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PAIRWISE STABILITY IN A TWO-SIDED MATCHING MARKET WITH INDIVISIBLE GOODS AND MONEY

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Abstract We consider a two-sided matching market in which the traders are partitioned into two sets; the set of sellers and the set of buyers. Each seller owns at most one indivisible good and each buyer owns a certain amount of money. Money is assumed to be an integer variable. Each trader can trade with at most one trader of the opposite side. The marriage model of Gale and Shapley is a special case of our model. We give a constructive proof to show the existence of a pairwise stable outcome.

Keywords: Game theory, stable matching, marriage model, indivisible goods

1. Introduction

Theoretical studies of two-sided matching markets started after the pioneering paper by Gale and Shapley [6]. In two-sided matching markets the set of participants-called players-is partitioned into two sets. The main purpose in such markets is to match the players of one side to the players of the opposite side. A matching consists of pairs such that both players of a pair belong to the opposite sides and each player appears at most once. A player is matched if he belongs to the matching, otherwise he is unmatched. A matching is called stable if each player is acceptable to his/her partner and there is no pair of players from opposite sides which are not matched to one another but prefer each other to their current partners.[1]

In the study of two-sided matching markets there are two standard models, the marriage model due to Gale and Shapley [6] and the assignment game by Shapley and Shubik [15]. In the marriage model a player of one side is matched with at most one player of the opposite side. Monetary transfer is not permitted in their model. Associated with each player is a strictly ordered finite list containing each player of the opposite side. Gale and Shapley described an algorithm which produces a stable matching in this model. In the one-to-one buyer-seller model by Shapley and Shubik [15], known as an assignment game, money plays an explicit role. In this model side payments are permitted. Shapley and Shubik showed that the core of the assignment game is a non-empty complete lattice. Gale and Shapley's marriage model and Shapley and Shubik's assignment game have been widely studied and several variations and extensions of these can be found in the literature.

Eriksson and Karlander [1] gave a model which is a common generalization of the marriage model and the assignment game. The existence of a stable outcome and the lattice property of the set of stable outcomes is shown in their paper. Sotomayor [16] further investigated the model of Eriksson and Karlander [1] and gave a non-constructive proof of the existence of pairwise stable outcome. The existence proof of Eriksson and Karlander holds

[1]If a player is unmatched, for convenience we say that it is self-matched.

when payoff matrices have integer entries. However, the proofs of Sotomayor are simple and cover both discrete and continuous markets.

Discrete convex analysis, proposed by Murota [11, 12] is a unified framework of discrete optimization. M-convex functions due to Murota [11, 12] and $M^\natural$-convex functions due to Murota and Shioura [13] are considered nice discrete convex functions from the point of view of mathematical economics. Fujishige and Tamura [3] proposed a common generalization of the marriage model and the assignment game by utilizing the framework of discrete convex analysis. They further extended their model in [4] by assuming possibly bounded side payments and proved the existence of a pairwise stable outcome.

In mathematical economics markets with indivisible goods have been widely studied. Kelso and Crawford [9] presented a two-sided matching model with money and introduced a Gross Substitute (GS) condition. Under GS they showed the non-emptiness of the core. Their model also includes the marriage model by Gale and Shapley [6] and the assignment game by Shapley and Shubik [15]. Gul and Stacchetti [8] further investigated the model of Kelso and Crawford and proposed two conditions: Single Improvement (SI) condition and No Complementarities (NC) condition. They showed that SI and NC are equivalent to GS for set functions. Fujishige and Yang [5] gave a relationship between GS and $M^\natural$-convexity for set functions. Quinzii [14] considered a model of an exchange economy with two kinds of goods. The first kind of good is perfectly divisible (money) and the other is indivisible. Each player has a certain amount of money and at most one indivisible good. She proved that the core of the economy is non-empty. Gale [7] considered the model of Quinzii [14] and gave a direct proof of the existence of equilibrium by using a generalization of the lemma of Knaster, Kuratowski and Mazurkewicz [10] in combinatorial topology. Recently, Farooq [2] gave a generalization of the models of Eriksson and Karlander [1] and Sotomayor [16] by identifying the preferences of the players by strictly increasing linear valuations. The money is modeled as a continuous variable in his work. He gave a constructive proof for the existence of a stable outcome.

The motivation of our current work is the model of Farooq [2]. Unlike the model of Farooq [2], we will use money as a discrete variable rather than as a continuous variable. Considering money as a discrete variable is useful in such auction markets where the market condition says that each bid should increase the price of the item by, say, 1 USD. Such conditions are empirical since very small increments in the price may not be worthwhile for the auctioneer. In our model the set of players is partitioned into two sets: the set of sellers and the set of buyers. Each seller owns at most one indivisible good and each buyer has a certain amount of money. Each player can trade with at most one player of the opposite side. The transfer of money from a buyer to a seller has an upper and a lower bound. When the upper and lower bounds on money are set to zero our model coincides with the marriage model of Gale and Shapley [6]. We give a constructive proof to show the existence of a stable outcome in our model.

This paper is organized as follows. In Section 2, we give a complete description of our model. Also, the pairwise stability is defined in this section. In Section 3, we devise an algorithm which finds a stable outcome in our model. In Section 4, we propose two problems and conclude the paper.

2. Model Description And Pairwise Stability

We consider a model with two kinds of players and two kinds of goods. The first kind of players are sellers and each seller owns at most one indivisible good. The other kind of players are buyers each of whom has a certain amount of money. We also assume that each player can trade with at most one player of the opposite side. It is also assumed that the transfer of money from buyer to seller has an upper bound and a lower bound and the money is given in integers.

Mathematically, we describe our model as follows. Denote the finite sets of the sellers and buyers by P and Q, respectively, and the set of all possible seller-buyer pairs by $E = P \times Q$. We express lower and upper bounds of prices by two vectors $\underline{\pi}, \overline{\pi} \in \mathbf{Z}^E$ where $\underline{\pi}_{ij} \leq \overline{\pi}_{ij}$ for each $(i, j) \in E$.[2] A vector $p = (p_{ij} \in \mathbf{Z} \mid (i, j) \in E)$ is called a *price vector* if $\underline{\pi}_{ij} \leq p_{ij} \leq \overline{\pi}_{ij}$ for each $(i, j) \in E$.

It is natural to think that each player has preferences over the players of opposite side. We give the preferences of the players by utility functions defined below. For each $(i, j) \in E$, $\nu_{ij}(x)$ denotes the utility[3] to the seller i if he/she trades with the buyer j (that is, matched with j) and receives an amount x of money. Similarly, $\nu_{ji}(-x)$ denotes the utility to the buyer j if he/she trades with the seller i and pays an amount x of money.[4] Furthermore, for each $(i, j) \in E$, we suppose that ν_{ij} and ν_{ji} are increasing and defined by

$$\nu_{ij}(x) = \alpha_{ij}x + \beta_{ij}, \quad \nu_{ji}(-x) = -\alpha_{ji}x + \beta_{ji}, \tag{2.1}$$

where α_{ij} and α_{ji} are given positive real numbers, β_{ij} and β_{ji} are any given real numbers and $x \in \mathbf{Z}$.

If $\nu_{ij}(x) \geq 0$ we shall say that the buyer j is *acceptable* to the seller i at x. This means that i is willing to trade with j at the amount x. Similarly, $\nu_{ji}(-x) \geq 0$ means that the seller i is *acceptable* to the buyer j at x. We remark that even if i and j are mutually acceptable they may not be matched with each other since both i and j have preference lists and a stable matching depends upon the preferences. We say that i *prefers* j to j' at x and x' if $\nu_{ij}(x) > \nu_{ij'}(x')$. Similarly, j *prefers* i to i' at x and x' if $\nu_{ji}(-x) > \nu_{ji'}(-x')$. A seller i is *indifferent* between j and j' at x and x' if $\nu_{ij}(x) = \nu_{ij'}(x')$. Moreover, a buyer j is said to be *indifferent* between i and i' at x and x' if $\nu_{ji}(-x) = \nu_{ji'}(-x')$. If $\nu_{ij}(x) \geq 0$ for some $x \in \mathbf{Z}$, then the buyer j is acceptable to the seller i at x by definition. However, if $\nu_{ij}(x) = 0$ then we say that the seller i is indifferent between the buyer j and himself at x.[5] Similarly, if $\nu_{ji}(-x) = 0$ for some $x \in \mathbf{Z}$, then we say that the buyer j is indifferent between the seller i and himself at x. Since the preferences of the players purely based on the monetary transfer, we can assume that preferences of the players are not strict.

A *matching*, denoted by X, is a subset of E such that each player appears at most once. Given a matching X, $k \in P \cup Q$ is said to be *unmatched* in X if it does not appear in X; otherwise *matched* in X. A matching X is *individually rational* if each player is acceptable to his/her partner in X. Let (i, j) be a seller-buyer pair who are not matched to one another in a matching X but prefer each other to their current partners in X. Then (i, j) will be said to *block* the matching X.[6]

A matching X is said to be *pairwise stable* if it is individually rational and is not blocked by any seller-buyer pair.

[2] Throughout the paper, $\mathbf{Z}$ and $\mathbf{R}$ stand for the set of integers and the set of real numbers, respectively. The notation $\mathbf{Z}^E$ stands for the integer lattice whose points are indexed by E.

[3] Due to the nature of our model, we depart from our former terminology of saying ν_{ij} and ν_{ji} as valuations.

[4] For convenience, we do not write a plus sign with x in $\nu_{ij}(x)$. However, it always means that i is a payee. The negative sign in $\nu_{ji}(-x)$ means j is a payer.

[5] Recall that if a player is unmatched, we say it self-matched.

[6] Assume that a seller-buyer pair (i, j) blocks a matching X and let us say that i is unmatched. This means that i prefers to be matched with j rather than staying single.

A 4-tuple $(X; p, q, r)$ is said to be an *outcome* if X is a matching, p is a price vector and $(q, r) \in \mathbf{R}^P \times \mathbf{R}^Q$ is defined by[7]

$$q_i = \begin{cases} \nu_{ij}(p_{ij}) & \text{if } (i,j) \in X \text{ for some } j \in Q \\ 0 & \text{otherwise} \end{cases} \quad (i \in P), \tag{2.2}$$

$$r_j = \begin{cases} \nu_{ji}(-p_{ij}) & \text{if } (i,j) \in X \text{ for some } i \in P \\ 0 & \text{otherwise} \end{cases} \quad (j \in Q). \tag{2.3}$$

Mathematically, we define the pairwise stability as follows. An outcome $(X; p, q, r)$ is *pairwise stable* if the following two conditions are satisfied:

(ps1) $q \geq \mathbf{0}$ and $r \geq \mathbf{0}$,

(ps2) $\nu_{ij}(c) \leq q_i$ or $\nu_{ji}(-c) \leq r_j$ for all $c \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ and for all $(i,j) \in E$.[8]

Condition (ps1) says that the matching X is individually rational. Condition (ps2) means $(X; p, q, r)$ is not blocked by any seller-buyer pair. A matching X is called *pairwise stable* if $(X; p, q, r)$ is pairwise stable.

3. Existence Of Pairwise Stable Outcome

The procedure adopted by Farooq [2] to show the existence of a pairwise stable outcome when money is a continuous variable does not work in our case. We will use different mathematical tools and give a constructive proof to show that there always exists a pairwise stable outcome in the model described in Section 2. Initially, we define $p_{ij} \in \mathbf{Z}$, for each $(i,j) \in E$, by[9]

$$p_{ij} = \begin{cases} \overline{\pi}_{ij} & \text{if } \nu_{ji}(-\overline{\pi}_{ij}) \geq 0 \\ \max\left\{\underline{\pi}_{ij}, \left\lfloor \frac{\beta_{ji}}{\alpha_{ji}} \right\rfloor\right\} & \text{otherwise.} \end{cases} \tag{3.1}$$

Then $\underline{\pi}_{ij} \leq p_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$. Note that if $\nu_{ji}(-p_{ij}) \geq 0$ for some $(i,j) \in E$, then p_{ij} is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which this inequality holds.

Before describing the algorithm mathematically, we first give an outline of the algorithm. Initially we will exclude all those seller-buyer pairs such that at least one of them is not acceptable to the other. Then, from the set of mutually acceptable seller-buyer pairs, we will find a matching X of all those seller-buyer pairs such that buyer is the most preferred for the seller and the sum of the utilities to the matched buyers is maximum. We define q and r by (2.2) and (2.3), respectively. Then the 4-tuple $(X; p, q, r)$ obviously satisfies (ps1). However, if there exists a mutually acceptable seller-buyer pair such that the seller is unmatched then it may form a blocking pair, that is, (ps2) may not hold. To eliminate all such pairs, we will modify the corresponding components of the price vector. It is worthwhile to note that the price vector will be non-increasing and the size of matching will be non-decreasing throughout the algorithm. If the price vector is decreased then the preference lists of the players may change. Therefore a matched player may change his partner according to the new preference list. We will make it certain that a matched buyer will remain matched. However, it is not required that a matched seller will remain matched. Throughout our procedure, we will exclude two kinds of unmatched pairs, if they appear; (i) those seller-buyer pairs such that the buyer is not acceptable to the seller and (ii) those seller-buyer pairs such that the corresponding component of the price vector is the lower bound and

[7] The notation $\mathbf{R}^P$(resp. $\mathbf{R}^Q$) stands for real vector space with coordinates indexed by P (resp. Q).

[8] For any $x, y \in \mathbf{Z}$, we define $[x, y]_{\mathbf{Z}} = \{a \in \mathbf{Z} \mid x \leq a \leq y\}$.

[9] $\lfloor x \rfloor = \sup\{n \in \mathbf{Z} \mid x \geq n\}$.

the payoff of the buyer is not greater than his/her utility to the seller. If the price vector remains unchanged in some iteration, then at least one kind of a seller-buyer pair mentioned above will be eliminated. Since the price vector is discrete and bounded and the number of players is finite, the algorithm will be terminated after a finite number of iterations.

Now we present our algorithm mathematically. First, we define some subsets of E which will help us to find a matching satisfying (ps1). Define two subsets L_0 and E_0 of E as follows:

$$L_0 = \{(i, j) \in E \mid \nu_{ji}(-p_{ij}) < 0\}, \tag{3.2}$$

$$E_0 = \{(i, j) \in E \mid \nu_{ij}(p_{ij}) < 0\}. \tag{3.3}$$

Then L_0 is the set of those seller-buyer pairs where the seller is not acceptable to the buyer, whereas, E_0 is the set of those seller-buyer pairs where buyer is not acceptable to the seller. The set of mutually acceptable seller-buyer pairs is defined by

$$\widetilde{E} = E \setminus \{L_0 \cup E_0\}. \tag{3.4}$$

We also define $\tilde{q}_i$, for each $i \in P$, by

$$\tilde{q}_i = \max\{\nu_{ij}(p_{ij}) \mid (i, j) \in \widetilde{E}\} \tag{3.5}$$

and

$$\widetilde{E}_P = \{(i, j) \in \widetilde{E} \mid \nu_{ij}(p_{ij}) = \tilde{q}_i\}. \tag{3.6}$$

The maximum over an empty set is taken to be zero by definition. The set $\widetilde{E}_P$ contains those seller-buyer pairs which are mutually acceptable and the buyer is most preferred for seller out of all acceptable buyers.

Initially we put $r = \mathbf{0}$ and define a subset $\widehat{E}_P$ of $\widetilde{E}_P$ by

$$\widehat{E}_P = \{(i, j) \in \widetilde{E}_P \mid \nu_{ji}(-p_{ij}) \geq r_j\}. \tag{3.7}$$

At this stage, obviously $\widetilde{E}_P = \widehat{E}_P$. However, in the subsequent iterations of the algorithm, $\widehat{E}_P$ may be a proper subset of $\widetilde{E}_P$. Also, define $\widetilde{Q} = \emptyset$. Let X be a matching in the bipartite graph $(P, Q; \widehat{E}_P)$ that satisfies the following conditions:

(a1) X matches all members of $\widetilde{Q}$,
(a2) X maximizes $\sum_{(i,j)\in X} \nu_{ji}(-p_{ij})$ among the matchings satisfying (a1).

Define the vectors q and r by (2.2) and (2.3), respectively. Then from (3.1) and (3.2), we obtain

$$p_{ij} = \underline{\pi}_{ij} \text{ and } \nu_{ji}(-p_{ij}) < r_j \quad ((i, j) \in L_0). \tag{3.8}$$

Let $\widetilde{Q}$ denote the set of matched buyers in X, that is,

$$\widetilde{Q} = \{j \in Q \mid j \text{ is matched in } X\}. \tag{3.9}$$

The main purpose of defining $\widehat{E}_P$ and putting condition (a1) on the matching is to keep the vector r non-decreasing throughout the algorithm. The condition (a2) will help us to prove (ps2) at termination of the algorithm. Next, define a set U by

$$U = \{(i, j) \in \widetilde{E}_P \mid i \text{ is unmatched in } X\}. \tag{3.10}$$

Then U is the set of all those seller-buyer pairs that are mutually acceptable and the buyer is most preferred for the seller but the seller is unmatched in X.

In each iteration of the algorithm, we will modify p in such a way that (ps1) and the feasibility of p (that is, $\underline{\pi}_{ij} \leq p_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$) are preserved. To modify p, we find an integer n_{ij}, for each $(i,j) \in U$, by[10]

$$n_{ij} = \max\left\{1, \left\lceil \frac{r_j - \nu_{ji}(-p_{ij})}{\alpha_{ji}} \right\rceil \right\}. \tag{3.11}$$

Define a subset L of U by

$$L = \{(i,j) \in U \mid p_{ij} - n_{ij} < \underline{\pi}_{ij}\}. \tag{3.12}$$

Now we modify the price vector p. The modified price vector is denoted by $\tilde{p}$ and is defined by

$$\tilde{p}_{ij} := \begin{cases} \max\{\underline{\pi}_{ij}, p_{ij} - n_{ij}\} & \text{if } (i,j) \in U \\ p_{ij} & \text{otherwise} \end{cases} \qquad ((i,j) \in E). \tag{3.13}$$

Then obviously $\underline{\pi}_{ij} \leq \tilde{p}_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$. Define a subset $\widetilde{E}_0$ of U as follows:

$$\widetilde{E}_0 := \{(i,j) \in U \mid \nu_{ij}(\tilde{p}_{ij}) < 0\}. \tag{3.14}$$

We finally propose our algorithm.

Algorithm:

Step 0: Put $r = \mathbf{0}$ and $\widetilde{Q} = \emptyset$. Initially define p, L_0, E_0, $\widetilde{E}$, $\tilde{q}$, $\widetilde{E}_P$ and $\widehat{E}_P$ by (3.1)−(3.7), respectively. Find a matching X in the bipartite graph $(P, Q; \widehat{E}_P)$ satisfying (a1) and (a2). Define r, $\widetilde{Q}$ and U by (2.3), (3.9) and (3.10), respectively.

Step 1: If $U = \emptyset$ then define q by (2.2) and stop.

Step 2: For each pair $(i,j) \in U$, calculate n_{ij} by (3.11) and find $\tilde{p}$ by (3.13). Define L and $\widetilde{E}_0$ by (3.12) and (3.14), respectively. Update E_0 by $E_0 := E_0 \cup \widetilde{E}_0$ and L_0 by $L_0 := L_0 \cup L$.

Step 3: Put $p := \tilde{p}$ and modify $\widetilde{E}$ by

$$\widetilde{E} := \widetilde{E} \setminus \{L_0 \cup E_0\}. \tag{3.15}$$

Define $\tilde{q}$ by (3.5). Modify $\widetilde{E}_P$ and $\widehat{E}_P$ by (3.6) and (3.7), respectively, for the updated $\widetilde{E}$ and p. Find a matching X in the bipartite graph $(P, Q; \widehat{E}_P)$ satisfying (a1) and (a2). Define r, $\widetilde{Q}$ and U by (2.3), (3.9) and (3.10), respectively. Go to Step 1.

In the rest of the work, we will show that our algorithm works correctly and terminates after a finite number of iterations. We will add prefixes $(old)*$ and $(new)*$ to sets/vectors/integers before and after update, respectively, in any iteration of the algorithm. When the context is clear we will not add these prefixes. We start with the following lemma.

Lemma 3.1. *In each iteration of the algorithm at Step 3, there exists a matching in the bipartite graph $(P, Q; \widehat{E}_P)$ satisfying (a1) and (a2).*

Proof. It is enough to prove that $(old)X \subseteq (new)\widehat{E}_P$ in each iteration at Step 3. Initially the set E_0 is defined by (3.3) at Step 0 before finding the matching. Then in each iteration at Step 2, E_0 is augmented if $\widetilde{E}_0$ is nonempty, otherwise, it remains the same. Also $L, \widetilde{E}_0 \subseteq U$ and $U \cap (old)X = \emptyset$ at Step 2. Therefore, (3.13) and (3.15) imply that $(old)X \subseteq (new)\widetilde{E}_P$ at Step 3. By (3.7), $(old)r_j$ is the lower bound of $\nu_{ji}(-(new)p_{ij})$ for each $(i,j) \in (new)\widehat{E}_P$. Therefore, $(old)X \subseteq (new)\widehat{E}_P$. □

[10] $\lceil x \rceil = \inf\{n \in \mathbf{Z} \mid x \leq n\}$.

The next lemma describes the important features of the algorithm. This lemma will be used to prove the subsequent lemmas. Specifically, the first two parts are crucial in proving the termination of the algorithm in a finite number of iterations.

Lemma 3.2. *In each iteration of the algorithm, the following hold:*

(i) *The price vector p decreases or remains the same. In particular, if $U \setminus \{L \cup \widetilde{E}_0\} \neq \emptyset$ at Step 2 then p_{ij} decreases at Step 3 for all $(i,j) \in U \setminus \{L \cup \widetilde{E}_0\}$.*

(ii) *$\widetilde{E}$ reduces or remains the same. In particular, if $L \neq \emptyset$ or $\widetilde{E}_0 \neq \emptyset$ at Step 2 then $\widetilde{E}$ reduces at Step 3.*

(iii) *The vector r increases or remains the same.*

Proof. (i) Initially the price vector p is defined by (3.1) and in each iteration it is modified by (3.13). From (3.13), one can easily see that p decreases or remains the same at Step 3. If $U \neq \emptyset$ then we find $\tilde{p}$ by (3.13) at Step 2. For each $(i,j) \in U$, n_{ij} is a positive integer. Now, if $U \setminus \{L \cup \widetilde{E}_0\} \neq \emptyset$ then one can easily see from (3.13) that $\tilde{p}_{ij} = p_{ij} - n_{ij}$ for all $(i,j) \in U \setminus \{L \cup \widetilde{E}_0\}$ at Step 2. This proves the assertion.

(ii) Initially $\widetilde{E}$ is defined by (3.4) at Step 0 and it is modified by (3.15) at Step 3 in each iteration. If $L = \widetilde{E}_0 = \emptyset$ at Step 2 then $\widetilde{E}$ remains the same at Step 3. If $L \neq \emptyset$ at Step 2 then $\widetilde{E}$ reduces at Step 3. If $\widetilde{E}_0 \neq \emptyset$ at Step 2 then E_0 enlarges at Step 2 and consequently, $\widetilde{E}$ reduces at Step 3.

(iii) By the proof of Lemma 3.1, we have $(old)X \subseteq (new)\widehat{E}_P$ at Step 3. Hence, for each $j \in (old)\widetilde{Q}$, there exists $(i,j) \in (new)\widehat{E}_P$ such that $\nu_{ji}(-(new)p_{ij}) \geq (old)r_j$. Also, $(old)r_j = 0$ for each $j \in Q \setminus (old)\widetilde{Q}$ by (2.3). Since $(new)X \subseteq (new)\widehat{E}_P$ and by (a1) we have $(old)\widetilde{Q} \subseteq (new)\widetilde{Q}$, it holds that $(new)r_j = \nu_{ji}(-(new)p_{ij}) \geq (old)r_j$ for each $(i,j) \in (new)X$. Moreover, $(new)r_j = (old)r_j = 0$ for each $j \in Q \setminus (new)\widetilde{Q}$. Hence, the vector r increases or remains the same. □

The next two lemmas do hold in each iteration of the algorithm at Step 3.

Lemma 3.3. *In each iteration of the algorithm at Step 3, we have $\nu_{ji}(-((old)p_{ij} - n_{ij})) \geq (old)r_j$ for each $(i,j) \in (old)U$, where n_{ij} is calculated at Step 2. Furthermore, if $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j$ for some $(i,j) \in (old)U$ then $(old)p_{ij} - n_{ij}$ is the maximum integer for which this inequality holds.*

Proof. Let $(i,j) \in (old)U$ at Step 3. Then

$$\begin{aligned} \nu_{ji}(-((old)p_{ij} - n_{ij})) &\geq \nu_{ji}(-(old)p_{ij}) + \alpha_{ji} \left\lceil \frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} \right\rceil \\ &\geq \nu_{ji}(-(old)p_{ij}) + (old)r_j - \nu_{ji}(-(old)p_{ij}) \\ &= (old)r_j. \end{aligned}$$

Next we prove that if $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j$ then $(old)p_{ij} - n_{ij}$ is the maximum integer for which this holds. Assume that $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j$. Since $n_{ij} \geq 1$ by definition (3.11), we first consider the case when $\frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} < 1$, that is, $n_{ij} = 1$. By (a2), we obtain $\nu_{ji}(-(old)p_{ij}) \leq (old)r_j$. But $(old)r_j < \nu_{ji}(-((old)p_{ij} - 1))$. This proves the assertion when $\frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} < 1$. Next we consider the case when $\frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} \geq 1$, that is, $n_{ij} = \left\lceil \frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} \right\rceil$. Therefore

$$\begin{aligned}\nu_{ji}(-((old)p_{ij} - n_{ij} + 1)) &= \nu_{ji}(-(old)p_{ij}) + \alpha_{ji}\left(\left\lceil \frac{(old)r_j - \nu_{ji}(-(old)p_{ij})}{\alpha_{ji}} \right\rceil - 1\right) \\ &< \nu_{ji}(-(old)p_{ij}) + (old)r_j - \nu_{ji}(-(old)p_{ij}) \\ &= (old)r_j.\end{aligned}$$

This means that $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j > \nu_{ji}(-((old)p_{ij} - n_{ij} + 1))$ which implies that $(old)p_{ij} - n_{ij}$ is the maximum integer for which $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j$. □

Lemma 3.4. *In each iteration of the algorithm at Step 3, we have $\nu_{ji}(-(new)p_{ij}) \geq (old)r_j$ for each $(i,j) \in (old)U \setminus L$, where L is defined at Step 2. Furthermore, if $\nu_{ji}(-(new)p_{ij}) > (old)r_j$ for some $(i,j) \in (old)U \setminus L$ then $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which this inequality holds.*

Proof. We prove the second part of the assertion. Assume that $\nu_{ji}\,(-(new)p_{ij}) > (old)r_j$ for some $(i,j) \in (old)U \setminus L$ at Step 3. Also by (a2), we obtain $(old)r_j \geq \nu_{ji}\,(-(old)p_{ij})$. Thus, $\nu_{ji}(-(new)p_{ij}) > \nu_{ji}(-(old)p_{ij})$. Therefore, (3.13) implies that $(new)p_{ij} = (old)p_{ij} - n_{ij}$, where n_{ij} is calculated at Step 2. The second part of the assertion follows from the second part of Lemma 3.3. □

Lemma 3.5. *In each iteration of the algorithm at Step 3, we have $(new)p_{ij} = \underline{\pi}_{ij}$ and $\nu_{ji}(-(new)p_{ij}) \leq (old)r_j$ for each $(i,j) \in L$, where L is defined at Step 2.*

Proof. Let $(i,j) \in L$ at Step 3. Then the first part of the assertion is true by (3.12) and (3.13). We prove the second part. Since $(new)p_{ij} > (old)p_{ij} - n_{ij}$, it holds that $\nu_{ji}(-(new)p_{ij}) < \nu_{ji}(-((old)p_{ij} - n_{ij}))$. Also $L \subseteq (old)U$. Thus, Lemma 3.3 implies $\nu_{ji}(-((old)p_{ij} - n_{ij})) \geq (old)r_j$. If $\nu_{ji}(-((old)p_{ij} - n_{ij})) = (old)r_j$ then the result trivially holds. If $\nu_{ji}(-((old)p_{ij} - n_{ij})) > (old)r_j$ then again by Lemma 3.3, $(old)p_{ij} - n_{ij}$ is the maximum integer for which this inequality holds. Since $(new)p_{ij} > (old)p_{ij} - n_{ij}$, it holds that $\nu_{ji}(-(new)p_{ij}) \leq (old)r_j$. □

The next lemma, in some sense, is more general than Lemma 3.4 and will play a critical role in proving (ps2).

Lemma 3.6. *In each iteration of the algorithm at Step 3, if $\nu_{ji}(-(new)p_{ij}) > (new)r_j$ for some $(i,j) \in E$ then $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which this inequality holds.*

Proof. Assume that $\nu_{ji}(-(new)p_{ij}) > (new)r_j$ for some $(i,j) \in E$ in the first iteration of the algorithm at Step 3. Then $\nu_{ji}(-(new)p_{ij}) > 0$ since $(new)r_j \geq 0$. If $(i,j) \notin (old)U$ then by (3.13), $(new)p_{ij}$ at Step 3 is the initial value defined by (3.1). The definition (3.1) yields that $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which $\nu_{ji}(-(new)p_{ij}) > 0$ at Step 3. Thus, the result holds in this case. Now, let $(i,j) \in (old)U$ at Step 3. Lemma 3.2 (iii) yields that $(new)r_j \geq (old)r_j$. Therefore, $\nu_{ji}(-(new)p_{ij}) > (old)r_j$. Hence, Lemma 3.5 implies that $(i,j) \in (old)U \setminus L$, where L is defined at Step 2. Thus, by Lemma 3.4, we conclude that $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which this inequality holds. This proves the result in the first iteration.

We suppose that the assertion holds in all iterations fewer than t, $t \geq 2$. We shall show that the assertion holds in t-th iteration. Assume that $\nu_{ji}(-(new)p_{ij}) > (new)r_j$ for some $(i,j) \in E$ at Step 3 in t-th iteration of the algorithm. Firstly, consider the case when

$(i,j) \notin (old)U$ at Step 3 in t-th iteration. Then $(old)p_{ij} = (new)p_{ij}$ by (3.13). Therefore, by assumption we get

$$\nu_{ji}(-(old)p_{ij}) = \nu_{ji}(-(new)p_{ij}) > (new)r_j. \tag{3.16}$$

Lemma 3.2 (iii) and (3.16) imply that $\nu_{ji}(-(old)p_{ij}) > (old)r_j$. By the induction hypothesis, $(old)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which this inequality holds. Thus, by (3.16), $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which $\nu_{ji}(-(new)p_{ij}) > (new)r_j$. Secondly, we consider the case when $(i,j) \in (old)U$ at Step 3 in t-th iteration. Lemma 3.2 (iii) gives $(new)r_j \geq (old)r_j$. Hence, $\nu_{ji}(-(new)p_{ij}) > (old)r_j$. Therefore, $(i,j) \notin L$ by Lemma 3.5. Thus, by Lemma 3.4, $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which $\nu_{ji}(-(new)p_{ij}) > (old)r_j$. Consequently, $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ for which $\nu_{ji}(-(new)p_{ij}) > (new)r_j$. □

Theorem 3.1. *If the algorithm terminates then $(X; p, q, r)$ satisfies (ps1) and (ps2).*

Proof. Suppose that the algorithm terminates at Step 1. Then $U = \emptyset$ and let $(X; p, q, r)$ be the 4-tuple obtained at termination. Initially we set $r = \mathbf{0}$ and find a matching in the bipartite graph $(P, Q; \widehat{E}_P)$ satisfying (a1) and (a2). Then we define the vector r by (2.3) at Step 0. Also, initially we define $\widetilde{E}$ by (3.4). Therefore

$$\nu_{ij}(p_{ij}) \geq 0 \text{ and } \nu_{ji}(-p_{ij}) \geq 0 \quad ((i,j) \in \widetilde{E}) \tag{3.17}$$

at Step 0. Since in each iteration we modify $\widetilde{E}$ by (3.15) at Step 3, therefore (3.17) holds in each iteration at Step 3. Also $\widehat{E}_P \subseteq \widetilde{E}$ in each iteration, the definitions (2.2) and (2.3) imply $q \geq \mathbf{0}$ and $r \geq \mathbf{0}$ at termination. Thus, (ps1) holds true.

Next we show that $(X; p, q, r)$ satisfies (ps2). We claim that for any $(i,j) \in E$, if there exists a $c \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbf{Z}}$ with $c > p_{ij}$ then the following inequality holds:

$$\nu_{ji}(-c) \leq r_j. \tag{3.18}$$

If the algorithm terminates in the first iteration then by (3.1) the inequality (3.18) obviously holds. Otherwise, we divide our argument in two cases: $\nu_{ji}(-p_{ji}) \leq r_j$ and $\nu_{ji}(-p_{ji}) > r_j$. If $\nu_{ji}(-p_{ji}) \leq r_j$ then the monotonicity of ν_{ji} implies (3.18). If $\nu_{ji}(-p_{ji}) > r_j$ then by Lemma 3.6, we obtain (3.18). Further, since $X \subseteq \widetilde{E}_P$, the definition (3.6) of $\widetilde{E}_P$ implies that for any matched seller i in X, the following inequality is satisfied:

$$\nu_{ij}(p_{ij}) \leq q_i \quad ((i,j) \in E). \tag{3.19}$$

Note that $U = \emptyset$. Also, $L_0 \cup E_0$ and $\widetilde{E}$ are disjoint and cover all of E. Therefore, if there exists some seller i unmatched in X then $(i,j) \in L_0 \cup E_0$ for each $j \in Q$. If $(i,j) \in E_0$ for some $j \in Q$ then

$$\nu_{ij}(p_{ij}) \leq 0 = q_i \tag{3.20}$$

by definition. From (3.19) and (3.20), we obtain

$$\nu_{ij}(p_{ij}) \leq q_i \quad ((i,j) \in E \setminus L_0). \tag{3.21}$$

Moreover, we assert that

$$p_{ij} = \underline{\pi}_{ij} \text{ and } \nu_{ji}(-p_{ij}) \leq r_j \quad ((i,j) \in L_0). \tag{3.22}$$

If the algorithm terminates in the first iteration then (3.22) is true at Step 1 by (3.8). Otherwise, Lemma 3.2 (iii) and Lemma 3.5 imply (3.22).

Thus for any $(i,j) \in E$ and $c \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ with $c \leq p_{ij}$, the inequalities (3.21) and (3.22) imply that

$$\nu_{ij}(c) \leq q_i \text{ or } [\nu_{ji}(-p_{ij}) \leq r_j \text{ and } p_{ij} = \underline{\pi}_{ij}]. \tag{3.23}$$

The inequalities (3.18) and (3.23) imply that (ps2) is satisfied. □

Theorem 3.2. *The algorithm terminates after a finite number of iterations.*

Proof. In each iteration of the algorithm at Step 2, either L and $\widetilde{E}_0$ are empty or at least one of them is non-empty. We first consider the case when $L = \widetilde{E}_0 = \emptyset$. Then by Lemma 3.2 (i), p_{ij} decreases for each $(i,j) \in U$. Note that p is discrete and bounded, and by Lemma 3.2 (i), it decreases or remains the same. Therefore, the vector p can be decreased a finite number of times.

Next we consider the case when $L \neq \emptyset$ or $\widetilde{E}_0 \neq \emptyset$. In either case, $\widetilde{E}$ reduces at Step 3 by Lemma 3.2 (ii). Furthermore, $\widetilde{E}$ reduces or remains the same in each iteration of the algorithm by Lemma 3.2 (ii). Therefore, this case is possible at most $|E|$ times. □

4. Concluding Remarks

We have presented a two-sided matching model where money is a discrete variable. More specifically, the money is given in integers. The preferences of the players are represented by increasing utility functions. The existence of a pairwise stable outcome is guaranteed in our model. It is not certain that one can get a pairwise stable outcome in a many-to-many version of our model by using the same mathematical apparatus. Further, one can observe from Theorem 3.2 that the complexity of our algorithm may depend on the size of those intervals where prices fall. It would be worthwhile for someone to devise an algorithm whose complexity is polynomial in the number of players.

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