An Adaptive Quasi-Notch Filter for a Biased Sinusoidal Signal Estimation

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Abstract—This note presents an alternative approach to classical adaptive notch filters designed to identify the frequency, magnitude, phase and offset of a biased sinusoidal signal. The algorithm takes advantages of an adaptive law for the resonant frequency of a third-order generalized integrator as a part of an orthogonal-signals generator system. The resulting estimator presents a dynamic order equal to 4. The strength of the discussed strategy results in a fast and accurate signal tracking capability and a good rejection to noise. The properties of the algorithm are verified in simulations in a range of signal conditions, such as step and sweep changes in frequency and voltage sag confirming the effectiveness of the strategy for estimation and tracking of time-varying parameters.

I. INTRODUCTION

The accurate tracking of a sinusoidal signal has attracted a great deal of attention and still is an active research area to date. Indeed, the problem of reliably estimating the parameters in a sinusoidal signal, i.e. amplitude, frequency, phase and bias, has drawn considerable attention in the last years in a wide range of applications such as control theory [1], [2], [3], [4], [5], signal processing [6], [7], [8], [9], [10], [11], biomedical engineering [12], instrumentation and measurements, power systems [13], [14], [15], [16], [17], [18], [19], [20], and so on. In particular, several problems arising in the power system area, such as grid-connected devices, require an accurate and fast detection of the phase angle, amplitude and frequency of the utility voltage to assure the correct generation of the reference signals. Moreover, as reported in [21], the presence of an offset may provide significant errors in parameters estimation process and therefore, the synthesis of an accurate method for the on-line estimation of the entire parameters set for a biased sinusoid signal represents a very attractive challenge. The foremost commonly used approaches can be divided into two main classes: phase-locked loop (PLL) topologies [22], [23], [24], which are based on the estimation of the difference between phase angle of the input signal and that of a generated output signal in order to regulate this value to zero by means of a control loop, and adaptive notch filters (ANFs) which are extensively employed due to low complexity. Generally an adaptive ANF is able to estimate only a subset of the unknown parameters (see [25], [26], [27], [28]). In [22] a PLL system has been proposed that is designed to estimate in-phase and quadrature-phase amplitudes of the desired signal. Advantages of the discussed PLL over the conventional ones are its capability of providing the fundamental component of the input signal which is not only locked in phase but also in amplitude to the actual signal while providing an estimate of its frequency. Numerical experiments have shown that such a method is also robust with respect to noisy input signals but it can not deal the case of biased sinusoid. In [27], the authors propose an adaptive notch filter where all signals are globally bounded, estimated frequency is asymptotically correct for all initial conditions and all frequency values and transient performance is considerably enhanced where a novel nonlinear time scaling is applied. However, this approach does not consider the case of biased sinusoid too and the tuned parameters depend on an upper bound of the sinusoid amplitude to guarantee the convergence of the method. In [29] a new approach to the problem of globally convergent frequency estimator is proposed where the estimator is represented by a fourth-order system for the case of biased sinusoidal signals and in [30] is presented an estimator for biased sinusoid with a dynamic order equal to 3 that is able to determine amplitude, frequency and bias for the signal of interest. Even if the method in [30] presents the smallest dynamical order, as far as our knowledge is concerned, numerical experiments show that the estimated results may not be very accurate due to the presence of noisy signals. Second-order generalized integrators, namely SOGI, have been considered in particular estimation topologies where it is necessary the generation of orthogonal reference signals and interesting methods have been discussed that are based on such integrators. In the previous paper [31], an estimation scheme is discussed based on a discrete-time least-square algorithm, used for the estimation of both offset and frequency. The remaining parameters are obtained via simple relationships between output generated signals. In the approach proposed in [31] no adaption of the SOGI resonant frequency has been considered while, in this paper, a new structure of orthogonal-signals generator (OSG) based on generalized integrators is proposed characterized by adjustable resonant frequency. A motivation which justifies the proposal of this new way to solve the problem is related to the difficulty, in some applications, to suitable tune the bandwidth of the filter. The aim of the paper is twofold. Firstly a new OSG is presented based on a third-order generalized integrator, (TOGI). TOGI is a new extension of the SOGI system which allows to cope with sinusoidal signals affected by a bias term, adaptively. Together with TOGI filter, an adaptive resonant frequency law is investigated to track the frequency of the unknown input signal. The orthogonal-signals generator based on TOGI (OSG-TOGI) system is then able to estimate the whole set of the four

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unknown parameters with a dynamic order equal to 4, which is less than the dynamic order in most results published in literature. Moreover the proposed algorithm requires only one parameter for the realization of the frequency estimator. The exponentially stability of the adaptive law is proved via averaging theory [32]. Simulations considering step and sweep changes in frequency and amplitude sag have been conducted to prove the reliability of the algorithm. The paper is organized as follows: Section II presents OSG-TOGI scheme; in Section III the estimation method is discussed; Section IV contains simulation results. Finally Section V is devoted to conclusions.

II. OSG-TOGI PROPERTIES

OSG-TOGI is depicted in Fig. 1. Let \( v(t) \) be a biased sinusoidal signal as

\[
v(t) = A_0 + A_c \sin(\omega_c t + \phi_c),
\]

where \( A_0, A_c, \omega_c \) and \( \phi_c \) are the unknown parameters to be estimated.

![Block diagram of the Orthogonal Signals Generator based on TOGI](image)

Output signals \( v_1(t), v_2(t) \) and \( v_3(t) \) are related to the input one \( v(t) \) by the following transfer functions

\[
F_1(s) = \frac{V_1(s)}{V(s)} = \frac{K_s \omega_s s}{s^2 + K_s \omega_s s + \omega_s^2},
\]

\[
F_2(s) = \frac{V_2(s)}{V(s)} = \frac{K_s \omega_s^2}{s^2 + K_s \omega_s s + \omega_s^2},
\]

and

\[
F_3(s) = \frac{V_3(s)}{V(s)} = \frac{K_s \omega_s \left(s^2 + \omega_s^2\right)}{(s + \omega_s)\left(s^2 + K_s \omega_s s + \omega_s^2\right)},
\]

where \( V_1(s), V_2(s) \) and \( V_3(s) \) represents the Laplace transform of the signals \( v_1(t), v_2(t) \) and \( v_3(t) \) respectively while \( V(s) \) is the Laplace transform of the input signal \( v(t) \). An immediate result is that, in steady-state conditions, the signal \( v_1(t) \) converges to an unbiased sinusoid, due to the derivative term in \( F_1(s) \), unlike signals \( v_2(t) \) and \( v_3(t) \) that are both affected by a bias term, as it can be proved by an investigation on the transfer functions in Eqs. (2)-(4).

The transfer functions \( F_1(s) \) and \( F_2(s) \) represent second order filters with a bandwidth depending on the gain \( K_s \) and a resonant frequency equal to \( \omega_s \). In particular, \( F_2(s) \) presents second order low-pass filtering characteristics with static gain \( K_s \) and \( F_1 \) behaves as a second order band-pass filter with no attenuation and no phase shift at the resonant frequency. If \( K_s \) decreases, the bandwidth of the filter \( F_1(s) \) becomes narrower resulting a heavy filtering (that justifies the term “quasi-notch filter”), nevertheless this entails a slowdown on the dynamic response of the system increasing oscillations and the stabilization time. The latter filter, represented by transfer function \( F_3(s) \), is a proper notch filter with a band stop centered at \( \omega_s \). Eq. (4) reveals that the filter gain at the frequency \( \omega_s \) is equal to zero, i.e. \( |F_3(j \omega_s)| = 0 \). This fact implies that every sinusoidal input term with frequency equal to the resonant one is completely removed. As a consequence, an adaptation of the resonant frequency to the unknown one \( \omega_c \) means a complete filtering of the signal \( v(t) \) that results in a constant \( v_3(t) \) signal depending on the bias term. For the proposed OSG-TOGI

![Bode diagrams of \( F_1(s) \) for different values of the gain \( K_s \).](image)

![Bode diagrams of \( F_2(s) \) for different values of the gain \( K_s \).](image)
With the controller parameters $K_s, \omega_s > 0$, the output signals converge exponentially fast to the following quantities

$$v_{1\infty}(t) = mA_c \sin(\omega_c t + \phi_c + \phi), \quad (5)$$

$$v_{2\infty}(t) = K_s A_0 - mA_c \frac{\omega_s}{\omega_c} \cos(\omega_c t + \phi_c + \phi) \quad (6)$$

and

$$v_{3\infty}(t) = K_s A_0 - K_s A_c \omega_s \sqrt{\frac{1 - m^2}{\omega_s^2 + \omega_c^2} \text{sign}[\omega_s - \omega_c] \times \cos\left(\omega_c t + \phi_c + \phi - \arctan\left(\frac{\omega_c}{\omega_s}\right)\right) \quad (7)$$

where

$$m = \frac{K_s \omega_c \omega_s}{\sqrt{(\omega_s^2 - \omega_c^2)^2 + K_s^2 \omega_s^4 \omega_c^2}}, \quad (8)$$

$$\phi = \text{sign}[\omega_s - \omega_c] \frac{\pi}{2} - \arctan\left(\frac{K_s \omega_c \omega_s}{\omega_s^2 - \omega_c^2}\right). \quad (9)$$

The \text{sign}[\cdot] function is defined as follows

$$\text{sign}[x] = \begin{cases} +1 & x \leq 0, \\ -1 & x < 0. \end{cases} \quad (10)$$

If the resonant frequency is tuned on the unknown input one, then the signal $v_{1\infty}(t)$ coincides with the unbiased sinusoid in input, $v_{2\infty}(t)$ is a biased sinusoidal signal with a phase shift of $\pi/2$ with respect to $v_{1\infty}(t)$ and finally $v_{3\infty}(t)$ represents the offset term of $v_{2\infty}(t)$. In such a case, OSG-TOGI output signals assume the following form

$$v_{1\infty}(t) = A_c \sin(\omega_c t + \phi_c), \quad (11)$$

$$v_{2\infty}(t) = K_s A_0 - A_c \cos(\omega_c t + \phi_c) \quad (12)$$

and

$$v_{3\infty}(t) = K_s A_0. \quad (13)$$

This idea suggests a simple method for biased sinusoidal signal estimation. An adaptive tuning of the OSG-TOGI resonant frequency, in fact, permits to obtain a set of three signals, as reported in Eqs. (11)-(13), that provide all the required informations to track the input $v(t)$. Moreover the aforementioned OSG-TOGI filtering properties permit to cope with noises that are eventually present in the input channel.

III. ADAPTIVE TUNING OF THE TOGI RESONANT FREQUENCY

In this section an adaptive resonant frequency law for the OSG-TOGI system, together with its exponential stability property, is investigated. In Fig. 5, the overall scheme based on OSG-TOGI is depicted where, in order to achieve an online estimation of all the unknown parameters, the following adaptive tuning law for the resonant frequency of the OSG-TOGI is proposed

$$\dot{\omega}_s = \gamma \left(v(t) - v_1(t) - \frac{v_3(t)}{K_s} \right) (v_3(t) - v_2(t)) \omega_s. \quad (14)$$

For small enough values of $\gamma$, with $\gamma > 0$, it is possible to proof the exponential convergence property of such a law by using the averaging theory [32]. Once $\omega_s$ reaches its equilibrium point, i.e. $\dot{\omega}_s$ is tuned on $\omega_c$, the bias term can be estimated as (see Eq. (13))

$$\dot{A}_0 = \frac{v_{3\infty}(t)}{K_s}. \quad (15)$$

Moreover, by considering the signal

$$\xi(t) = v_{3\infty}(t) - v_{2\infty}(t) + jv_{1\infty}(t), \quad (16)$$

the unknown amplitude $A_c$ and the phase angle $\Theta(t) = \omega_c t + \phi_c$ are estimated respectively as

$$\dot{A}_c = |\xi(t)|, \quad (17)$$

$$\dot{\Theta}(t) = \ar g\xi(t). \quad (18)$$

A. Averaging method and stability analysis

The averaging method applies to a system of the form

$$\dot{x} = \gamma f(t, x, \gamma) \quad (19)$$

where $\gamma$ is a small positive parameter and $f(t, x, \gamma)$ is a $T$-periodic function. The evolution of such a system is supposed to be determinate by two time-scaled dynamics: a
fast oscillatory dynamic and a slow one due to the presence of parameter \( \gamma \). The averaging method associates at the original system in Eq. (19) an autonomous average system

\[
\dot{x} = \gamma f_{av}(x)
\]  
(20)

where

\[
f_{av}(x) = \frac{1}{T} \int_{0}^{T} f(t, x, 0) dt.
\]  
(21)

Standard method may be employed to analyze the equilibrium points and their stability of such a system. Stability of the averaged and the original systems are related by the following theorem (see [32]).

Theorem 1: Let \( f(t, x, \gamma) \) and its partial derivatives with respect to \((x, \gamma)\) up to the second order be continuous and bounded for \((t, x, \gamma) \in [0, \infty) \times D_{0} \times [0, \gamma_{0}])\), for every compact set \(D_{0} \in D\), where \(D \in \mathbb{R}^{n}\) is a domain. Suppose \( f \) is \( T\)-periodic in \( t \) for some \( T > 0 \) and \( \gamma \) is a positive parameter. Let \( x(t, \gamma) \) and \( x_{av}(t, \gamma) \) denote the solutions for the original and averaged systems, respectively. Then

- If \( x_{av}(t, \gamma) \in D \forall t \in [0, b/\gamma] \) with finite \( b > 0 \) and \( x(0, \gamma) - x_{av}(0) = O(\gamma) \), then there exists \( \gamma^{*} > 0 \) such that for all \( 0 < \gamma < \gamma^{*} \), \( x(t, \gamma) \) is defined and \( x(t, \gamma) = x_{av}(t, \gamma) = O(\gamma) \) on \([0, b/\gamma]\).

- If the origin \( x = 0 \in D \) is an exponentially stable equilibrium point of the average system in Eq. (20), \( \Omega \in D \) is a compact subset of its region of attraction, \( x_{av}(0) \in \Omega \), and \( x(0, \gamma) - x_{av}(0) = O(\gamma) \), then there exists \( \gamma^{*} > 0 \) such that for all \( 0 < \gamma < \gamma^{*} \), \( x(t, \gamma) \) is defined and \( x(t, \gamma) = x_{av}(t, \gamma) = O(\gamma) \) for all \( t \in [0, \infty) \).

B. Stability analysis

At the limiting case \( \gamma \rightarrow 0 \), the resonant frequency \( \omega_{s} \) can be assumed frozen. Therefore it is possible to consider the steady state signals \( v_{1}(t) \), \( v_{2}(t) \) and \( v_{3}(t) \) of \( v_{1}(t) \), \( v_{2}(t) \) and \( v_{3}(t) \) respectively in Eq. (14). Therefore, the averaging method, applied to the adaptive law (14), provides the averaged system

\[
\dot{\omega}_{s, av} = -\frac{\gamma}{2} A_{c} m_{av} \sqrt{1 - m_{av}^{2}} \text{sign}[\omega_{s, av} - \omega_{c}] \frac{\omega_{c} \omega_{s, av}^{2}}{\omega_{c}^{2} + \omega_{s, av}^{2}}
\]  
(24)

where the notation \( m_{av} \) stands for the expression of \( m \) when \( \omega_{s} \) is replaced by \( \omega_{s, av} \).

To prove the stability of the averaged system, a candidate quadratic Lyapunov equation is

\[
V(\omega_{s, av}) = (\omega_{s, av} - \omega_{c})^{2}.
\]  
(25)

From Eq. (25), it follows that

\[
\dot{V}(\omega_{s, av}) = -\gamma A_{c}^{2} m_{av} \sqrt{1 - m_{av}^{2}} |\omega_{s, av} - \omega_{c}| \frac{\omega_{c} \omega_{s, av}^{2}}{\omega_{c}^{2} + \omega_{s, av}^{2}}.
\]  
(26)

\( \dot{V}(\omega_{s, av}) \) is negative definite for all possible values of \( \omega_{s, av} \). Therefore for all initial condition \( \omega_{s, av} > 0 \) the stability of the averaged system is proved and, by the statement of Theorem 1, the original system has the exponentially stable solution \( \{v_{1, \infty}, v_{2, \infty}, v_{3, \infty}, \omega_{c}\} \).

IV. NUMERICAL RESULTS

Some significant simulation experiments are conducted in order to highlight the properties of the proposed method.

The cases of frequency step, frequency sweep and voltage sag have been considered. Input signals, sampled with a period of \( T_{s} = 10^{-6} \) s, are affected by two sources of noise: the harmonic signal \( 0.01 \sin(2 \pi 150 t) \) and a Gaussian noise with zero mean and standard deviation equal to 0.01. To put in evidence the performances of the proposed method, a comparison with PLL-based estimation technique proposed in [21], namely CTA, and the linear scheme presented in [30], namely AKBNS, has been provided. In [21] the PLL schema depicted in Fig. 6 is used.

![General structure of a single-phase PLL](image)

Fig. 6. General structure of a single-phase PLL.

The \( PI \) controller depends on the gain parameters \( K_{i} \) and \( K_{p} \) while the orthogonal system generator is based on a modified version of standard SOGI structure that is able to deal with biased sinusoid and characterized by gain \( K_{PLL} \) and resonant frequency \( \omega_{r} \). In Fig. 6 the signal \( \omega_{ff} \) represents the initial condition for the frequency estimate.

The second method that will be considered in the comparison is based on the differential system [30]

\[
\dot{x}(t) = K_{p} \dot{x}(t)(-2 \alpha \dot{x}(t) - \alpha^{2} \ddot{x}(t)) - K_{i} \dot{x}(t) - K_{i} \dot{x}(t) v(t) - \bar{K}_{a} \ddot{x}(t)
\]  
(27)

and

\[
\hat{\theta}(t) = x(t) + K_{i} \dot{x}(t) v(t)
\]  
(28)

where

\[
\bar{K}_{a} = \bar{K}_{a}(\hat{\theta}) = \begin{cases} 0, & |\hat{\theta}(t)| < \theta_{0}, \\ \theta_{0}, & \theta_{0} \leq |\hat{\theta}(t)| \leq 2 \theta_{0}, \\ 1, & |\hat{\theta}(t)| > 2 \theta_{0}. \end{cases}
\]  
(29)

In the equations above, \( \hat{\theta} \equiv -\omega_{r}^{2} \) is the estimate of the unknown frequency, \( \alpha \) and \( K_{i} \) are free positive parameters.
and $\sigma(t)$ is the solution of the differential system
\[
\begin{align*}
\dot{\sigma}_1(t) &= \sigma_2(t), \\
\dot{\sigma}_2(t) &= -2\alpha \sigma_2(t) - \alpha^2 \sigma_1(t) + v(t), \\
\sigma(t) &= \sigma_1(t).
\end{align*}
\]

The parameter $\theta_0$ is a positive parameter that permits to avoid parameter estimates drift in case of noisy signals. For all the cases, the OSG-TOGI parameters are $K_s = 1$ and $\gamma = 100$. The PLL-based method is tuned with the following parameters $K_{PLL} = 1$, $K_i = 986.96$ and $K_p = 62.83$. The parameters $K_s = 10^8$, $\alpha = 6 \times 10^2$ and $\theta_0 = 10^5$ are chosen for the ABKNS technique.

**Frequency step.** The first experiment deals with the input signal
\[
v(t) = 1 + \sin \left(2\pi f_c(t) t + \frac{\pi}{4}\right),
\]
where the frequency $f_c(t)$ is defined as
\[
f_c(t) = \begin{cases}
47, & 0 \leq t < 0.4, \\
52, & 0.2 \leq t < 0.8.
\end{cases}
\]

The same initial condition for all the methods is $\omega_s(0) = 266 \text{ rad/s}$. As shown in Fig. 7 the frequency estimation of the OSG-TOGI method reaches the reference value with satisfactory precision. The noise heavily affects the estimates of CTA and ABKNS methods while FF is characterized by a better filtering action on the input signal.

![Fig. 7. Example 1. Frequency step case.](image)

**Frequency sweep.** In this simulation the case of frequency sweep has been considered. The input signal is the following
\[
v(t) = \sin \left(2\pi f_c(t) t + \frac{\pi}{4}\right)
\]
where the frequency is defined as
\[
f_c(t) = \begin{cases}
50, & 0 \leq t < 0.2667, \\
3.75t + 49, & 0.2667 \leq t < 0.5333, \\
51, & 0.5333 \leq t < 0.8.
\end{cases}
\]

The initial choice of $\omega_s(0) = 283 \text{ rad/s}$ has been done for all methods. Also in this case, as depicted in Fig. 8 the proposed method performs a better filtering action on the estimated frequency with respect to other methods. Note however that the filtering characteristics of CTA and ABKNS methods can be improved by increasing the gain parameters at the expense of longer transient periods. The tuning must be carefully done in order to not filter too much the input signal so a trade-off between filtering action and estimation response must be taken into account.

![Fig. 8. Example 2. Frequency sweep case.](image)

**Voltage sag.** The last simulation is devoted to the voltage sag case. The methods are initialized by choosing $\omega_s(0) = 565 \text{ rad/s}$. The input signal is the following
\[
v(t) = 1 + A_c(t) \sin \left(2\pi 100t + \frac{\pi}{4}\right)
\]
where
\[
A_c(t) = \begin{cases}
1, & 0 \leq t < 0.16, \\
-2.5t + 1.4, & 0.16 \leq t < 0.32, \\
0.6, & 0.32 \leq t < 0.48, \\
2.5t - 0.6, & 0.48 \leq t < 0.64, \\
1, & 0.64 \leq t < 0.8.
\end{cases}
\]

The results are depicted in Fig. 9.

![Fig. 9. Example 2. Voltage sag case.](image)
V. Conclusions

In this paper, an adaptive filter has been proposed for the biased sinusoid estimation problem. The discussed methodology makes use of an orthogonal-signals generator based on a third-order generalized integrator filter. Such a filter is an extension of the classical second-order generalized integrator, already known in literature. Unlike previous works based on generalized integrators which are not designed for biased sinusoidal signals, the proposed one is able to provide the estimation of the whole parameters set adapting the resonant frequency of the filter. The averaging theory has been used to prove the exponential stability of such a law and simulations have been conducted to highlight the reliability of the method: the cases of voltage sag, step and sweep changes in frequency of noisy signals have been considered.

References
