtained positions of the platform which were far away from the home position. This result may corroborate the statement that bearing clearances at the U-joints have great influence on the position of the platform and it might be seen as a verification of the results discussed in [HKKP02].

4.3. The manipulator’s operation modes. Until now d_1, d_2 and d_3 were treated as fixed design parameters. In this subsection they will be treated as parameters which are allowed to change, i.e. we will study the behavior of this mechanism when the prismatic joints are actuated. Computation of the Hilbert dimension of each ideal \mathcal{K}_i with d_1, d_2, d_3 used as unknowns shows that

$$\overline{\dim}(\mathcal{K}_i) = 3, \quad i = 1, \dots, 9$$

where $\overline{\dim}$ denotes the dimension over $\mathbb{C}[h_1, h_2]$, in contrast to \dim which denotes the dimension over $\mathbb{C}[h_1, h_2, d_1, d_2, d_3]$ as in the previous subsections. It follows that in general the 3-UPU manipulator has 3 DOFs.

In [ZBG02] the DYMO 3-URU parallel robot is discussed which is very similar to the SNU 3-UPU robot. This mechanism can be obtained by replacing each P-joint by an R-joint, where its axis is parallel to the second and third axis of the U-joint-combination. Both mechanisms can reach the same positions of the platform, the only difference is that the distance between \mathbf{A}_i and \mathbf{B}_i is adjusted in different ways. As studied in the paper cited above the DYMO 3-URU robot has some essentially different operation modes which can be changed only at special positions of the platform, e.g. a purely translational mode and a purely rotational mode. In the following we will show how these operation modes can be deduced from the subsystems describing the SNU 3-UPU robot by analyzing each system \mathcal{K}_i regarding the special type of motion its solutions describe.

We solve each system \mathcal{J}_i , substitute the solution into the matrix \mathbf{M} and denote the result by \mathbf{M}_i from which we can deduce statements about the solutions of the sub-system \mathcal{K}_i and with it about the motion of the platform. It is absolutely not necessary to use equations (3.2)-(3.4) for this inspection, because they describe only the limb lengths which are now treated as free. Equation (3.9) is used to simplify \mathbf{M}_i , if possible, i.e. if three unknowns of (3.9) are 0 we set the remaining unknown to 1. Furthermore we describe, if necessary, the position of the platform by a series of simpler transformations, starting from the “planar home position”.

System \mathcal{K}_1 : $\{y_0 = 0, y_1 = 0, y_2 = 0, y_3 = 0\}$

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1 x_2 - x_0 x_3) & 2(x_1 x_3 + x_0 x_2) \\ 0 & 2(x_1 x_2 + x_0 x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2 x_3 - x_0 x_1) \\ 0 & 2(x_1 x_3 - x_0 x_2) & 2(x_2 x_3 + x_0 x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

As we can see each solution of \mathcal{K}_1 describes a pure spatial rotation of the platform, its centre lies in the centre of the platform and does not move. To parameterize this operation mode one could choose x_1, x_2, x_3 as parameters (x_0 can then be obtained using (3.9)).

System \mathcal{K}_2 : $\{x_0 = 0, y_1 = 0, y_2 = 0, y_3 = 0\}$

$$\mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2x_1 y_0 & x_1^2 - x_2^2 - x_3^2 & 2x_1 x_2 & 2x_1 x_3 \\ 2x_2 y_0 & 2x_1 x_2 & -x_1^2 + x_2^2 - x_3^2 & 2x_2 x_3 \\ 2x_3 y_0 & 2x_1 x_3 & 2x_2 x_3 & -x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

Each solution of the system \mathcal{K}_2 corresponds to a rotation of the platform about an axis (x_1, x_2, x_3) by 180 degrees and subsequent translation along this axis, given by $2y_0$. This can easily be seen by computing the eigenspace of the rotational part of \mathbf{M}_2 and the angle of rotation. For a parametrization x_2, x_3, y_0 could be chosen as parameters. This operation mode corresponds to the second part of the mixed mode discussed in [ZBG02].

System \mathcal{K}_3 : $\{x_1 = 0, y_0 = 0, y_2 = 0, y_3 = 0\}$

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2x_0 y_1 & x_0^2 - x_2^2 - x_3^2 & -2x_0 x_3 & 2x_0 x_2 \\ -2x_3 y_1 & 2x_0 x_3 & x_0^2 + x_2^2 - x_3^2 & 2x_2 x_3 \\ 2x_2 y_1 & -2x_0 x_2 & 2x_2 x_3 & x_0^2 - x_2^2 + x_3^2 \end{pmatrix}$$

Here every solution corresponds to a rotation of the platform about an axis N normal to the platform passing through the center of the platform by 180 degrees, then a rotation about the axis $(-x_0, -x_3, x_2)$ by 180 degrees and subsequent translation along that axis, given by $2y_1$. This can again be seen by computing the eigenspace of the rotational part of \mathbf{M}_3 and its rotation angle. For a parametrization x_2, x_3, y_1 could be chosen as parameters. This operation mode corresponds to the first part of the mixed mode mentioned in the article by Zlatanov et al.

All together systems \mathcal{K}_2 and \mathcal{K}_3 describe similar operation modes, basically the platform is rotated about an axis by a fixed angle and translated along the same axis. The only difference is that in the second case the platform is rotated about normal axis N by 180 degrees before the essential transformation.

System \mathcal{K}_4 : $\{x_0 = 0, x_1 = 0, y_2 = 0, y_3 = 0\}$

$$\mathbf{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2(x_2 y_0 - x_3 y_1) & 0 & x_2^2 - x_3^2 & 2x_2 x_3 \\ 2(x_2 y_1 + x_3 y_0) & 0 & 2x_2 x_3 & -x_2^2 + x_3^2 \end{pmatrix}$$

Solutions of \mathcal{K}_4 correspond to positions of the platform where it is turned upside down and coplanar to the base. To parameterize this planar operation mode one

could use x_3, y_0, y_1 , where x_3 is responsible for the rotation of the platform about its normal axis N and y_0, y_1 for the translation in the base-plane.

System \mathcal{K}_5 : $\{x_2 = 0, x_3 = 0, y_0 = 0, y_1 = 0\}$

$$\mathbf{M}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2(-x_0 y_2 + x_1 y_3) & 0 & x_0^2 - x_1^2 & -2x_0 x_1 \\ 2(-x_0 y_3 - x_1 y_2) & 0 & 2x_0 x_1 & x_0^2 - x_1^2 \end{pmatrix}$$

The operation mode which is described here is basically the same as the previous planar mode, except that the platform is not turned upside down. For a parametrization x_1, y_2, y_3 could be used as parameters.

System \mathcal{K}_6 : $\{x_0 = 0, x_2 = 0, x_3 = 0, y_1 = 0\}$

$$\mathbf{M}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2y_0 & 1 & 0 & 0 \\ 2y_3 & 0 & -1 & 0 \\ -2y_2 & 0 & 0 & -1 \end{pmatrix}$$

Here $x_1 = 1$ was used to simplify \mathbf{M}_6 . Each solution of system \mathcal{K}_6 corresponds to a rotation of the platform about its normal axis N by 180 degrees and a subsequent translation. It follows that the described operation mode is basically a pure translation. To parameterize it we have to use y_0, y_2, y_3 as parameters.

System \mathcal{K}_7 : $\{x_1 = 0, x_2 = 0, x_3 = 0, y_0 = 0\}$

$$\mathbf{M}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2y_1 & 1 & 0 & 0 \\ -2y_2 & 0 & 1 & 0 \\ -2y_3 & 0 & 0 & 1 \end{pmatrix}$$

Here we have the other purely translational operation mode discussed in [ZBG02] without any rotation of the platform. The transformation matrix was simplified by substituting $x_0 = 1$. It can be parameterized using y_1, y_2, y_3 .

Systems \mathcal{K}_8 and \mathcal{K}_9 :

As we have already seen these systems only lead to real solutions if $d_3 = d_1$ and $d_2 = d_1$. Inspection of \mathcal{J}_8 resp. \mathcal{J}_9 shows that for all solutions the four unknowns x_2, x_3, y_2, y_3 have to be 0. We get the following transformation matrix:

$$\mathbf{M}_8 = \mathbf{M}_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2(-x_0 y_1 + x_1 y_0) & 1 & 0 & 0 \\ 0 & 0 & x_0^2 - x_1^2 & -2x_0 x_1 \\ 0 & 0 & 2x_0 x_1 & x_0^2 - x_1^2 \end{pmatrix}$$

Using the remaining equations $\{x_0 y_0 = 0, x_1 y_1 = 0\}$ of \mathcal{J}_8 resp. \mathcal{J}_9 it can be shown that the only operation modes the systems \mathcal{K}_8 and \mathcal{K}_9 allow are three different 1-DOF motions which are: pure translation along the platform's normal axis N with the platform in normal position resp. rotated about N by 180 degrees and pure rotation about the normal axis N . These modes can be seen as special cases of the purely translational resp. rotational mode.

All together we get exactly all seven different operation modes which are studied in [ZBG02]. In addition to that we found two types of complex 3-DOF motions whose restriction to real positions lead to 1-DOF motions which are already covered by other operation modes.

We already mentioned that there exist positions where the mechanism can change from one mode to another mode. One of them is e.g. the “home position” given in the previous subsection. To reach this position it is necessary that all limbs have equal length. In the following we try to give an overview about the positions where such a mode-change can happen by inspecting each pair of ideals $\{\mathcal{K}_i, \mathcal{K}_j\}$ with respect to common real solutions. Because the real parts of \mathcal{K}_8 and \mathcal{K}_9 are already contained in other ideals we only look at the first seven ideals. For each pair we compute the dimension of the intersection of the corresponding varieties and we get the following results. As we can see there are six possible combinations

TABLE 1

	\mathcal{K}_1	\mathcal{K}_2	\mathcal{K}_3	\mathcal{K}_4	\mathcal{K}_5	\mathcal{K}_6	\mathcal{K}_7
\mathcal{K}_1	3	2	2	1	1	0	0
\mathcal{K}_2	2	3	1	2	0	1	-1
\mathcal{K}_3	2	1	3	2	0	-1	1
\mathcal{K}_4	1	2	2	3	-1	-1	-1
\mathcal{K}_5	1	0	0	-1	3	2	2
\mathcal{K}_6	0	1	-1	-1	2	3	-1
\mathcal{K}_7	0	-1	1	-1	2	-1	3

of operation modes which have no position in common. To change from one to the other a detour has to be made. Now we are able to describe the positions in which a mode-change can take place. In the following list a pair $\{\mathcal{K}_i, \mathcal{K}_j\}$ stands for the change from one corresponding operation mode to the other, and the necessary position is described as a series of simple transformations.

Change $\{\mathcal{K}_1, \mathcal{K}_7\}$ or $\{\mathcal{K}_3, \mathcal{K}_5\}$: Platform is in the “planar home position”, no further transformations.

Change $\{\mathcal{K}_1, \mathcal{K}_6\}$ or $\{\mathcal{K}_2, \mathcal{K}_5\}$: Platform is rotated about its normal axis N by 180 degrees.

Change $\{\mathcal{K}_3, \mathcal{K}_7\}$: Platform is translated along its normal axis N .

Change $\{\mathcal{K}_2, \mathcal{K}_6\}$: Platform is rotated about N by 180 degrees, followed by translation along N .

Change $\{\mathcal{K}_1, \mathcal{K}_5\}$: Platform is rotated arbitrarily about its normal axis N .

Change $\{\mathcal{K}_1, \mathcal{K}_4\}$ or $\{\mathcal{K}_2, \mathcal{K}_3\}$: Platform is turned upside down, follow by an arbitrary rotation about N .

Change $\{\mathcal{K}_5, \mathcal{K}_7\}$: Platform is translated arbitrarily in the base plane.

Change $\{\mathcal{K}_5, \mathcal{K}_6\}$: Platform is rotated about its normal axis N by 180 degrees, followed by an arbitrary translation in the base plane.

Change $\{\mathcal{K}_2, \mathcal{K}_4\}$: Platform is rotated about an arbitrary axis in the base plane by 180 degrees, followed by a translation along that axis.

Change $\{\mathcal{K}_3, \mathcal{K}_4\}$: Platform is rotated about N by 180 degrees, rotated about an arbitrary axis in the base plane by 180 degrees, followed by a translation along that axis.

Change $\{\mathcal{K}_1, \mathcal{K}_2\}$: Platform is rotated about an arbitrary axis by 180 degrees.

Change $\{\mathcal{K}_1, \mathcal{K}_3\}$: Platform is rotated about its normal axis N by 180 degrees, followed by a rotation about an arbitrary axis by 180 degrees.

All other combinations which are $\{\mathcal{K}_2, \mathcal{K}_7\}$, $\{\mathcal{K}_3, \mathcal{K}_6\}$, $\{\mathcal{K}_4, \mathcal{K}_5\}$, $\{\mathcal{K}_4, \mathcal{K}_6\}$, $\{\mathcal{K}_4, \mathcal{K}_7\}$ and $\{\mathcal{K}_6, \mathcal{K}_7\}$ do not allow any operation mode swap.

All in all we have seen that the decomposition of the original system into ten smaller sub-systems led us to a decomposition of the manipulator's workspace, which can be covered now with seven essentially different operation modes.

5. Mobile mechanisms

Finally we want to find all sets of design parameters h_1, h_2, d_1, d_2, d_3 where the SNU 3-UPU robot allows self-motion. Until now the dimension of each ideal \mathcal{K}_i was 0. The question is if there are design parameters where the dimension of at least one ideal is greater than 0. It is reasonable to claim that all design parameters are strictly positive and that the mobile mechanism allows real assembly.

To find such parameters each sub-system $\mathcal{K}_1, \dots, \mathcal{K}_7$ is examined separately where the modus operandi is always the same. Sub-systems \mathcal{K}_8 and \mathcal{K}_9 are not examined because for each of the corresponding vanishing sets $\mathcal{V}(\mathcal{J}_8)$ and $\mathcal{V}(\mathcal{J}_9)$ the restriction to real solutions leads to

$$\mathcal{V}(\langle x_2, x_3, y_2, y_3, x_0 y_0, x_1 y_1 \rangle)$$

which can be written as

$$\mathcal{V}(\mathcal{J}_1 \cup \langle x_2, x_3 \rangle) \cup \mathcal{V}(\mathcal{J}_2 \cup \langle x_2, x_3 \rangle) \cup \mathcal{V}(\mathcal{J}_3 \cup \langle x_2, x_3 \rangle) \cup \mathcal{V}(\mathcal{J}_4 \cup \langle x_2, x_3 \rangle)$$

and it follows that all real solutions of systems \mathcal{K}_8 and \mathcal{K}_9 are contained in one of the solution sets of $\mathcal{K}_1, \dots, \mathcal{K}_4$.

The main argument for the search for mobile mechanisms is that if the dimension of an ideal shall change from 0 to a higher dimension for some special design parameters, it is necessary that at least one univariate polynomial in the original ideal has to vanish identically by these parameters.

In the following we show exemplarily for the sub-system \mathcal{K}_7 the algorithm that is employed in this search:

- Elimination of unknowns appearing linearly in \mathcal{K}_7 , i.e. y_0, x_1, x_2, x_3
- Computation of all univariate polynomials over $\mathbb{C}[x_0, y_1, y_2, y_3]$, leading to

$$\begin{aligned} u_1(x_0) &= x_0^2 - 1 \\ u_2(y_1) &= 36(h_1 - h_2)^2 y_1 + d_1^4 + d_2^4 + d_3^4 + 9(h_1 - h_2)^4 - \\ &\quad - 3(h_1 - h_2)^2 (d_1^2 + d_2^2 + d_3^2) - d_1^2 d_2^2 - d_1^2 d_3^2 - d_2^2 d_3^2 \\ u_3(y_2) &= 48(h_1 - h_2)^2 y_2^2 - (d_1 - d_2)^2 (d_1 + d_2)^2 \\ u_4(y_3) &= 144(h_1 - h_2)^2 y_3^2 - (d_1^2 + d_2^2 - 2d_3^2)^2 \end{aligned}$$

- Checking if all these polynomials are also contained in $\mathbb{C}[h_1, h_2, d_1, d_2, d_3, x_0, y_1, y_2, y_3]$
- For each univariate polynomial u_i :
 - Computation of a basis of $\mathcal{K}_7 \cup C \subset \mathbb{C}[h_1, h_2, d_1, d_2, d_3, x_0, y_1, y_2, y_3]$ where C is the ideal generated by the coefficients of the univariate u_i
 - Elimination of all remaining unknowns, i.e. x_0, y_1, y_2, y_3
 - Solving the resulting system and removing solutions which contain non-vanishing expressions like e.g. $d_1 + d_2$

All these computations can be done with *Maple* using the procedures `Basis`, `Solve` and `UnivariatePolynomial` from the package `Groebner` and `EliminationIdeal` from the package `PolynomialIdeals`.

Is clear that univariates where at least one coefficient is constant need not to be considered. The only feasible solution remaining after this procedure is

$$(5.1) \quad \{h_1 = h_2, d_2 = d_1, d_3 = d_1\}.$$

Computation of the dimension of \mathcal{K}_7 using (5.1) shows that it has dimension 2 and the corresponding motion is a pure translation where the distance of the circumcentres remains constant and the platform is always parallel to the base. The motion of this mechanism looks rather trivial.

Sub-systems $\mathcal{K}_6, \mathcal{K}_4, \mathcal{K}_3, \mathcal{K}_1$ are treated the same way but in each of these cases all univariates contain coefficients like $h_1^3 h_2^3 (h_1 + 2 h_2)^3 (2 h_1 + h_2)^3$ and it follows that none of the univariates can vanish identically under the assumption of strictly positive design parameters.

Sub-system \mathcal{K}_5 leads to the same mobility-conditions as \mathcal{K}_7 and finally \mathcal{K}_2 leads to 1-dimensional motion which is not trivial.

$$(5.2) \quad \{h_1 = 2 h_2 \ d_1 = 3 h_2 \ d_2 = d_3\}$$

$$(5.3) \quad \{h_1 = 2 h_2 \ d_2 = 3 h_2 \ d_1 = d_3\}$$

$$(5.4) \quad \{h_1 = 2 h_2 \ d_3 = 3 h_2 \ d_1 = d_2\}$$

$$(5.5) \quad \{h_2 = 2 h_1 \ d_1 = 3 h_1 \ d_2 = d_3\}$$

$$(5.6) \quad \{h_2 = 2 h_1 \ d_2 = 3 h_1 \ d_1 = d_3\}$$

$$(5.7) \quad \{h_2 = 2 h_1 \ d_3 = 3 h_1 \ d_1 = d_2\}$$

Substantially these six sets of conditions describe the same type of motion for reasons of symmetry and they can be summarized by saying that one circumradius has to be twice the other one, one limb length has to be three times the shorter circumradius and the remaining two limb lengths have to be equal. It is remarkable that when all lengths are equal there exists a position where the motion processes corresponding to (5.2)-(5.4) resp. (5.5)-(5.7) can pass into each other, namely the “planar home position”.

All in all there are only two essentially different types of the SNU 3-UPU parallel robot which are “legally” mobile without actuation of a prismatic joint.

6. Conclusion

Using methods from algebraic geometry a complete analysis of the kinematic behavior of the SNU 3-UPU parallel manipulator was given. The main tool was primary decomposition of the ideal of the polynomials describing the manipulator. It turned out that from theoretical kinematic point of view the manipulator having

generic design parameters should be stiff. Especially the direct kinematics yields 78 solutions. Most of the solutions are complex. The biggest number of real solutions found was 16. On the other hand we have shown that two special sets of design parameters lead to two and one parametric self-motions. Furthermore we have shown that the primary decomposition of the set of algebraic equations has an interesting kinematic interpretation: the workspace of the manipulator decomposes into different parts, each of them corresponds to one of the primary-components of the ideal describing the manipulator.

The practical application of the SNU 3-UPU manipulator is very doubtful; all models show the same poor properties having a quite large mobility without changing the input. Also the workspace is decomposed into different parts having different kinematic properties. There are always poses where the manipulator can bifurcate into one part or the other. This means that even theoretically it is not controllable in these poses. Furthermore it has to be mentioned that our results show that this parallel manipulator contradicts some of the always highly acclaimed properties of parallel manipulators like e.g. high stiffness. Furthermore it seems to be quite likely that other parallel manipulators having the property of simplified (e.g. pure translational) operation modes could have similar behavior. The investigation of such manipulators with the developed methods is subject of future research.

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