Factorization algebras in quantum field theory

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CHAPTER 1

Introduction and overview

1.1. Introduction

This book will provide the analog, in quantum field theory, of the deformation quantization approach to quantum mechanics. In this introduction, we will start by recalling how deformation quantization works in quantum mechanics.

The collection of observables in quantum mechanics form an associative algebra. The observables of a classical mechanical system form a Poisson algebra. In the deformation quantization approach to quantum mechanics, one starts with a Poisson algebra A^{cl}, and attempts to construct an associative algebra A^q, which is an algebra flat over the ring $\mathbb{C}[[\hbar]]$, together with an isomorphism of associative algebras $A^q/\hbar \cong A^{cl}$. In addition, if $a, b \in A^{cl}$, and $\widetilde{a}, \widetilde{b}$ are any lifts of a, b to A^q, then

$$\lim_{\hbar \to 0} \frac{1}{\hbar}[\widetilde{a}, \widetilde{b}] = \{a, b\} \in A^{cl}.$$

We will describe an analogous approach to studying perturbative quantum field theory. In order to do this, we need to explain the following.

- The structure present on the collection of observables of a *classical* field theory. This structure is the analog, in the world of field theory, of the commutative algebra which appears in classical mechanics. This structure we call a commutative factorization algebra (section 2.2).
- The structure present on the collection of observables of a *quantum* field theory. This structure is that of a factorization algebra (section 3.1). We view our definition of factorization algebra as a C^∞ analog of a definition introduced by Beilinson and Drinfeld. However, the definition we use is very closely related to other definitions in the literature, in particular to the Segal axioms.

- The extra structure on the commutative factorization algebra associated to a classical field theory which makes it "want" to quantize. This is the analog, in the world of field theory, of the Poisson bracket on the commutative algebra of observables.
- The quantization theorem we prove. This states that, provided certain obstruction groups vanish, the classical factorization algebra associated to a classical field theory admits a quantization. Further, the set of quantizations is parametrized (order by order in $\hbar$) by the space of deformations of the Lagrangian describing the classical theory.

This quantization theorem is proved using the physicists' techniques of perturbative renormalization, as developed mathematically in [**Cos11c**]. We claim that this theorem is a mathematical encoding of the perturbative methods developed by physicists.

This quantization theorem applies to many examples of physical interest, including pure Yang-Mills theory and σ-models. For pure Yang-Mills theory, it is shown in [**Cos11c**] that the relevant obstruction groups vanish, and that the deformation group is one-dimensional; so that there exists a one-parameter family of quantizations. A certain two-dimensional σ-model was constructed in this language in [**Cos10**, **Cos11a**]. Other examples are considered in [**GG11**] and [**CL11**].

Finally, we will explain how (under certain additional hypotheses) the factorization algebra associated to a perturbative quantum field theory encodes the correlation functions of the theory. This justifies the assertion that factorization algebras encode a large part of quantum field theory.

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1.2. The motivating example of quantum mechanics

The model problems of classical and quantum mechanics involve a particle moving in some Euclidean space $\mathbb{R}^n$ under the influence of some fixed field. Our main goal in this section is to describe these model problems in a way that makes the idea of a factorization algebra (section 3.1) emerge naturally, but we also hope to give mathematicians some feeling for the physical meaning of terms like "field" and "observable." We will not worry about making precise definitions, since that's what this book aims to do. As a narrative strategy, we describe a kind of cartoon of a physical experiment, and we ask that physicists accept this cartoon as a friendly caricature elucidating the features of physics we most want to emphasize.

1.2.1. A particle in a box. For the general framework we want to present, the details of the physical system under study are not so important. However, for concreteness, we will focus attention on a very simple system: that of a single particle confined to some region of space. We confine our particle inside some box and occasionally take measurements of this system. The set of possible trajectories of the particle around the box constitute all the imaginable behaviors of this particle; we might write this mathematically as $\text{Maps}(I, box)$, where $I \subset \mathbb{R}$ denotes the time interval over which we conduct the experiment. In other words, the set of possible behaviors forms a space of *fields* on the timeline of the particle.

The behavior of our theory is governed by the action functional. The simplest case is the action of the massless free field theory, whose value on a function $f : I \to box$ is

$$S(f) = \int_I \langle f, \Delta f \rangle .$$

The aim of this section is to outline the structure one would expect the observables – that is, the possible measurements one can make – should satisfy.

1.2.2. Classical mechanics. Let us start by considering the much simpler case, where our particle is treated as a classical system. In that case, the trajectory of the particle is constrained to be in a solution to the Euler-Lagrange equations of our theory. For example, if the action functional governing our theory is that of the massless free theory, then a map $f : I \to box$ satisfies the Euler-Lagrange equation if it is a straight line.

We are interested in the observables for this classical field theory. Since the trajectory of our particle is constrained to be a solution to the Euler-Lagrange equation, the only measurements one can make are functions on the space of solutions to the Euler-Lagrange equation.

If $U \subset \mathbb{R}$ is an open subset, we will let Fields(U) denote the space of fields on U, that is, the space of maps $f : U \to box$. We will let

$$\mathrm{EL}(U) \subset \mathrm{Fields}(U)$$

denote the subspace consisting of those maps $f : U \to box$ which are solutions to the Euler-Lagrange equation. As U varies, $\mathrm{EL}(U)$ forms a sheaf of spaces on $\mathbb{R}$.

We will let $\mathrm{Obs}^{cl}(U)$ denote the space of functions on $\mathrm{EL}(U)$ (the precise class of functions we will consider will be discussed later). As U varies, the spaces $\mathrm{Obs}^{cl}(U)$ form a cosheaf of commutative algebras on $\mathbb{R}$. We will think of $\mathrm{Obs}^{cl}(U)$ as the space observables for our classical system which only consider the behavior of the particle on times contained in U.

Note that $\mathrm{Obs}^{cl}(U)$ is a cosheaf of commutative algebras on $\mathbb{R}$.

1.2.3. Measurements in quantum mechanics. The notion of measurement is fraught in quantum theory, but we will take a very concrete view. Taking a measurement means that we have physical measurement device (e.g., a camera) that we allow to interact with our system for a period of time. The measurement is then how our measurement device has changed due to the interaction. In other words, we *couple* the two physical systems, then decouple them and record how the measurement device has modified from its initial condition. (Of course, there is a symmetry in this situation: both systems are affected by their interaction, so a measurement inherently disturbs the system under study.)

The *observables* for a physical system are all the imaginable measurements we could take of the system. Instead of considering all possible observables, we might also consider those observables which occur within a specified time period. This period can be specified by an open interval $U \subset \mathbb{R}$.

Thus, we arrive at the following principle.

Principle 1. For every open subset $U \subset \mathbb{R}$, we have a set $\mathrm{Obs}(U)$ of observables one can make on U.

The superposition principle tells us that quantum mechanics (and quantum field theory) is fundamentally linear. This leads to

Principle 2. The set $\mathrm{Obs}(U)$ is a complex vector space.

We think of $\mathrm{Obs}(U)$ as being the space of ways of coupling a measurement device to our system on the region U. Thus, there is a natural map $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ if $U \subset V$ is an open subset. This means that the space $\mathrm{Obs}(U)$ forms a pre-cosheaf.

1.2.4. Combining observables. Measurements (and so observables) differ qualitatively in the classical and quantum settings. If we study a classical particle, the system is not noticeably disturbed by measurements, and so we can do multiple measurements at the same time. Hence, on each interval J we have a commutative multiplication map $Obs(J) \otimes Obs(J) \to Obs(J)$, as well as the maps that let us combine observables on disjoint intervals.

For a quantum particle, however, a measurement disturbs the system significantly. Taking two measurements simultaneously is incoherent, as the measurement devices are coupled to each other and thus also affect each other, so that we are no longer measuring just the particle. Quantum observables thus do not form a cosheaf of commutative algebras on the interval. However, there are no such problems with combining measurements occurring at different times. Thus, we find the following.

Principle 3. If U, U' are disjoint open subsets of $\mathbb{R}$, and $U, U' \subset V$ where V is also open, then there is a map

$$\star : \mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V).$$

If $O \in \mathrm{Obs}(U)$ and $O' \in \mathrm{Obs}(U')$, then $O \star O'$ is defined by coupling our system to measuring device O for $t \in U$, and to device O' for $t \in U'$.

Further, these maps are commutative, associative, and compatible with the maps $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ associated to inclusions $U \subset V$ of open subsets. (The precise meaning of these terms is detailed in section 2.1.)

1.2.5. Perturbative theory and the correspondence principle. In the bulk of this paper, we will be considering perturbative quantum theory. For us, this means that we work over the base ring $\mathbb{C}[[\hbar]]$, where at $\hbar = 0$ we find the classical theory. In perturbative theory, therefore, the space $\mathrm{Obs}(U)$ of observables on an open subset U is a $\mathbb{C}[[\hbar]]$-module, and the product maps are $\mathbb{C}[[\hbar]]$-linear.

The correspondence principle states that the quantum theory, in the $\hbar \to 0$ limit, must reproduce the classical theory. Applied to observables, this leads to the following principle.

> **Principle 4**. The vector space $\mathrm{Obs}^q(U)$ of quantum observables is a flat $\mathbb{C}[[\hbar]]$-module that, modulo $\hbar$, is the space $\mathrm{Obs}^{cl}(U)$ of classical observables.

These simple principles are at the heart of our approach to quantum field theory. They say, roughly, that the observables of a quantum field theory form a factorization algebra, which is a quantization of the factorization algebra associated to a classical field theory. The main theorem presented in this paper is that one can use the techniques of perturbative renormalization to construct factorization algebras perturbatively quantizing a certain class of classical field theories (including many classical field theories of physical and mathematical interest).

1.2.6. Associative algebras in quantum mechanics. The principles we have described so far indicate that the observables of a quantum mechanical system should assign, to every open subset $U \subset \mathbb{R}$, a vector space $\mathrm{Obs}(U)$, together with a product map

$$\mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V)$$

if U, U' are disjoint open subsets of an open subset V. This is the basic data of a factorization algebra (section 2.1).

It turns out that the factorization algebra produced by our quantization procedure applied to quantum mechanics has a special property: it is *locally constant* (section 3.3). This means that the map $\mathrm{Obs}((a,b)) \to \mathrm{Obs}(\mathbb{R})$ is an isomorphism for any interval (a,b). Let A be denote the vector space $\mathrm{Obs}(\mathbb{R})$; note that A is canonically isomorphic to $\mathrm{Obs}((a,b))$ for any interval (a,b).

The product map

$$\mathrm{Obs}((a,b)) \otimes \mathrm{Obs}((c,d)) \to \mathrm{Obs}((a,d))$$

(defined when $a < b < c < d$) becomes, when we perform this identification, a product map

$$m : A \otimes A \to A.$$

The axioms of a factorization algebra imply that this multiplication turns A into an associative algebra.

This should be familiar to topologists: associative algebras are algebras over the operad of little intervals in $\mathbb{R}$, and this is precisely what we have described. (As we will see later (section 2.5), this associative algebra is the Weyl algebra one expects to find as the algebra of observables of quantum mechanics.)

One important point to take away from this discussion is that *associative algebras appear in quantum mechanics because associative algebras are connected with the geometry of* $\mathbb{R}$. There is no fundamental connection between associative algebras and any concept of "quantization": associative algebras only appear when one considers one-dimensional quantum field theories. As we will see later, when one considers quantum field theories on n-dimensional space times, one finds a structure reminiscent of an E_n-algebra instead of an E_1-algebra.

1.3. A preliminary definition of prefactorization algebras

Below (see section 2.1) we give a more formal definition, but here we provide the basic idea. Let M be a topological space (which, in practice, will be a smooth manifold).

1.3.0.1 Definition. *A prefactorization algebra $\mathcal{F}$ on M, taking values in cochain complexes, is a rule that assigns a cochain complex $\mathcal{F}(U)$ to each open set $U \subset M$ along with*

(1) a cochain map $\mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$;
(2) a cochain map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for every finite collection of open sets where each $U_i \subset V$ and the U_i are pairwise disjoint;
(3) the maps are compatible in the obvious way (e.g. if $U \subset V \subset W$ is a sequence of open sets, the map $\mathcal{F}(U) \to \mathcal{F}(W)$ agrees with the composition through $\mathcal{F}(V)$).

Remark: A prefactorization algebra resembles a precosheaf, except that we tensor the cochain complexes rather than taking their direct sum.

The observables of a field theory (whether classical or quantum) form a prefactorization algebra on the spacetime manifold M. In fact, they satisfy a kind of local-to-global principle in the sense that the observables on a large open set are determined by the observables on small open sets. The notion of a factorization algebra (section 3.1) makes this local-to-global condition precise.

1.4. Prefactorization algebras in quantum field theory

The (pre)factorization algebras of interest in this paper arise from perturbative quantum field theories. We have already discussed (section 1.2) how factorization algebras appear in quantum mechanics. In this section we will see that this picture extends in a very natural way to quantum field theory.

The manifold M on which the prefactorization algebra is defined is the space-time manifold of the quantum field theory. If $U \subset M$ is an open subset, we will interpret $\mathcal{F}(U)$ as the space of observables (or measurements) that we can make, which only depend on the behavior of the fields on U. Performing a measurement involves coupling a measuring device to the quantum system in the region U.

One can bear in mind the example of a particle accelerator. In that situation, one can imagine the space-time M as being of the form $M = A \times (0, t)$, where A is the interior of the accelerator and t is the duration of our experiment.

In this situation, performing a measurement on some open subset $U \subset M$ is something concrete. Let us take $U = V \times (\varepsilon, \delta)$, where $V \subset A$ is some small region in the accelerator, and (ε, δ) is a short time interval. Performing a measurement on U amounts to coupling a measuring device to our accelerator in the region V, starting at time ε and ending at time δ. For example, we could imagine that there is some piece of equipment in the region V of the accelerator, which is switched on at time ε and switched off at time δ.

1.4.1. Interpretation of the prefactorization algebra axioms. Suppose that we have two different measuring devices, O_1 and O_2. We would like to set up our accelerator so that we measure both O_1 and O_2.

There are two ways we can do this. Either we insert O_1 and O_2 into disjoint regions V_1, V_2 of our accelerator. Then we can turn O_1 and O_2 on at any times we like, including for overlapping time intervals.

If the regions V_1, V_2 overlap, then we can not do this. After all, it doesn't make sense to have two different measuring devices at the same point in space at the same time.

However, we could imagine inserting O_1 into region V_1 during the time interval (a, b); and then removing O_1, and inserting O_2 into the overlapping region V_2 for the disjoint time interval (c, d).

These simple considerations immediately suggest that the possible measurements we can make of our physical system form a prefactorization algebra. Let $\mathrm{Obs}(U)$ denote the space of measurements we can make on an open subset $U \subset M$. Then, by combining measurements in the way outlined above, we would expect to have maps

$$\mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V)$$

whenever U, U' are disjoint open subsets of an open subset V. The associativity and commutativity properties of a prefactorization algebra are evident.

1.4.2. The cochain complex of observables. In the approach to quantum field theory considered in this paper, the factorization algebra of observables will be a factorization algebra of cochain complexes. One can ask for the physical meaning of the cochain complex $\mathrm{Obs}(U)$.

It turns out that the "physical" observables will be $H^0(\mathrm{Obs}(U))$. If $O \in \mathrm{Obs}^0(U)$ is an observable of cohomological degree 0, then the equation $\mathrm{d}O = 0$ can often be interpreted as saying that O is compatible with the gauge symmetries of the theory. Thus, only those observables $O \in \mathrm{Obs}^0(U)$ which are closed are physically meaningful.

The equivalence relation identifying $O \in \mathrm{Obs}^0(U)$ with $O + \mathrm{d}O'$, where $O' \in \mathrm{Obs}^{-1}(U)$, also has a physical interpretation, which will take a little more work to describe. Often, two observables on U are physically indistinguishable (that is, they can not be distinguished by any measurement one can perform). In the example of an accelerator outlined above, two measuring devices are equivalent if they always produce the same expectation values, no matter how we prepare our system, or no matter what boundary conditions we impose.

As another example, in the quantum mechanics of a free particle, the observable measuring the momentum of a particle at time t is equivalent to that measuring the momentum of a particle at another time t'. This is because, even at the quantum level, momentum is preserved (as the momentum operator commutes with the Hamiltonian).

From the cohomological point of view, if $O, O' \in \text{Obs}^0(U)$ are observables which are in the kernel of d (and thus "physically meaningful"), then they are equivalent in the sense described above if they differ by an exact observable.

It is a little more difficult to provide a physical interpretation for the non-zero cohomology groups $H^n(\text{Obs}(U))$. The first cohomology group $H^1(\text{Obs}(U))$ is the recipient of any anomalies to defining observables at the quantum level. For example, in a gauge theory, one might have a classical observable which respects gauge symmetry. However, it may not lift to a quantum observable respecting gauge symmetry; this happens if there is a non-zero anomaly in $H^1(\text{Obs}(U))$.

The cohomology groups $H^n(\text{Obs}(U))$, when $n < 0$, are best interpreted as symmetries, and higher symmetries, of observables. Indeed, we have seen that the physically meaningful observables are the closed degree 0 elements of $\text{Obs}(U)$. One can construct a simplicial set, whose n-simplices are closed and degree 0 elements of $\text{Obs}(U) \otimes \Omega^*(\Delta^n)$. The vertices of this simplicial set are observables, the edges are equivalences between observables, the faces are equivalences between equivalences, and so on.

The Dold-Kan correspondence tells us that the nth homotopy group of this simplicial set is $H^{-n}(\text{Obs}(U))$. This allows us to interpret $H^{-1}(\text{Obs}(U))$ as being the group of symmetries of the trivial observable $0 \in H^0(\text{Obs}(U))$, and $H^{-2}(\text{Obs}(U))$ as the symmetries of the identity symmetry of $0 \in H^0(\text{Obs}(U))$, and so on.

Although the cohomology groups $H^n(\text{Obs}(U))$ where $n > 1$ do not have such a clear physical interpretation, they are mathematically very natural objects and it is important not to discount them.

1.5. Classical field theory and factorization algebras

The main aim of this book is to present a deformation-quantization approach to quantum field theory. In this section we will outline how a classical field theory gives rise to the classical algebraic structure we consider.

We use the Lagrangian formulation throughout. Thus, classical field theory means the study of the critical locus of an action functional. In fact, we use the language of derived geometry, in which it becomes clear that functions on a derived critical locus (section 4.8) should form a P_0 algebra (section 2.3), that is, a commutative algebra with a Poisson bracket of cohomological degree 1. (For an overview of these ideas, see the final section of this chapter.)

Applying these ideas to infinite-dimensional spaces, such as the space of smooth functions on a manifold, one runs into analytic problems. Although there is no difficulty in constructing a commutative algebra Obs^{cl} of classical observabes, we find that the Poisson bracket on Obs^{cl} is not always well-defined. However, we show the following.

1.5.0.1 Theorem. *For a classical field theory (section 4.10) on a manifold M, there is a sub-commutative factorization algebra $\widetilde{\mathrm{Obs}}^{c}l$ of the commutative factorization algebra Obs^{cl} on which the Poisson bracket is defined, so that $\widetilde{\mathrm{Obs}}^{cl}$ forms a P_0 factorization algebra. Further, the inclusion $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ is a quasi-isomorphism of factorization algebras.*

Remark: Our approach to field theory involves both cochain complexes of infinite-dimensional vector spaces and families over manifolds (and dg manifolds). Instead of working with topological vector spaces, we use *differentiable* vector spaces. For a careful discussion, see Appendix A. As a gloss, a differentiable vector space is a vector space V with a smooth structure, meaning that we have a well-defined set of smooth maps from any manifold X into V. In fact, going a bit further, we work with what we call *differentiable* vector spaces. This is a differentiable vector space with a flat connection, so that one knows how to take derivatives of smooth maps into the vector space. These notions make it possible to efficiently study cochain complexes of vector spaces in families over manifolds.

1.5.1. A gloss of the main ideas. In the rest of this section, we will outline why one would expect that classical observables should form a P_0 algebra. More details are available in section 4.1.

The idea of the construction is very simple: if $U \subset M$ is an open subset, we will let $\mathcal{EL}(U)$ be the derived space of solutions to the Euler-Lagrange equation on U. Since we are dealing with perturbative field theory, we are interested in those solutions to the equations of motion which are infinitely close to a given solution.

The differential graded algebra $\mathrm{Obs}^{cl}(U)$ is defined to be the space of functions on $\mathcal{EL}(U)$. (Since $\mathcal{EL}(U)$ is an infinite dimensional space, it takes some work to define $\mathrm{Obs}^{cl}(U)$. Details will be presented later (chapter 4).

On a compact manifold M, the solutions to the Euler-Lagrange equations are the critical point of the action functional. If we work on an open subset $U \subset M$, this is no longer strictly true, because the integral of the action functional over U is not defined. However, fields on U have a natural foliation, where tangent vectors lying in the leaves of the foliation correspond to variations $\phi \to \phi + \delta\phi$, where $\delta\phi$ has compact support. In this case, the Euler-Lagrange equations are the critical points of a closed one-form $\mathrm{d}S$ defined along the leaves of this foliation.

Any derived scheme which arises as the derived critical locus (section 4.8) of a function acquires an extra structure: it's ring of functions is equipped with the structure of a P_0 algebra. The same holds for a derived scheme arising as the derived critical locus of a closed one-form define along some foliation. Thus, we would expect that $\mathrm{Obs}^{cl}(U)$ is equipped with a natural structure of P_0 algebra; and that, more generally, the commutative factorization algebra Obs^{cl} should be equipped with the structure of P_0 factorization algebra.

1.6. Quantum field theory and factorization algebras

Another aim of the book is to relate quantum field theory, as developed in [**Cos11c**], to factorization algebras. We give a natural definition of an *observable* of a quantum field theory, which leads to the following theorem.

1.6.0.1 Theorem. *For a classical field theory (section 4.10) and a choice of BV quantization (section 5.4), the quantum observables* Obs^q *form a factorization algebra over the ring* $\mathbb{R}[[\hbar]]$.

Moreover, the factorization algebra of classical observables Obs^{cl} *is homotopy equivalent to the quotient* $\mathrm{Obs}^q/\hbar$ *as a factorization algebra.*

Thus, the quantum observables form a factorization algebra and, in a very weak sense, are related to the classical observables. The quantization theorems will sharpen the relationship between classical and quantum observables.

1.7. The weak quantization theorem

We have explained how a classical field theory gives rise to a lax P_0 factorization algebra Obs^{cl}, and how a quantum field theory (in the sense of [**Cos11c**]) gives rise to a factorization algebra Obs^q over $\mathbb{R}[[\hbar]]$, which specializes at $\hbar = 0$ to the factorization algebra Obs^{cl} of classical observables. In this section we will state our *weak quantization theorem*, which says that the Poisson bracket on Obs^{cl} is compatible, in a certain sense, with the quantization given by Obs^q.

This statement is the analog, in our setting, of a familiar statement in quantum-mechanical deformation quantization. Recall (section 1.1) that in that setting, we require that the associative product on the algebra A^q of quantum observables is related to the Poisson bracket on the Poisson algebra A^{cl} of classical observables by the formula

$$\{a,b\} = \lim_{\hbar\to 0} \hbar^{-1}[\widetilde{a},\widetilde{b}]$$

where $\widetilde{a},\widetilde{b}$ are any lifts of the elements $a,b \in A^{cl}$ to A^q.

One can make a similar definition in the world of P_0 algebras. If A^{cl} is any commutative differential graded algebra, and A^q is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$, then we can define a cochain map

$$\{-,-\}_{A^q} : A^{cl} \otimes A^{cl} \to A^{cl}$$

which measures the failure of the commutative product on A^{cl} to lift to a product on A^q, to first order in $\hbar$. (A precise definition is given in section 2.3).

Now, suppose that A^{cl} is a P_0 algebra (that is, a commutative dga equipped with a Poisson bracket of cohomological degree 1). Let A^q be a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$. We say that A^q is a *weak quantization* of A^{cl} if the bracket $\{-,-\}_{A^q}$ on A^{cl}, induced by A^q, is homotopic to the given Poisson bracket on A^{cl}.

This is a very weak notion, because the bracket $\{-,-\}_{A^q}$ on A^{cl} need not be a Poisson bracket; it is simply a bilinear map. When we discuss the notion of strong quantization (section 1.8), we will explain how to put a certain operadic structure on A^q which guarantees that this induced bracket is a Poisson bracket.

1.7.1. The weak quantization theorem. Now that we have the definition of weak quantization at hand, we can state our weak quantization theorem.

For every open subset $U \subset M$, $\mathrm{Obs}^{cl}(U)$ is a lax P_0 algebra. Given a BV quantization of our classical field theory, $\mathrm{Obs}^q(U)$ is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which coincides, modulo $\hbar$, with $\mathrm{Obs}^{cl}(U)$. Our definition of weak quantization makes sense with minor modifications for lax P_0 algebras as well as for ordinary P_0 algebras.

1.7.1.1 Theorem (The weak quantization theorem). *For every $U \subset M$, the cochain complex* $\mathrm{Obs}^q(U)$ *of classical observables on U is a weak quantization of the lax P_0 algebra* $\mathrm{Obs}^{cl}(U)$*.*

1.8. The strong quantization conjecture

We have seen (section 1.7) how the observables of a quantum field theory are a quantization, in a weak sense, of the lax P_0 algebra of observables of a quantum field theory. The definition of quantization appearing in this theorem is unsatisfactory, however, because the bracket on the classical observables arising from the quantum observables is not a Poisson observable.

In this section we will explain a stronger notion of quantization. We would like to show that the quantization of the classical observables of a field theory we construct lifts to a strong quantization. However, this is unfortunately is still a conjecture (except for the case of free fields).

1.8.0.2 Definition. *A* BD algebra *is a cochain complex A, flat over $\mathbb{R}[[\hbar]]$, equipped with a commutative product and a Poisson bracket of cohomological degree* 1*, satisfying the identity*

$$\mathrm{d}(a \cdot b) = a \cdot (\mathrm{d}b) \pm (\mathrm{d}a) \cdot b + \hbar\{a,b\}.$$

The BD operad is investigated in detail in section 2.4. Note that, modulo $\hbar$, a BD algebra is a P_0 algebra.

1.8.0.3 Definition. *A* quantization *of a P_0 algebra A^{cl} is a BD algebra A^q, flat over $\mathbb{R}[[\hbar]]$, together with an equivalence of P_0 algebras between $A^q/\hbar$ and A^{cl}.*

More generally, one can (using standard operadic techniques) define a concept of *homotopy BD algebra*. This leads to a definition of a homotopy quantization of a P_0 algebra.

Recall that the classical observables Obs^{cl} of a classical field theory have the structure of a P_0 factorization algebra on our space-time manifold M.

1.8.0.4 Definition. *Let $\mathcal{F}^{cl}$ be a P_0 factorization algebra on M. Then, a* strong quantization *of $\mathcal{F}^{cl}$ is a lift of $\mathcal{F}^{cl}$ to a homotopy BD factorization algebra $\mathcal{F}^q$, such that $\mathcal{F}^q(U)$ is a quantization (in the sense described above) of $\mathcal{F}^{cl}$.*

We conjecture that our construction of the factorization algebra of quantum observables associated to a quantum field theory has this structure. More precisely,

Conjecture. *Suppose we have a classical field theory on M, and a BV quantization of the theory. Then, Obs^q has the structure of a homotopy BD factorization algebra quantizing the P_0 factorization algebra Obs^{cl}.*

CHAPTER 2

Prefactorization algebras and basic examples

In this chapter we will give a formal definition of the notion of prefactorization algebra, and construct some relatively simple examples, related to free fields and to the Kac-Moody vertex algebra.

2.1. Prefactorization algebras

Let M be a topological space and let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We are particularly interested in the case where M is a smooth manifold and $\mathcal{C}$ is Vect or dgVect with the usual tensor product as the monoidal product. In this section we will give a formal definition of the notion of a prefactorization algebra.

2.1.1. The essential idea of a prefactorization algebra. A prefactorization algebra $\mathcal{F}$ on M, taking values in cochain complexes, is a rule that assigns a cochain complex $\mathcal{F}(U)$ to each open set $U \subset M$ along with

- a cochain map $\mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$;
- a cochain map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for every finite collection of open sets where each $U_i \subset V$ and the U_i are pairwise disjoint;
- the maps are compatible in the obvious way (e.g. if $U \subset V \subset W$ is a sequence of open sets, the map $\mathcal{F}(U) \to \mathcal{F}(W)$ agrees with the composition through $\mathcal{F}(V)$). It resembles a precosheaf, except that we tensor the cochain complexes rather than taking their direct sum.

2.1.2. Prefactorization algebras in the style of algebras over an operad.

2.1.2.1 Definition. *Let* Fact_M *denote the following multicategory associated to* M*.*

- *The objects consist of all connected open subsets of* M*.*

- *For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps* $\mathrm{Fact}_M(\{U_\alpha\}_{\alpha \in A}, V)$*. If the U_α are pairwise disjoint and all are contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty.*
- *The composition of maps is defined in the obvious way.*

Remark: By "multicategory" we mean what is also called a colored operad or a pseudotensor category. In [**Lei04**], there is an accessible discussion of multicategories; in Leinster's terminology, we work with "fat symmetric multicategories."

2.1.2.2 Definition. *A* prefactorization algebra *on M taking values in $\mathcal{C}$ is a functor (of multicategories) from* Fact_M *to $\hat{\mathcal{C}}$.*

Remark: In other words, a prefactorization algebra *is* an algebra over the colored operad Fact_M.

Remark: When the monoidal product on $\mathcal{C}$ is the coproduct, then a precosheaf on M defines a prefactorization algebra. Hence, our definition broadens the idea of "inclusion of open sets leads to inclusion of sections" by allowing more general monoidal structures to "combine" the sections on disjoint open sets. Something analogous happens when we equip the category of abelian groups with the tensor product as its monoidal structure.

Note that if $\mathcal{F}$ is any prefactorization algebra, then $\mathcal{F}(\emptyset)$ is a commutative algebra object of $\mathcal{C}$.

2.1.2.3 Definition. *We say a prefactorization algebra $\mathcal{F}$ is* unital *if the commutative algebra $\mathcal{F}(\emptyset)$ is unital.*

2.1.3. Prefactorization algebras in the style of precosheaves.

Any multicategory $\mathscr{C}$ has an associated symmetric monoidal category $S\mathscr{C}$, which is defined to be the universal symmetric monoidal category equipped with a functor of multicategories $\mathscr{C} \to S\mathscr{C}$. Concretely, an object of $S\mathscr{C}$ is a formal tensor product $a_1 \otimes \cdots \otimes a_n$ of objects of $\mathscr{C}$. Morphisms in $S\mathscr{C}$ are characterized by the property that for any object b in $\mathcal{C}$, the set of maps $S\mathscr{C}(a_1 \otimes \cdots \otimes a_n, b)$ in the symmetric monoidal category is exactly the set of maps $\mathscr{C}(\{a_1, \ldots, a_n\}, b)$ in the multicategory category $S\mathscr{C}$.

We can give an alternative definition of prefactorization algebra by working with the symmetric monoidal category $S\,\mathrm{Fact}_M$ rather than the multicategory Fact_M.

2.1.3.1 Definition. *Let $S\,\mathrm{Fact}_M$ denote the following symmetric monoidal category.*

- *The objects of $S\,\mathrm{Fact}_M$ consist of topological spaces U equipped with a map $U \to M$ which, on each connected component of U, is an open embedding embedding.*
- *A map from $U \to M$ to $V \to M$ is a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ \downarrow & \swarrow & \\ M & & \end{array}$$

  where the map i is an embedding.
- *The symmetric monoidal structure on $S\,\mathrm{Fact}_M$ is given by disjoint union.*

2.1.3.2 Lemma. *$S\,\mathrm{Fact}_M$ is the universal symmetric monoidal category containing the multicategory Fact_M.*

The alternative definition of prefactorization algebra is as follows.

2.1.3.3 Definition. *A prefactorization algebra with values in a symmetric monoidal category $\mathscr{C}$ is a symmetric monoidal functor*

$$S\,\mathrm{Fact}_M \to \mathscr{C}.$$

Remark: Although “algebra” appears in its name, a prefactorization algebra only allows one to ”multiply” elements that live on disjoint open sets. The category of prefactorization algebras (taking values in some fixed target category) has a symmetric monoidal product, so we can study commutative algebra objects in that category. As an example, we will consider the observables for a classical field theory (chapter 4).

2.2. Structured factorization algebras

In this section we will define what it means to have a factorization algebra endowed with the structure of an algebra over an operad. Since, in this paper, we are principally concerned with factorization algebras taking values in the category of differentiable cochain complexes we will restrict attention to this case in the present section.

Not all operads work for this construction: only operads endowed with an extra structure – that of a *Hopf operad* can be used.

2.2.0.4 Definition. *A Hopf operad is an operad in the category of differential graded cocommutative coalgebras.*

Any Hopf operad P is, in particular, a differential graded operad. In addition, the cochain complexes $P(n)$ are endowed with the structure of differential graded commutative coalgebra. The operadic composition maps

$$\circ_i : P(n) \otimes P(m) \to P(n+m-1)$$

are maps of coalgebras, as are the maps arising from the symmetric group action on $P(n)$.

If P is a Hopf operad, then the category of dg P-algebras becomes a symmetric monoidal category. If A, B are P-algebras, the tensor product $A \otimes_{\mathbb{C}} B$ is also a P-algebra. The structure map

$$P_{A \otimes B} : P(n) \otimes (A \otimes B)^{\otimes n} \to A \otimes B$$

is defined to be the composition

$$P(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{c(n)} P(n) \otimes P(n) \otimes A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{P_A \otimes P_B} A \otimes B.$$

In this diagram, $c(n) : P(n) \to P(n)^{\otimes 2}$ is the comultiplication on $c(n)$.

Any dg operad which is the homology operad of an operad in topological spaces is a Hopf operad (because topological spaces are automatically cocommutative coalgebras, with comultiplication defined by the diagonal map). For example, the commutative operad Com is a Hopf operad, with coproduct defined on the generator $\star \in \mathrm{Com}(2)$ by

$$c(\star) = \star \otimes \star.$$

With the comultiplication defined in this way, the tensor product of commutative algebras is the usual one. If A and B are commutative algebras, the product on $A \otimes B$ is defined by

$$(a \otimes b) \star (a' \otimes b') = (-1)^{|a'||b|} (a \star a') \otimes (b \star b').$$

The Poisson operad is also a Hopf operad, with coproduct defined (on the generators $\star, \{-,-\}$ by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

If A, B are Poisson algebras, then the tensor product $A \otimes B$ is a Poisson algebra with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|}(a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

2.2.1. Structured factorization algebras. Let dgDiff denote the dg multicategory of differentiable cochain complexes. If F_i, G are differentiable cochain complexes, then an element of

$$\mathrm{Hom}^r(F_1, \dots, F_k \mid G)$$

is a smooth multilinear map

$$F_1 \times \cdots \times F_k \to G$$

of cohomological degree r. The differential arises in the usual way from the differentials on F_i and on G; closed elements of degree 0 are multilinear cochain maps.

If P is a dg Hopf operad, then we can talk about P-algebras in the multicategory of differentiable cochain complexes. If F is a differentiable cochain complex, then we can define the endomorphism dg operad $\mathrm{End}(F)$ whose n'th component is

$$\mathrm{End}(F)(n) = \mathrm{Hom}(F, \dots, F \mid F).$$

Then, a P-structure on F is a map of dg operads

$$P \to \mathrm{End}(F).$$

We call such an object a differentiable P-algebra.

Such P-algebras themselves form a multicategory, in a natural way. To see this, let $S\,\mathrm{dgDiff}$ denote the universal symmetric monoidal dg category containing the dg multicategory dgDiff of differentiable cochain complexes. If $F_1, \dots, F_k$ are P-algebras in DVS, then they are P-algebras in $S\,\mathrm{DVS}$. Since P is a Hopf operad $F_1 \otimes \cdots \otimes F_k$ is then a P-algebra in $S\,\mathrm{DVS}$. A morphism in the multicategory of P-algebras is then a map of P algebras

$$F_1 \otimes \cdots \otimes F_k \to G.$$

Note that forgetting the the forgetful functor from the multicategory of differentiable P-algebras to that of differentiable cochain complexes is faithful. That is, the map of sets

$$\mathrm{Hom}_P(F_1, \ldots, F_k \mid G) \to \mathrm{Hom}_{\mathrm{dgDiff}}(F_1, \ldots, F_k \mid G)$$

is injective. Further, the image of this map lies in the space of closed degree 0 multi-morphisms; these are the same as multilinear cochain maps.

Thus, if F_i, G are differentiable P-algebras, and if

$$\phi : F_1 \times \cdots \times F_k \to G$$

is a smooth multilinear cochain map, we can ask whether ϕ is a map of P-algebras.

2.2.1.1 Definition. *Let P be a differential graded Hopf operad. A prefactorization differentiable P-algebra is a prefactorization algebra with values in the multicategory of differentiable P-algebras. A factorization P-algebra is a prefactorization P-algebra, such that the underlying prefactorization algebra with values in differentiable cochain complexes is a factorization algebra.*

We can unpack this definition as follows. Suppose that $\mathcal{F}$ is a factorization P-algebra. Then, $\mathcal{F}$ is a factorization algebra; and, in addition, for all $U \subset M$, $\mathcal{F}(U)$ is a P-algebra. The structure maps

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

(defined when $U_1, \ldots, U_n$ are disjoint open subsets of V) are required to be P-algebra maps in the sense defined above.

2.2.2. Commutative factorization algebras. One of the most important examples is when P is the operad Com of commutative algebras. Then, we find that $\mathcal{F}(U)$ is a commutative algebra for each U. Further, if $U_1, \ldots, U_k \subset V$ are as above, the product map

$$m : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$

is compatible with the commutative algebra structures, in the following sense.

(1) If $1 \in \mathcal{F}(U_i)$ is the unit for the commutative product on each $F(U_i)$, then

$$m(1, \ldots, 1) = 1.$$

(2) If $\alpha_i, \beta_i \in \mathcal{F}(U_i)$, then

$$m(\alpha_1\beta_1, \ldots, \alpha_k\beta_k) = \pm m(\alpha_1, \ldots, \alpha_k) m(\beta_1, \ldots, \beta_k)$$

where $\pm$ indicates the usual Koszul rule of signs.

Note that the axioms of a factorization algebra imply that $\mathcal{F}(\emptyset)$ is the ground ring k (which we normally take to be $\mathbb{R}$ or $\mathbb{C}$ for classical theories and $\mathbb{R}[[\hbar]]$ or $\mathbb{C}[[\hbar]]$ for quantum field theories. The axioms above, in the case that $k = 1$ and $U_1 = \emptyset$, imply that the map

$$\mathcal{F}(\emptyset) \to \mathcal{F}(U)$$

is a map of unital commutative algebras.

If $\mathcal{F}$ is a commutative prefactorization algebra, then we can recover $\mathcal{F}$ uniquely from the underlying cosheaf of commutative algebras. Indeed, the maps

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$

can be described in terms of the commutative product on $\mathcal{F}(V)$ and the maps

$$\mathcal{F}(U_i) \to \mathcal{F}(V).$$

2.3. The P_0 operad

Recall that the collection of observables in quantum mechanics form an associative algebra. The observables of a classical mechanical system form a Poisson algebra. In the deformation quantization approach to quantum mechanics, one starts with a Poisson algebra A^{cl}, and attempts to construct an associative algebra A^q, which is an algebra flat over the ring $\mathbb{C}[[\hbar]]$, together with an isomorphism of associative algebras $A^q/\hbar \cong A^{cl}$. In addition, if $a, b \in A^{cl}$, and $\widetilde{a}, \widetilde{b}$ are any lifts of a, b to A^q, then

$$\lim_{\hbar \to 0} \frac{1}{\hbar}[\widetilde{a}, \widetilde{b}] = \{a, b\} \in A^{cl}.$$

This book concerns the analog, in quantum field theory, of the deformation quantization picture in quantum mechanics. We have seen that the sheaf of solutions to the Euler-Lagrange equation of a classical field theory can be encoded by a commutative factorization algebra. A commutative factorization algebra is the analog, in our setting, of the commutative algebra appearing in deformation quantization. We have argued (section 1.6) that the observables of a quantum field theory should form a factorization algebra . This factorization algebra is the analog of the associative algebra appearing in deformation quantization.

In deformation quantization, the commutative algebra of classical observables has an extra structure – a Poisson bracket – which makes it “want” to deform into an associative algebra. In this section we will explain the analogous structure on a commutative factorization algebra which makes it want to deform into a factorization algebra. Later (section 4.13) we will see that the commutative factorization algebra associated to a classical field theory has this extra structure.

2.3.1. The E_0 operad.

2.3.1.1 Definition. *Let E_0 be the operad defined by*

$$E_0(n) = \begin{cases} 0 & \text{if } n > 0 \\ \mathbb{R} & \text{if } n = 0 \end{cases}$$

Thus, an E_0 algebra in the category of real vector spaces is a real vector space with a distinguished element in it. More generally, an E_0 algebra in a symmetric monoidal category $\mathcal{C}$ is the same thing as an object A of $\mathcal{C}$ together with a map $1_{\mathcal{C}} \to A$

The reason for the terminology E_0 is that this operad can be interpreted as the operad of little 0-discs.

The inclusion of the empty set into every open set implies that, for any factorization algebra $\mathcal{F}$, there is a unique map from the unit factorization algebra $\mathbb{R} \to \mathcal{F}$.

2.3.2. The P_0 operad. The Poisson operad is an object interpolating between the commutative operad and the associative (or E_1) operad. We would like to find an analog of the Poisson operad which interpolates between the commutative operad and the E_0 operad.

Let us define the P_k operad to be the operad whose algebras are commutative algebras equipped with a Poisson bracket of degree $1-k$. With this notation, the usual Poisson operad is the P_1 operad.

Recall that the homology of the E_n operad is the P_n operad, for $n > 1$. Thus, just as the semi-classical version of an algebra over the E_1 operad is a Poisson algebra in the usual sense (that is, a P_1 algebra), the semi-classical version of an E_n algebra is a P_n algebra.

Thus, we have the following table:

Classical	Quantum
?	E_0 operad
P_1 operad	E_1 operad
P_2 operad	E_2 operad
$\vdots$	$\vdots$

This immediately suggests that the P_0 operad is the semi-classical version of the E_0 operad.

Note that the P_0 operad is a Hopf operad: the coproduct is defined by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

In concrete terms, this means that if A and B are P_0 algebras, their tensor product $A \otimes B$ is again a P_0 algebra, with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|} (a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

2.3.3. P_0 **factorization algebras.** Since the P_0 operad is a Hopf operad, it makes sense to talk about P_0 factorization algebras. We can give an explicit description of this structure. A P_0 factorization algebra is a commutative factorization algebra $\mathcal{F}$, together with a Poisson bracket of cohomological degree 1 on each commutative algebra $\mathcal{F}(U)$, with the following additional properties. Firstly, if $U \subset V$, the map

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

must be a homomorphism of P_0 algebras.

The second condition is that observables coming from disjoint sets must Poisson commute. More precisely, let U_1, U_2 be disjoint subsets f V. . Let $j_i : \mathcal{F}(U_i) \to \mathcal{F}(V)$

be the natural maps. Let $\alpha_i \in \mathcal{F}(U_i)$, and $j_i(\alpha_i) \in \mathcal{F}(V)$. Then, we require that

$$\{j_1(\alpha_1), j_2(\alpha_2)\} = 0 \in \mathcal{F}(V)$$

where $\{-,-\}$ is the Poisson bracket on $\mathcal{F}(V)$.

2.3.4. Quantization of P_0 algebras. We know what it means to quantize an Poisson algebra in the ordinary sense (that is, a P_1 algebra) into an E_1 algebra.

There is a similar notion of quantization for P_0 algebras. A quantization is simply an E_0 algebra over $\mathbb{R}[[\hbar]]$ which, modulo $\hbar$, is the original P_0 algebra, and for which there is a certain compatibility between the Poisson bracket on the P_0 algebra and the quantized E_0 algebra.

Let A be a commutative algebra in the category of cochain complexes. Let A_1 be an E_0 algebra flat over $\mathbb{R}[[\hbar]]/\hbar^2$, and suppose that we have an isomorphism of chain complexes

$$A_1 \otimes_{\mathbb{R}[[\hbar]]/\hbar^2} \mathbb{R} \cong A.$$

In this situation, we can define a bracket on A of degree 1, as follows.

We have an exact sequence

$$0 \to \hbar A \to A_1 \to A \to 0.$$

The boundary map of this exact sequence is a cochain map

$$D : A \to A$$

(well-defined up to homotopy).

Let us define a bracket on A by the formula

$$\{a, b\} = D(ab) - (-1)^{|a|} a Db - (Da)b.$$

Because D is well-defined up to homotopy, so is this bracket. However, unless D is an order two differential operator, this bracket is simply a cochain map $A \otimes A \to A$, and not a Poisson bracket of degree 1.

In particular, this bracket induces one on the cohomology $H^*(A)$ of A. The cohomological bracket is independent of any choices.

2.3.4.1 Definition. *Let A be a P_0 algebra in the category of cochain complexes. Then a quantization of A is an E_0 algebra $\widetilde{A}$ over $\mathbb{R}[[\hbar]]$, together with a quasi-isomorphism of E_0 algebras*

$$\widetilde{A} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A,$$

which satisfies the following correspondence principle*: the bracket on $H^*(A)$ induced by $\widetilde{A}$ must coincide with that given by the P_0 structure on A.*

In the next section we will consider a more sophisticated, operadic notion of quantization, which is strictly stronger than this one. To distinguish between the two notions, one could call the definition of quantization presented here a *weak quantization*, while the definition introduced later will be called a *strong quantization*.

2.4. The Beilinson-Drinfeld operad

Beilinson and Drinfeld [**BD04**] constructed an operad over the formal disc which generically is equivalent to the E_0 operad, but which at 0 is equivalent to the P_0 operad. We call this operad the Beilinson-Drinfeld operad.

The operad P_0 is generated by a commutative associative product $- \star -$, of degree 0; and a Poisson bracket $\{-,-\}$ of degree $+1$.

2.4.0.2 Definition. *The Beilinson-Drinfeld (or BD) operad is the differential graded operad over the ring $\mathbb{R}[[\hbar]]$ which, as a graded operad, is simply*

$$BD = P_0 \otimes \mathbb{R}[[\hbar]];$$

but with differential defined by

$$\mathrm{d}(- \star -) = \hbar\{-,-\}.$$

If M is a flat differential graded $\mathbb{R}[[\hbar]]$ module, then giving M the structure of a BD algebra amounts to giving M a commutative associative product, of degree 0, and a Poisson bracket of degree 1, such that the differential on M is a derivation of the Poisson bracket, and the following identity is satisfied:

$$\mathrm{d}(m \star n) = (\mathrm{d}m) \star n + (-1)^{|m|} m \star (\mathrm{d}n) + \hbar\{m, n\}$$

2.4.0.3 Lemma. *There is an isomorphism of operads,*

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong P_0,$$

and a quasi-isomorphism of operads over $\mathbb{R}((\hbar))$,

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)) \simeq E_0 \otimes \mathbb{R}((\hbar)).$$

Thus, the operad BD interpolates between the P_0 operad and the E_0 operad.

BD is an operad in the category of differential graded $\mathbb{R}[[\hbar]]$ modules. Thus, we can talk about BD algebras in this category, or in any symmetric monoidal category enriched over the category of differential graded $\mathbb{R}[[\hbar]]$ modules.

The BD algebra is, in addition, a Hopf operad, with coproduct defined in the same way as in the P_0 operad. Thus, one can talk about BD factorization algebras.

2.4.1. BD quantization of P_0 algebras.

2.4.1.1 Definition. *Let A be a P_0 algebra (in the category of cochain complexes). A BD quantization of A is a flat $\mathbb{R}[[\hbar]]$ module A^q, flat over $\mathbb{R}[[\hbar]]$, which is equipped with the structure of a BD algebra, and with an isomorphism of P_0 algebras*

$$A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A.$$

Similarly, an order k BD quantization of A is a differential graded $\mathbb{R}[[\hbar]]/\hbar^{k+1}$ module A^q, flat over $\mathbb{R}[[\hbar]]/\hbar^{k+1}$, which is equipped with the structure of an algebra over the operad

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/\hbar^{k+1},$$

and with an isomorphism of P_0 algebras

$$A^q \otimes_{\mathbb{R}[[\hbar]]/\hbar^{k+1}} \mathbb{R} \cong A.$$

This definition applies without any change in the world of factorization algebras.

2.4.1.2 Definition. *Let $\mathcal{F}$ be a P_0 factorization algebra on M. Then a BD quantization of $\mathcal{F}$ is a BD factorization algebra $\widetilde{\mathcal{F}}$ equipped with a quasi-isomorphism*

$$\widetilde{\mathcal{F}} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \simeq \mathcal{F}$$

of P_0 factorization algebras on M.

2.4.2. Operadic description of ordinary deformation quantization. We will finish this section by explaining how the ordinary deformation quantization picture can be phrased in similar operadic terms.

Consider the following operad BD_1 over $\mathbb{R}[[\hbar]]$. BD_1 is generated by two binary operations, a product $*$ and a bracket $[-,-]$. The relations are that the product is associative; the bracket is antisymmetric and satisfies the Jacobi identity; the bracket and the product satisfy a certain Leibniz relation, expressed in the identity

$$[ab, c] = a[b, c] \pm [b, c]a$$

(where $\pm$ indicates the Koszul sign rule); and finally the relation

$$a * b \mp b * a = \hbar[a, b]$$

holds. This operad was introduced by Ed Segal [Seg10].

Note that, modulo $\hbar$, BD_1 is the ordinary Poisson operad P_1. If we set $\hbar = 1$, we find that BD_1 is the operad E_1 of associative algebras. Thus, BD_1 interpolates between P_1 and E_1 in the same way that BD_0 interpolates between P_0 and E_0.

Let A be a P_1 algebra. Let us consider possible lifts of A to a BD_1 algebra.

2.4.2.1 Lemma. *A lift of A to a BD_1 algebra, flat over $\mathbb{R}[[\hbar]]$, is the same as a deformation quantization of A in the usual sense.*

PROOF. We need to describe BD_1 structures on $A[[\hbar]]$ compatible with the given Poisson structure. To give such a BD_1 structure is the same as to give an associative product on $A[[\hbar]]$, linear over $\mathbb{R}[[\hbar]]$, and which modulo $\hbar$ is the given commutative product on A. Further, the relations in the BD_1 operad imply that the Poisson bracket on A is related to the associative product on $A[[\hbar]]$ by the formula

$$\hbar^{-1}(a * b \mp b * a) = \{a, b\} \mod \hbar.$$

□

2.5. The factorization algebras associated to free scalar field theories

In this section, we will construct the factorization algebra associated to the free scalar field theory on a Riemannian manifold. We will show that, for one-dimensional manifolds, this factorization algebra recovers the familiar Weyl algebra, the algebra observables for quantum mechanics.

2.5.1. Before we describe these basic examples, we need to introduce some notation.

Let E be a graded vector bundle on a manifold M, and let U be an open subset of M. We let $\mathscr{E}(U)$ denote the cochain complex of smooth sections of E on U, and we let $\mathscr{E}_c(U)$ denote the cochain complex of compactly supported sections of E on U. Throughout this book, we will often use the notation $\overline{\mathscr{E}}(U)$ to denote the distributional sections on U, defined by

$$\overline{\mathscr{E}}(U) = \mathscr{E}(U) \otimes_{C^\infty(U)} \mathcal{D}(U),$$

where $\mathcal{D}(U)$ is the space of distributions on U. Similarly, let $\overline{\mathscr{E}}_c(U)$ denote the compactly supported distributional sections of E on U. There are natural inclusions

$$\mathscr{E}_c(U) \hookrightarrow \overline{\mathscr{E}}_c(U) \hookrightarrow \overline{\mathscr{E}}(U),$$

$$\mathscr{E}_c(U) \hookrightarrow \mathscr{E}(U) \hookrightarrow \overline{\mathscr{E}}(U),$$

by viewing smooth functions as distributions.

If, as above, E is a graded vector bundle on M, let $E^! = E^\vee \otimes \mathrm{Dens}_M$. We give $\mathscr{E}^!$ a differential that is the formal adjoint to Q on E. Let $\mathscr{E}^!(U)$, $\mathscr{E}_c^!(U)$ denote the cochain complexes of smooth and compactly supported sections of $E^!$, and let $\overline{\mathscr{E}}^!(U)$ and $\overline{\mathscr{E}}_c^!(U)$ denote the cochain complexes of distributional and compactly-supported distributional sections of $E^!$.

Note that $\overline{\mathscr{E}}_c(U)$ is the continuous dual to $\mathscr{E}^!(U)$, and that $\mathscr{E}_c(U)$ is the continuous dual to $\overline{\mathscr{E}}^!(U)$.

The prefactorization algebras we will discuss are constructed from symmetric algebras on the vector spaces $\mathscr{E}_c(U)$ and $\overline{\mathscr{E}}_c(U)$. This symmetric algebra can be defined in two ways: either using the completed projective tensor product of topological vector spaces, or in terms of sections of bundles on U^n. For concreteness, we will discuss the latter construction.

Thus, let us define $(\mathscr{E}_c(U))^{\otimes n}$ to be the tensor power defined using the completed projective tensor product on the topological vector space $\mathscr{E}_c(U)$. Concretely, if $E^{\boxtimes n}$

denotes the vector bundle on M^n obtained as the external tensor product, then

$$(\mathscr{E}_c(U))^{\otimes n} = \Gamma_c(U^n, E^{\boxtimes n})$$

is the compactly supported smooth sections of $E^{\boxtimes n}$ on U^n. Similarly, we have

$$(\overline{\mathscr{E}}_c(U))^{\otimes n} = \overline{\Gamma}_c(U^n, E^{\boxtimes n})$$

is the compactly supported distributional sections of $E^{\boxtimes n}$ on M^n.

Symmetric (or exterior) powers of $\mathscr{E}_c(U)$ and $\overline{\mathscr{E}}_c(U)$ are defined by taking coinvariants of $\mathscr{E}_c(U)^{\otimes n}$ with respect to the action of the symmetric group S_n. Thus, we can define, for example, the symmetric algebra

$$\begin{aligned} \mathrm{Sym}^* \mathscr{E}_c^!(U) &= \bigoplus_n \mathrm{Sym}^n \mathscr{E}_c^!(U) \\ \mathrm{Sym}^* \overline{\mathscr{E}}_c^!(U) &= \bigoplus_n \mathrm{Sym}^n \overline{\mathscr{E}}_c^!(U) \end{aligned}$$

Note that, since $\overline{\mathscr{E}}_c^!(U)$ is dual to $\mathscr{E}(U)$, we can view $\mathrm{Sym}\, \overline{\mathscr{E}}_c^!(U)$ as the algebra of polynomial functions on $\mathscr{E}(U)$. Thus, we often write

$$\mathscr{O}(\mathscr{E}(U)) = \mathrm{Sym}\, \overline{\mathscr{E}}_c^!(U)$$

for this algebra of functions. Similarly, $\mathrm{Sym}\, \mathscr{E}_c^!(U)$ is the algebra of polynomial functions on $\overline{\mathscr{E}}(U)$.

2.5.2. Defining the prefactorization algebra. After this preliminary discussion, we can introduce the the prefactorization algebras associated to free field theories. Let M be a Riemannian manifold, and so M is equipped with a natural density, arising from the metric. We will use this natural density to integrate functions, and also to provide an isomorphism between functions and densities that we use implicitly from hereon. The field theory we will discuss has as fields $\phi \in C^\infty(M)$ and has as action functional

$$S(\phi) = \int_M \phi \triangle \phi,$$

where $\triangle$ is the Laplacian on M. (Normally we will reserve the symbol $\triangle$ for the Batalin-Vilkovisky Laplacian, but that's not necessary in this section.)

If $U \subset M$ is an open subset, then the space of solutions to the equation of motion on U is the space of harmonic functions on U.

In this book, we will always consider the *derived* space of solutions of the equation of motion. For more details about the derived philosophy, the reader should consult Chapter 4. In this simple situation, the derived space of solutions to the free field equations, on an open subset $U \subset M$, is the two-term complex

$$\mathscr{E}(U) = \left(C^\infty(U)^0 \xrightarrow{\triangle} C^\infty(U)^1 \right),$$

where the superscripts indicate the cohomological degree.

The classical observables of a field theory on an open subset $U \subset M$ are functions on the derived space of solutions to the equations of motion on U. We will take our functions to be polynomial functions, and thus the symmetric algebra of the dual. The dual to the two-term complex $\mathscr{E}(U)$ above is the complex

$$\mathscr{E}(U)^\vee = \left(\mathcal{D}_c(U)^{-1} \xrightarrow{\triangle} \mathcal{D}_c(U)^0 \right),$$

where $\mathcal{D}_c(U)$ indicates the space of compactly supported distributions on U.

Thus, as a first pass, one would want to define the classical observables as the symmetric algebra on $\mathscr{E}(U)^\vee$. (When we say symmetric algebra, we use the completed projective tensor product of the topological vector space $\mathcal{D}_c(U)$.)

However, this will lead to difficulties defining the quantum observables. When we work with an interacting theory, these difficulties can only be surmounted using the techniques of renormalization. For a free field theory, though, there is a much simpler solution.

We have

$$\mathscr{E}_c^!(U) \cong \left(C_c^\infty(U)^{-1} \to C_c^\infty(U)^0 \right),$$

using our identification between densities and functions. Note that there is a natural map of cochain complexes $\mathscr{E}_c^!(U) \to \mathscr{E}(U)^\vee$, given by viewing a compactly supported function as a distribution.

2.5.2.1 Lemma. *The map $\mathscr{E}_c^!(U) \to \mathscr{E}(U)^\vee$ is a continuous homotopy equivalence.*

PROOF. This is a special case of a general result proved in the appendix A.9. Note that by continuous homotopy equivalence we mean that there is a continuous inverse map $\mathscr{E}(U)^\vee \to \mathscr{E}_c^!(U)$ and continuous cochain homotopies between the two composed maps and the identity maps. □

This lemma says that at the level of cohomology, we can replace a distributional linear observable by a smooth linear observable. In more physical language, it says that it suffices to work with "smeared observables."

Thus, we will define our classical observables to be

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^*(\mathscr{E}_c^!(U)) = \mathrm{Sym}^*(C_c^\infty(U)^{-1} \xrightarrow{\triangle} C_c^\infty(U)^0),$$

the symmetric algebra on $\mathscr{E}_c^!(U)$, defined using the completed projective tensor product of nuclear spaces. In section A.7 we describe some generalities about symmetric algebras on spaces of sections of a vector bundle. In this case, the symmetric algebra is quite simple.

For example,

$$\mathrm{Sym}^2\left(C_c^\infty(U)^{-1} \xrightarrow{\triangle} C_c^\infty(U)^0\right) = \left(C_c^\infty(U^2)\right)^{-2}_{\mathrm{Alt}_2} \to C_c^\infty(U^2)^{-1} \to \left(C_c^\infty(U^2)_{S_2}\right). \tag{2.5.2.1}$$

Here $C_c^\infty(U^2)_{\mathrm{Alt}_2}$ indicates the coinvariants for the alternating action of the symmetric group S_2 on $C_c^\infty(U^2)$.

2.5.3. The one-dimensional case, in detail. This space is particularly simple in dimension 1.

2.5.3.1 Lemma. *If $U = (a,b) \subset \mathbb{R}$ is an interval in $\mathbb{R}$, then the algebra of classical observables for the free field has cohomology*

$$H^*(\mathrm{Obs}^{cl}((a,b))) = \mathbb{R}[p,q],$$

the polynomial algebra in two variables.

PROOF. We will show that the complex

$$\mathscr{E}_c^!((a,b)) = \left(C_c^\infty((a,b))^{-1} \xrightarrow{\triangle} C_c^\infty((a,b))^0\right)$$

is continuously homotopy equivalent to the complex $\mathbb{R}^2$ situated in degree 0.

Let us view the complex $\mathscr{E}_c^!((a,b))$ as a subspace of the dual of the complex of fields

$$\mathscr{E}((a,b)) = \left(C^\infty((a,b))^0 \xrightarrow{\triangle} C^\infty((a,b))^1\right).$$

We denote an arbitrary field of degree 0 by $\phi \in C^\infty((a,b))^0$ and an arbitrary field of degree 1 by $\psi \in C^\infty((a,b))^1$.

Choose a function $f \in C_c^\infty(a,b)$ with the property that $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$. Then, let us define momentum and position observables $p, q \in \mathscr{E}_c^!((a,b))$ by

$$q(\phi,\psi) = \int \phi(x) f(x)\mathrm{d}x,$$
$$p(\phi,\psi) = -\int \phi(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x.$$

These are the natural distributions associated to f and f' respectively. Note that p, q are in degree 0. We need to show that the cochain map

$$I : \mathbb{R}\{p,q\} \to \mathscr{E}_c^!((a,b))$$

is a continuous homotopy equivalence (where $\mathbb{R}\{p,q\}$ refers to the vector space spanned by p and q).

To define the inverse map, it suffices to write down a closed subspace $K \subset C_c^\infty((a,b))$ such that the quotient map

$$\pi : C_c^\infty((a,b)) \to C_c^\infty((a,b))/K$$

has dimension 2, with basis the images of f and f'. For K, we take the space of functions g with the following properties.

(1) $\int_a^b g(x)\mathrm{d}x = 0$.
(2) The first property implies that g has a compactly supported antiderivative G. We then require that $\int_a^b G(x)\mathrm{d}x = 0$.

It is easy to verify that $C_c^\infty((a,b))$ is spanned by functions $g \in K$ and by f and f'. Indeed, for $h \in C_c^\infty((a,b))$, let

$$c_1 = \int_a^b h(x)\mathrm{d}x$$

and set $h_1 = h - c_1 f$. Thus $\int_a^b h_1(x)\mathrm{d}x = 0$ and hence has a compactly supported antiderivative H. Repeating this construction, we find the constant c_2 such that $H - c_2 f$ satisfies $\int_a^b H(x) - c_2 f(x)\mathrm{d}x = 0$. Hence,

$$h(x) = h_2(x) + c_1 f(x) + c_2 f'(x),$$

where h_2 lives in K.

Next, we need to define a homotopy between the identity and the endomorphism $i \circ \pi$ of $\mathscr{E}_c^!((a,b))$. Equivalently, we need a homotopy that contracts the two-term complex

$$C_c^\infty((a,b))^{-1} \xrightarrow{\triangle} K^0.$$

The homotopy S is defined by sending $g \in K$ to

$$S(g)(x) = \int_{y=a}^{x} \int_{u=a}^{y} g(u) \mathrm{d}u \mathrm{d}y$$

The result of performing this iterated integral has compact support, by the assumption that $g \in K$.

Thus we have constructed the desired continuous homotopy equivalence. $\square$

It is clear that classical observables form a prefactorization algebra. Indeed, $\mathrm{Obs}^{cl}(U)$ is a commutative differential graded algebra. If $U \subset V$, there is a natural algebra homomorphism

$$i_V^U : \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(V),$$

which on generators is just the natural map $C_c^\infty(U) \to C_c^\infty(V)$, extension by zero.

If $U_1, \dots, U_n \subset V$ are disjoint open subsets, the prefactorization structure map is the continuous multilinear map

$$\begin{array}{ccc} \mathrm{Obs}^{cl}(U_1) \times \cdots \times \mathrm{Obs}^{cl}(U_n) & \to & \mathrm{Obs}^{cl}(V) \\ (\alpha_1, \dots, \alpha_n) & \mapsto & \prod_{i=1}^n i_V^{U_i} \alpha_i. \end{array}$$

2.5.4. The Poisson bracket.

We now return to the general case.

2.5.4.1 Lemma. *There is a unique continuous Poisson bracket on* $\mathrm{Obs}^{cl}(U)$ *of cohomological degree* 1*, with the property that for* $\alpha \in C_c^\infty(U)^{-1}$ *and* $\beta \in C_c^\infty(U)^{-1}$*, we have*

$$\{\alpha, \beta\} = \int_U \alpha\beta.$$

PROOF. Uniqueness is immediate, since the dg algebra $\mathrm{Obs}^{cl}(U)$ is topologically generated by $\alpha \in C_c^\infty(U)^{-1}$ and $\beta \in C_c^\infty(U)^{-1}$

Existence follows from the fact that the multilinear map

$$\begin{array}{ccc} C_c^\infty(U^n) \times C_c^\infty(U) \times C_c^\infty(U) & \to & C_c^\infty(U^n) \\ (f, \alpha, \beta) & \mapsto & f \int_U \alpha\beta \end{array}$$

is continuous, and so extends to a map from the completed projective tensor product

$$C_c^\infty(U^{n+2}) = C_c^\infty(U^n) \otimes C_c^\infty(U) \otimes C_c^\infty(U) \to C_c^\infty(U^n).$$

□

Note that for U_1, U_2 disjoint open subsets of V and for observables $\alpha_i \in \mathrm{Obs}^{cl}(U_i)$, we have

$$\{i_V^{U_1}\alpha_1, i_V^{U_2}\alpha_2\} = 0.$$

That is, observables coming from disjoint open subsets "commute" with respect to the Poisson bracket. This means that $\mathrm{Obs}^{cl}(U)$ defines a P_0 prefactorization algebra. (We will see later in section 3.2 that this prefactorization algebra is actually a factorization algebra.)

2.5.5. Quantum observables. As we explained in section 1.8, our philosophy is that we should take a P_0 factorization algebra and deform it into a BD factorization algebra. In the situation we are considering in this section, we will construct a factorization algebra of quantum observables Obs^q with the property that, as a vector space,

$$\mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)[\hbar],$$

but with a differential d such that

(1) modulo $\hbar$, d coincides with the differential on $\mathrm{Obs}^{cl}(U)$, and
(2) the equation

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot b + \hbar\{a, b\}$$

holds.

Here, $\cdot$ indicates the commutative product on $\mathrm{Obs}^{cl}(U)$. These properties imply that Obs^q defines a BD factorization algebra quantizing the P_0 factorization algebra Obs^{cl}.

Our construction starts with a certain graded Heisenberg Lie algebra. Let

$$\mathcal{H}(U) = \left(C_c^\infty(U)^0 \xrightarrow{\triangle} C_c^\infty(U)^1 \right) \oplus (\mathbb{R}\hbar)^1$$

where $\mathbb{R}\hbar$ is situated in degree 1. We give $\mathcal{H}(U)$ a Lie bracket by saying that, for $\alpha \in C_c^\infty(U)^0$ and $\beta \in C_c^\infty(U)^1$, the bracket is

$$[\alpha, \beta] = \hbar \int_U \alpha\beta.$$

Thus, $\mathcal{H}(U)$ is a graded version of a Heisenberg algebra, centrally extending the abelian dg Lie algebra $C_c^\infty(U)^0 \to C_c^\infty(U)^1$.

Let

$$\mathrm{Obs}^q(U) = C_*(\mathcal{H}(U)),$$

where C_* denotes the Chevalley-Eilenberg complex for the Lie algebra homology of $\mathcal{H}(U)$, defined using the completed projective tensor product. Thus,

$$\begin{aligned} \mathrm{Obs}^q(U) &= (\mathrm{Sym}^* \mathcal{H}(U)[1], \mathrm{d}) \\ &= \left(\mathrm{Obs}^{cl}(U)[\hbar], \mathrm{d}\right) \end{aligned}$$

where the differential arises from the Lie bracket on $\mathcal{H}(U)$. (Note that we always work with cochain complexes, so our grading convention of C_* is the negative of one popular convention.)

Remark: Those readers who are operadically inclined might notice that the Lie algebra chain complex of a Lie algebra $\mathfrak{g}$ is the E_0 version of the universal enveloping algebra of a Lie algebra. Thus, our construction is an E_0 version of the familiar construction of the Weyl algebra as a universal enveloping algebra of a Heisenberg algebra.

Next, we need to give $\mathrm{Obs}^q(U)$ the structure of a prefactorization algebra. Observe that if $U \subset V$, there is a natural map of dg Lie algebras $\mathcal{H}(U) \to \mathcal{H}(V)$ arising from the natural map $C_c^\infty(U) \to C_c^\infty(V)$, extended to the identity on the central extension. Similarly, if $U_1, \ldots, U_n \subset V$ are disjoint, there is a natural map of dg Lie algebras

$$\mathcal{H}(U_1) \oplus \cdots \oplus \mathcal{H}(U_n) \to \mathcal{H}(V).$$

Applying Chevalley-Eilenberg chains to this map yields the factorization algebra structure maps

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_n) \to \mathrm{Obs}^q(V).$$

It is easy to verify that Obs^q satisfies the axioms of a prefactorization algebra in BD algebras. It follows from the fact that Obs^{cl} is a factorization algebra (which we prove in section 3.2) that Obs^q is a factorization algebra over $\mathbb{R}[\hbar]$.

2.5.6. Quantum mechanics and the Weyl algebra. Next, we will calculate the cohomology of the prefactorization algebra of the free scalar field theory in dimension 1, and we will show that it recovers the familiar Weyl algebra, which is the algebra of observables in quantum mechanics.

First, we need to explain why one can recover an associative algebra from a factorization algebra on the real line.

2.5.6.1 Definition. *Let $\mathcal{F}$ be a prefactorization algebra on $\mathbb{R}$ taking values in the category of vector spaces (without any grading). We say $\mathcal{F}$ is* locally constant *if the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism whenever the inclusion of opens $U \subset V$ is a homotopy equivalence.*

2.5.6.2 Lemma. *Let $\mathcal{F}$ be a locally constant, unital prefactorization algebra on $\mathbb{R}$ taking values in vector spaces. Let $A = \mathcal{F}(\mathbb{R})$. Then A has a natural structure of an associative algebra.*

Remark: Recall that $\mathcal{F}$ being unital means that the commutative algebra $\mathcal{F}(\emptyset)$ is equipped with a unit. We will find that A is an associative algebra over $\mathcal{F}(\emptyset)$.

PROOF. For any interval $(a, b) \subset \mathbb{R}$, the map

$$\mathcal{F}((a, b)) \to \mathcal{F}(\mathbb{R}) = A$$

is an isomorphism. Thus, we have a canonical isomorphism

$$A = \mathcal{F}((a, b))$$

for all intervals (a, b).

Notice that if $(a, b) \subset (c, d)$ then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\cong} & \mathcal{F}((a,b)) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow {\scriptstyle i_{(c,d)}^{(a,b)}} \\ A & \xrightarrow{\cong} & \mathcal{F}((c,d)) \end{array}$$

commutes.

The product map $m : A \otimes A \to A$ is defined as follows. Let $a < b < c < d$. Then, the prefactorization structure on $\mathcal{F}$ gives a map

$$\mathcal{F}((a, b)) \otimes \mathcal{F}((c, d)) \to \mathcal{F}((a, d)),$$

and so, after identifying $\mathcal{F}((a,b))$, $\mathcal{F}((c,d))$ and $\mathcal{F}((a,d))$ with A, we get a map

$$A \otimes A \to A.$$

This is the multiplication in our algebra.

It remains to check the following.

(1) This multiplication doesn't depend on the intervals $(a,b) \amalg (c,d) \subset (a,d)$ we chose, as long as $(a,b) < (c,d)$.
(2) This multiplication is associative and unital.

This is an easy (and instructive) exercise. □

Finally, we can prove that our construction of the factorization algebra for a free field theory, when restricted to dimension 1, reconstructs the Weyl algebra associated to quantum mechanics.

2.5.6.3 Proposition. *For* Obs^q *denote the factorization algebra on* $\mathbb{R}$ *constructed from the free field theory, as above, we have*

(1) the cohomology $H^*(\mathrm{Obs}^q)$ *is locally constant, and*
(2) the corresponding associative algebra is the Weyl algebra, the associative algebra generated by $p, q, \hbar$ *with the relation* $[p,q] = \hbar$ *and all other commutators being zero.*

PROOF. First we will show that the cohomology $H^*(\mathrm{Obs}^q)$ is locally constant. Recall that $\mathrm{Obs}^q(U)$ is the Lie algebra chains on the Heisenberg Lie algebra $\mathcal{H}(U)$. Let us filter $\mathrm{Obs}^q(U)$ by saying that

$$F^{\leq i}\,\mathrm{Obs}^q(U) = \mathrm{Sym}^{\leq i}(\mathcal{H}(U)[1]).$$

The associated graded for this filtration is $\mathrm{Obs}^{cl}(U)[\hbar]$. Thus, to show that $H^*\,\mathrm{Obs}^q$ is locally constant, it suffices (by considering the spectral sequence associated to this filtration) to show that $H^*\,\mathrm{Obs}^{cl}$ is locally constant. We have already seen that $H^*(\mathrm{Obs}^{cl}(a,b)) = \mathbb{R}[p,q]$ for any interval (a,b), and that the inclusion maps $(a,b) \to (a',b')$ induces isomorphisms. Thus, the cohomology of Obs^{cl} is locally constant.

Next, we will show that the associative algebra corresponding to $H^* \operatorname{Obs}^q$ is the Weyl algebra. Recall that

$$\operatorname{Obs}^q((a,b)) = \operatorname{Sym}^* \left(C_c^\infty((a,b))^{-1} \oplus C_c^\infty((a,b))^0 \right) [\hbar]$$

with a certain differential. We view $\operatorname{Obs}^q((a,b))$ as a space of functionals on the space $C^\infty((a,b))^0 \xrightarrow{\triangle} C^\infty((a,b))^1$ of fields. We will denote an arbitrary field of cohomological degree 0 by $\phi \in C^\infty((a,b))^0$ and an arbitrary field of cohomological degree 1 by $\psi \in C^\infty((a,b))^1$.

Choose a function $f_0 \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ with the property that $\int_{-\infty}^{\infty} f_0(x)\mathrm{d}x = 1$. Let $f_t \in C_c^\infty((t - \frac{1}{2}, t + \frac{1}{2}))$ be $f_t(x) = f(x - t)$. We define observables P_t, Q_t by

$$\begin{aligned} Q_t(\phi, \psi) &= \int_{\mathbb{R}} \phi(x) f_t(x) \mathrm{d}x \\ P_t(\phi, \psi) &= \int_{\mathbb{R}} \phi'(x) f_t(x) \mathrm{d}x. \end{aligned}$$

Note that Q_t and P_t represent measurements of positions and momenta of the field ϕ in a neighborhood of t.

Because the cohomology classes $[P_0], [Q_0]$ generate the commutative algebra $H^*(\operatorname{Obs}^{cl}(\mathbb{R}))$, it is automatic that they still generate the associative algebra $H^0 \operatorname{Obs}^q(\mathbb{R})$. We thus need to show that they satisfy the Heisenberg commutation relation

$$[[P_0], [Q_0]] = \hbar$$

for the associative product on $H^0 \operatorname{Obs}^q(\mathbb{R})$, which is an associative algebra by virtue of the fact that $H^* \operatorname{Obs}^q$ is locally constant.

The first thing we need to show is that the cohomology class $[P_t] \in H^0 \operatorname{Obs}^q(\mathbb{R})$ is independent of t. (This is conservation of momentum.)

To see this, let

$$h_{s,t}(x) = \int_{-\infty}^{x} \left(f_s(u) - f_t(u) \right) \mathrm{d}u.$$

Let us define an observable

$$H_{s,t}(\phi, \psi) = \int_{\mathbb{R}} h_{s,t}(x) \psi(x) \mathrm{d}x.$$

Thus $H_{s,t}$ is an element of Sym^2 of cohomological degree -1. Let d denote the differential on $\mathrm{Obs}^q(\mathbb{R})$. We have

$$\begin{aligned}(\mathrm{d}H_{s,t})(\phi,\psi) &= \int_{\mathbb{R}} h_{s,t}(x)\phi''(x)\mathrm{d}x \\ &= -\int_{\mathbb{R}} h'_{s,t}(x)\phi'(x)\mathrm{d}x \\ &= P_t(\phi) - P_s(\phi).\end{aligned}$$

Thus, $[P_t] = [P_s]$.

Note that if $|t| > 1$, the observables P_t and Q_0 have disjoint support. This means that we can use the factorization structure map

$$\mathrm{Obs}^q((-\tfrac{1}{2},\tfrac{1}{2})) \otimes \mathrm{Obs}^q((t-\tfrac{1}{2}, t+\tfrac{1}{2})) \to \mathrm{Obs}^q(\mathbb{R})$$

to define a product observable

$$Q_0 \cdot P_t \in \mathrm{Obs}^q(\mathbb{R}).$$

Concretely,

$$(Q_0 \cdot P_t)(\phi,\psi) = Q_0(\phi,\psi)P_t(\phi,\psi).$$

We will let $\star$ denote the associative multiplication on $H^0\,\mathrm{Obs}^q(\mathbb{R})$. We defined this multiplication by

$$\begin{aligned}[Q_0]\star[P_0] &= [Q_0 \cdot P_t] \text{ if } t > 1 \\ [P_0]\star[Q_0] &= [Q_0 \cdot P_t] \text{ if } t < -1.\end{aligned}$$

Thus, it remains to show that, if $t > 1$,

$$[Q_0 \cdot P_t] - [Q_0 \cdot P_{-t}] = \hbar.$$

We will construct an observable whose differential is the difference between the left and right hand sides. Consider the observable

$$\begin{aligned}S(\phi,\psi) &= Q_0(\phi,\psi)H_{-t,t}(\phi,\psi) \\ &= \left(\int_{\mathbb{R}} f_0(x)\phi(x)\mathrm{d}x\right)\left(\int_{\mathbb{R}} h_{-t,t}(y)\psi(y)\mathrm{d}y\right),\end{aligned}$$

where the functions f_0 and $h_{-t,t}$ were defined above.

Recall that the differential on $\mathrm{Obs}^q(\mathbb{R})$ has two terms: one coming from the Laplacian $\triangle$ mapping $C_c^\infty(\mathbb{R})^{-1}$ to $C_c^\infty(\mathbb{R})^0$, and one arising from the bracket of the Heisenberg Lie algebra. The second term maps

$$\mathrm{Sym}^2\left(C_c^\infty(\mathbb{R})^{-1} \oplus C_c^\infty(\mathbb{R})^0\right) \to \mathbb{R}\hbar.$$

Applying this differential to the observable S, we find that

$$\begin{aligned}(\mathrm{d}S)(\phi,\psi) = &\left(\int_{\mathbb{R}} f_0(x)\phi(x)\mathrm{d}x\right)\left(\int_{\mathbb{R}} h_{-t,t}(y)\phi''(y)\mathrm{d}y\right) \\ &+ \hbar\int_{\mathbb{R}} h_{-t,t}(x) f_0(x)\mathrm{d}x \\ =&Q_0(\phi,\psi)\left(P_t(\phi,\psi) - P_{-t}(\phi,\psi)\right) + \hbar.\end{aligned}$$

Thus, $[\mathrm{d}S] = 0$ so $[Q_0 P_{-t}] - [Q_0 P_t] = \hbar$, as desired. □

2.6. Standard quantum mechanics as a prefactorization algebra

This section is a brief digression from the central theme of the book. Throughout this book we take the path integral formalism as fundamental, and hence we do not focus on the Hamiltonian, or operator, approach to quantum physics. In this section, however, we will explain how to express the standard formalism of quantum mechanics in the language of prefactorization algebras.

2.6.1. Quantum mechanics.

Remark: Our goal here is to sketch the formal aspects of quantum mechanics, so we avoid technical issues (such as boundedness of operators or whether they are trace-class) by describing the simpler finite-dimensional setting.

Let V denote a finite-dimensional Hilbert space, A the continuous endomorphisms, and $H \in A$ the Hamiltonian operator. We view V as a state space for our system, A as where the observables live, and H as determining the time evolution of our system. We seek to describe the following experimental situation, which one might view as a scattering experiment:

- at time $t = 0$, we prepare our system in the initial state $v_0 \in V$;

- we modify the governing Hamiltonian over some finite time interval (i.e., apply an operator, i.e., an observable);
- at time $t = T$, we measure whether our system is in the final state $v_1 \in V$. If we run this experiment many times, with the same initial and final states and the same operator, we should find a statistical pattern in our data. To put this in the usual Dirac notation, if we denote the operator by $\mathcal{O}$ and idealize it as happening at a fixed moment t_0 in time, then we are trying to compute the number
$$\langle v_1 | e^{i(T-t_0)H} \mathcal{O} e^{it_0 H} | v_0 \rangle.$$

Remark: A state is actually a ray, or line, in V. We address this issue in the next subsection.

We want to describe this situation using a prefactorization algebra F on the interval $[0, T]$. Before jumping into details, here's the guiding idea. Interior open intervals describe moments when operators can act on our system. An interval that contains 0 (but not the other end) should describe a state of the system (and dually for intervals containing the other endpoint). But not only do we assign V to a connected interval containing exactly one endpoint and A to a connected interior interval; we also have a distinguished vector in each of these vector spaces. It is important to recall that a unital prefactorization algebra F always assigns a "pointed" vector space to an open set: $F(\emptyset) = \mathbb{C}$ and so the inclusion of the empty set into an open U always gives a map $\mathbb{C} \to F(U)$. Since the empty set has empty intersection with itself, we see that we have a distinguished map $F(\emptyset) \otimes F(\emptyset) \to F(\emptyset)$, namely multiplication. Hence we assign $\mathbb{C}$ *with multiplication* to the empty set, and so we have a distinguished element in $F(\emptyset)$, namely 1, which picks out a distinguished vector in every $F(U)$. Thus, a factorization algebra assigns a pair $(F(U), u \in F(U))$ to each open set.

Returning to quantum mechanics, we fix a vector $v_0 \in V$ and a vector $v_1 \in V$. To open subintervals, our functor assigns the following vector spaces:

- $[0, t) \mapsto (V, e^{itH} v_0)$
- $(s, t) \mapsto (A, e^{i(t-s)H})$
- $(t, T] \mapsto (V, e^{-i(T-t)H} v_1)$.

The first type of interval describes how the initial condition has evolved up to time t. The interior interval describes the possible operators that can act during that time interval, and the evolution operator is the natural distinguished operator. The final type of interval describes the state that will evolve to the final condition. We defer describing $F([0, T]$.

We must now describe the maps coming from inclusion of intervals. Hopefully, we give enough examples to pin down the idea.

First, we describe inclusions of connected intervals.

- For $[0, s) \subset [0, t)$, we use the map $v \mapsto e^{i(t-s)H}v$. This is an automorphism of V sending $e^{isH}v_0$ to $e^{itH}v_0$, and so it respects our marked points.
- For $(s, t) \subset (s', t')$, we use $\mathcal{O} \mapsto e^{i(t'-t)H}\mathcal{O}e^{i(s-s')H}$.
- For $(s, t) \subset [0, t')$, we use $\mathcal{O} \mapsto e^{i(t'-t)H}\mathcal{O}e^{isH}v_0$. This sends an operator in A to how it acts on the input state v_0.
- For $(s, t) \subset (s', T]$, we use $\mathcal{O} \mapsto e^{-i(s-s')H}\mathcal{O}^\dagger e^{-i(T-t)H}v_1$.

Next we describe the three simplest types of "multiplication," namely a pair of disjoint intervals maps into a bigger, connected interval.

- For $[0, s) \cup (s', t) \subset [0, t)$, we use $v \otimes \mathcal{O} \mapsto \mathcal{O}e^{i(s'-s)H}v$. This map describes how an operator acts on a state.
- For $(s, s') \cup (t, t') \subset (s, t')$, we use $\mathcal{O} \otimes \mathcal{P} \mapsto \mathcal{P}e^{i(t-s')H}\mathcal{O}$.
- For $(s, s') \cup (t, T] \subset (s, T]$, we use $\mathcal{O} \otimes v \mapsto \mathcal{O}^\dagger e^{-i(t-s')H}v$.

Finally, we describe what our functor assigns to the whole interval $[0, T]$. By the gluing axiom (described in the chapter on factorization algebras), we see that it must assign the vector space $V \otimes_A V$, where the left-hand V corresponds to the incoming states (and A acts by multiplication) and the right-hand V corresponds to the outgoing state (and A acts by multiplication of the adjoint). Hence we can view this as $V \otimes_{\mathrm{End}(V)} V^\vee$, which is isomorphic to the ground field $\mathbb{C}$: we have the map

$$\begin{aligned} V \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} V &\to V \otimes_A V \\ v \otimes \mathcal{O} \otimes w &\mapsto \langle w | \mathcal{O} v \rangle, \end{aligned}$$

which is precisely the usual inner product.

The maps in this factorization algebra thus allow us to compute scattering as follows. Suppose we apply some operator during the interval (s, t). The inclusion of intervals $(s,t) \hookrightarrow [0,T]$ yields a map $A \to V \otimes_A V \cong \mathbb{C}$. If we pick an operator $\mathcal{O}$ acting during the interval (s,t), then its image in $\mathbb{C}$ is $\langle v_1 | e^{i(T-t)H} \mathcal{O} e^{isH} | v_0 \rangle$.

2.6.2. Some subtleties. Our construction above captures much of the standard formalism of quantum mechanics, but there are a few loose ends we need to address.

First, in standard quantum mechanics, a state is not a vector in V but a line. Above, however, we fixed vectors v_0 and v_1, so there seems to be a discrepancy. The observation that rescues us is a natural one, from the mathematical viewpoint. Consider scaling v_0 and v_1 by elements of $\mathbb{C}^\times$. This defines a new factorization algebra, but it is isomorphic to what we described above, and the expectation value "$\langle v_0 | \mathcal{O} v_1 \rangle$" of an operator depends linearly in the rescaling of the input and output vectors. More precisely, there is a natural equivalence relation we can place on the factorization algebras described above that corresponds to the usual notion of state in quantum mechanics.

Another issue that might bother the reader is that our formalism only matches nicely with experiments that resemble scattering experiments. It does not seem well-suited to descriptions of systems like bound states (e.g., an atom sitting quietly, minding its own business). For such systems, we might consider running over the whole space of states (as described in the previous paragraph). Alternatively, we might drop the endpoints and simply work with the factorization algebra on the open interval, which focuses on the algebra of operators A.

2.7. Translation-invariant and holomorphic prefactorization algebras

In this section we will analyze in detail the notion of *translation-invariant* prefactorization algebras on $\mathbb{R}^n$ or $\mathbb{C}^n$. On $\mathbb{C}^n$ we can ask for a translation-invariant prefactorization algebra to have a holomorphic structure; this implies that all structure maps of the prefactorization algebra are (in a sense we will explain shortly) holomorphic. There are many natural field theories where the corresponding prefactorization algebra is holomorphic: for instance, chiral conformal field theories in complex dimension 1, and minimal twists [**Cos11c**] of supersymmetric field theories in complex dimension 2.

2.7.1. We now turn to the definition of a translation-invariant prefactorization algebra. If $U \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, let

$$T_x(U) := \{y \,:\, y - x \in U\}$$

denote the translate of U by x.

2.7.1.1 Definition. *A prefactorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ is* discretely translation invariant *if we have isomorphisms*

$$T_x : \mathcal{F}(U) \cong \mathcal{F}(T_x(U))$$

for all $x \in \mathbb{R}^n$ and all open subsets $U \subset \mathbb{R}^n$. These isomorphisms must satisfy a few conditions. First, we require that $T_x \circ T_y = T_{x+y}$ for every $x, y \in \mathbb{R}^n$. Second, for all disjoint open subsets $U_1, \dots, U_k$ in V, the diagram

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) & \xrightarrow{T_x} & \mathcal{F}(T_x U_1) \otimes \cdots \otimes \mathcal{F}(T_x U_k) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{T_x} & \mathcal{F}(T_x V) \end{array}$$

commutes. (Here the vertical arrows are the structure maps of the prefactorization algebra.)

We are interested in a refined version of this notion, where the structure maps of the prefactorization algebra depend smoothly on the position of the open sets. It is a bit subtle to talk about "smoothly varying an open set," and in order to do this, we introduce some notation.

Firstly, we need to introduce the notion of a *derivation* of a prefactorization algebra on a manifold M. We will construct a differential graded Lie algebra of derivations of any prefactorization algebra.

2.7.1.2 Definition. *A degree k derivation of a prefactorization algebra $\mathcal{F}$ is a collection of maps $D_U : \mathcal{F}(U) \to \mathcal{F}(U)$ of cohomological degree k for each open subset $U \subset M$, with the property that, if $U_1, \dots, U_n \subset V$ are disjoint, and $\alpha_i \in \mathcal{F}(U_i)$, then*

$$D_V m_V^{U_1,\dots,U_n}(\alpha_1, \dots, \alpha_n) = \pm \sum m_V^{U_1,\dots,U_n}(\alpha_1, \dots, D_{U_i}\alpha_i, \dots, \alpha_n),$$

where $\pm$ indicates the usual Koszul rule of signs.

Let $\mathrm{Der}^k(\mathcal{F})$ denote the derivations of degree k; it is easy to verify that $\mathrm{Der}^*(\mathcal{F})$ forms a differential graded Lie algebra. The differential is defined by $(\mathrm{d}D)_U = [\mathrm{d}_U, D_U]$, where d_U is the differential on $\mathcal{F}(U)$. The Lie bracket is defined by

$$[D, D']_U = [D_U, D'_U].$$

The concept of derivation allows us to talk about the action of a dg Lie algebra on a prefactorization algebra $\mathcal{F}$. Such an action is simply a homomorphism of differential graded Lie algebras

$$\mathfrak{g} \to \mathrm{Der}^*(\mathcal{F}).$$

Next, let us introduce some notation which will help us describe the smoothness conditions for a discretely translation-invariant prefactorization algebra.

Let $U_1, \dots, U_k \subset V$ be disjoint open subsets. Let $W \subset (\mathbb{R}^n)^k$ be the set of those $x_1, \dots, x_k$ such that the sets $T_{x_1}(U_1), \dots, T_{x_k}(U_k)$ are all disjoint and contained in V. It parametrizes the way we can move the open sets without causing overlaps. Let us assume that W has non-empty interior, which happens when the closure of the U_i are disjoint and contained in V.

Let $\mathcal{F}$ be any discretely translation-invariant prefactorization algebra. Then, for each $(x_1, \dots, x_k) \in W$, we have a multilinear map obtained as a composition

$$m_{x_1,\dots,x_k} : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_k) \to \mathcal{F}(T_{x_1}U_1) \times \cdots \times \mathcal{F}(T_{x_k}U_k) \to \mathcal{F}(V),$$

where the second map arises from the inclusion

$$T_{x_1}U_1 \amalg \cdots \amalg T_{x_k}U_k \hookrightarrow V.$$

2.7.1.3 Definition. *A discretely translation invariant prefactorization algebra $\mathcal{F}$ is* smoothly translation invariant *if the following conditions hold.*

1. *(1) The map $m_{x_1,\dots,x_k}$ above depends smoothly on $(x_1, \dots, x_k) \in W$.*
2. *(2) The prefactorization algebra $\mathcal{F}$ is equipped with an action of the Abelian Lie algebra $\mathbb{R}^n$ of translations. If $v \in \mathbb{R}^n$, we will denote the corresponding action maps by*

$$\frac{\mathrm{d}}{\mathrm{d}v} : \mathcal{F}(U) \to \mathcal{F}(U).$$

We view this Lie algebra action as an infinitesimal version of the global translation invariance.

(3) *The infinitesimal action is compatible with the global translation invariance in the following sense. If $v \in \mathbb{R}^n$, let $v_i \in (\mathbb{R}^n)^k$ denote the vector with v placed in the ith position and 0 in the other $k-1$ slots. If $\alpha_i \in \mathcal{F}(U_i)$, then we require that*

$$\frac{\mathrm{d}}{\mathrm{d}v_i} m_{x_1,\dots,x_k}(\alpha_1,\dots,\alpha_k) = m_{x_1,\dots,x_k}\left(\alpha_1,\dots,\frac{\mathrm{d}}{\mathrm{d}v}\alpha_i,\dots,\alpha_k\right).$$

When we refer to a translation invariant prefactorization algebra without further qualification, we will always mean a smoothly translation invariant prefactorization algebra.

Remark: As always, we work with prefactorization algebras taking values in the category of differentiable cochain complexes. Generalities about differentiable cochain complexes are developed in appendix A. There we explain what it means for a smooth multilinear map between differentiable cochain complexes to depend smoothly on some parameters.

2.7.2. Next, we will explain how to think of the structure of a translation-invariant prefactorization algebra on $\mathbb{R}^n$ in more operadic terms. This description has a lot in common with the E_n algebras familiar from topology.

Let $r_1,\dots,r_k,s \in \mathbb{R}_{>0}$. Let

$$\mathrm{Discs}_n(r_1,\dots,r_k \mid s) \subset (\mathbb{R}^n)^k$$

be the (possibly empty) open subset consisting of $x_1,\dots,x_k \in \mathbb{R}^n$ with the property that the closures of the balls $B_{r_i}(x_i)$ are all disjoint and contained in $B_s(0)$ (where $B_r(x)$ denotes the open ball of radius r around x).

2.7.2.1 Definition. *Let* Discs_n *be the* $\mathbb{R}_{>0}$*-colored operad in the category of smooth manifolds whose space of k-ary morphisms is the space* $\mathrm{Discs}_n(r_1,\dots,r_k \mid s)$ *between* $r_i, s \in \mathbb{R}_{>0}$ *described above.*

Note that a colored operad is the same thing as a multicategory (recall remark 2.1.2). An $\mathbb{R}_{>0}$-colored operad is thus a multicategory whose set of objects is $\mathbb{R}_{>0}$.

The essential data of the colored operad structure on the spaces $\mathrm{Discs}_n(r_1, \ldots, r_k \mid s)$ is the following. We have maps

$$\circ_i : \mathrm{Discs}_n(r_1, \ldots, r_k \mid t_i) \times \mathrm{Discs}_n(t_1, \ldots, t_m \mid s) \to \mathrm{Discs}_n(t_1, \ldots, t_{i-1}, r_1, \ldots, r_k, t_{i+1}, \ldots, t_m \mid s).$$

This map is defined by inserting the outgoing ball (of radius t_i) of a configuration $x \in \mathrm{Discs}_n(r_1, \ldots, r_k \mid t_i)$ into the ith incoming ball of a point $y \in \mathrm{Discs}_n(t_1, \ldots, t_k \mid s)$.

These maps satisfy the natural associativity and commutativity properties of a multicategory.

2.7.3. Next, let $\mathcal{F}$ be a translation-invariant prefactorization algebra on $\mathbb{R}^n$. Let

$$\mathcal{F}_r = \mathcal{F}(B_r(0))$$

denote the cochain complex $\mathcal{F}$ that assigns to a ball of radius r. This notation is reasonable because translation invariance gives us an isomorphism between $\mathcal{F}(B_r(0))$ and $\mathcal{F}(B_r(x))$ for any $x \in \mathbb{R}^n$.

The structure maps for a translation invariant prefactorization algebra yield, for each $p \in \mathrm{Discs}_n(r_1, \ldots, r_k \mid s)$, multiplication operations

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

The map $m[p]$ is a smooth multilinear map of differentiable spaces; and furthermore, this map depends smoothly on p.

These operations make the complexes $\mathcal{F}_r$ into an algebra over the $\mathbb{R}_{>0}$-colored operad $\mathrm{Discs}_n(r_1, \ldots, r_k \mid s)$, valued in the multicategory of differentiable cochain complexes. In addition, the complexes $\mathcal{F}_r$ are endowed with an action of the Abelian Lie algebra $\mathbb{R}^n$. This action is by derivations of the Discs_n-algebra $\mathcal{F}$ compatible with the action of translation on Discs_n, as described above.

2.7.4. Now, let us unravel explicitly what it means to be such a Discs_n algebra.

The first property is that, for each $p \in \mathrm{Discs}_n(r_1, \ldots, r_k \mid s)$, the map $m[p]$ is a multilinear map, of cohomological degree 0, compatible with differentials.

Second, let N be a manifold and let $f_i : N \to \mathcal{F}_{r_i}^{d_i}$ be smooth maps into the space $\mathcal{F}_{r_i}^{d_i}$ of elements of degree d_i. The smoothness properties of the map $m[p]$ mean that the map

$$\begin{aligned} N \times \mathrm{Discs}_n(r_1, \dots, r_k \mid s) &\to \mathcal{F}_s \\ (x, p) &\mapsto m[p](f_1(x), \dots, f_k(x)) \end{aligned}$$

is smooth.

Next, note that a permutation $\sigma \in S_k$ gives an isomorphism

$$\sigma : \mathrm{Discs}_n(r_1, \dots, r_k \mid s) \to \mathrm{Discs}_n(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s).$$

We require that, for each $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$ and each $\alpha_i \in \mathcal{F}_{r_i}$,

$$m[\sigma(p)](\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}) = m[p](\alpha_1, \dots, \alpha_k).$$

Finally, we require that the maps $m[p]$ are compatible with composition, in the following sense. For $p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i)$, $q \in \mathrm{Discs}_n(t_1, \dots, t_l \mid s)$, $\alpha_i \in F_{r_i}$, and $\beta_j \in \mathcal{F}_{t_j}$, we require that

$$\begin{aligned} m[q]\,(\beta_1, \dots, \beta_{i-1}, m[p](\alpha_1, \dots, \alpha_k), \beta_{i+1}, \dots, \beta_l) \\ = m[q \circ_i p](\beta_1, \dots, \beta_{i-1}, \alpha_1, \dots, \alpha_k, \beta_{i+1}, \dots, \beta_l). \end{aligned}$$

In addition, the action of $\mathbb{R}^n$ on each $\mathcal{F}^r$ is compatible with these multiplication maps, in the way described above.

2.7.5. Let us give one more equivalent way of rewriting these axioms, which will be useful when we discuss the holomorphic context. These alternative axioms will say that the spaces $C^\infty(\mathrm{Discs}_n(r_1, \dots, r_k \mid s))$ form an $\mathbb{R}_{>0}$-colored co-operad when we use the appropriate completed tensor product. Since we know how to tensor a differentiable vector space with the space of smooth functions on a manifold, it makes sense to talk about an algebra over this colored co-operad in the category of differentiable cochain complexes.

The smoothness axiom for the product map

$$m[p] : \mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_k} \to \mathcal{F}_s,$$

where$p \in \mathrm{Discs}_n(r_1, \dots, r_k \mid s)$, can be rephrased as follows. For any differentiable vector space V and smooth manifold M, we use the notation $V \otimes C^\infty(M)$ interchangeably with the notation $C^\infty(M, V)$; both indicate the differentiable vector space of smooth maps $M \to V$. The smoothness axiom states that the map above extends to a smooth map of differentiable spaces

$$\mu(r_1, \dots, r_k \mid s) : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s \otimes C^\infty(\mathrm{Discs}_n(r_1, \dots, r_k \mid s)).$$

In general, if $V_1, \dots, V_k, W$ are differentiable vector spaces and if X is a smooth manifold, let

$$C^\infty(X, \mathrm{Hom}(V_1, \dots, V_k \mid W))$$

denote the space of smooth multilinear maps

$$V_1 \times \cdots \times V_k \to C^\infty(X, W).$$

Note that there is a natural gluing map

$$\circ_i : C^\infty(X, \mathrm{Hom}(V_1, \dots, V_k \mid W_i)) \times C^\infty(Y, \mathrm{Hom}(W_1, \dots, W_l \mid T)) \\ \to C^\infty(XxY, \mathrm{Hom}(W_1, \dots, W_{i-1}, V_1, \dots, V_k, W_{i+1}, \dots, W_l \mid T)).$$

With this notation in hand, there are elements

$$\mu(r_1, \dots, r_k \mid s) \in C^\infty\left(\mathrm{Discs}(r_1, \dots r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)\right)$$

with the following properties.

(1) $\mu(r_1, \dots, r_k \mid s)$ is closed under the natural differential, arising from the differentials on the cochain complexes $\mathcal{F}_{r_i}$.

(2) If $\sigma \in S_k$, then

$$\sigma_* \mu(r_1, \dots, r_k \mid s) = \mu(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s)$$

where

$$\sigma_* : C^\infty(\mathrm{Discs}(r_1, \dots r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)) \\ \to C^\infty(\mathrm{Discs}(r_{\sigma(1)}, \dots r_{\sigma(k)} \mid s), \mathrm{Hom}(\mathcal{F}_{r_{\sigma(1)}}, \dots, \mathcal{F}_{r_{\sigma(k)}} \mid \mathcal{F}_s))$$

is the natural isomorphism.

(3) As before, let

$$\circ_i : \mathrm{Discs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{Discs}_n(t_1, \dots, t_m \mid s) \\ \to \mathrm{Discs}_n(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s).$$

denote the gluing map. Then, we require that

$$\circ_i^* \mu(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_l) = \mu(r_1, \dots, r_k \mid t_i) \circ_i \mu(t_1, \dots, t_l \mid s).$$

These elements equip the $\mathcal{F}_r$ with the structure of an algebra over the colored co-operad, as stated earlier.

2.7.6. Our next focus explain what it means for a (smoothly) translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ to be holomorphically translation invariant. For this definition to make sense, we require that $\mathcal{F}$ is defined over $\mathbb{C}$: that is, the vector spaces $\mathcal{F}(U)$ are complex vector spaces and all structure maps are complex linear.

Recall that such a factorization algebra has, as part of its structure, an action of the real Lie algebra $\mathbb{R}^{2n} = \mathbb{C}^n$ by derivations. This action is as a real Lie algebra; since $\mathcal{F}$ is defined over $\mathbb{C}$, the action extends to an action of the complexified translation Lie algebra $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. We will denote the action maps by

$$\frac{\mathrm{d}}{\mathrm{d}z_i}, \frac{\mathrm{d}}{\mathrm{d}\bar{z}_j} : \mathcal{F}(U) \to \mathcal{F}(U).$$

2.7.6.1 Definition. *A translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ is* holomorphically translation invariant *if it is equipped with derivations $\eta_i : \mathcal{F} \to \mathcal{F}$ of cohomological degree -1, for $i = 1 \dots n$, with the property that*

$$\mathrm{d}\eta_i = \frac{\mathrm{d}}{\mathrm{d}\bar{z}_i} \in \mathrm{Der}(\mathcal{F}).$$

Here, d *refers to the differential on the dg Lie algebra* $\mathrm{Der}(\mathcal{F})$.

We should understand this definition as saying that the vector fields $\frac{\mathrm{d}}{\mathrm{d}\bar{z}_i}$ act homotopically trivially on $\mathcal{F}$.

2.7.7. Now we will interpret holomorphically translation invariant prefactorization algebras in the language of $\mathbb{R}_{>0}$-colored operads. When we work in complex geometry, it is better to use polydiscs instead of balls, as is standard in complex analysis.

Thus, if $z \in \mathbb{C}^n$, let

$$PD_r(z) = \{w \in \mathbb{C}^n \mid |w_i - z_i| < r\}$$

be the polydisc of radius r around z. Let

$$\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{C}^n)^k$$

be the space of $z_1, \dots, z_k \in \mathbb{C}^n$ with the property that the closures of the polydiscs $PD_{r_i}(z_i)$ are disjoint and contained in the polydisc $PD_s(0)$.

It is clear that the spaces $\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)$ form a $\mathbb{R}_{>0}$-colored operad in the category of complex manifolds.

Now, let $\mathcal{F}$ be a holomorphically translation invariant prefactorization algebra on $\mathbb{C}^n$. Let $\mathcal{F}_r$ denote the differentiable cochain complex $\mathcal{F}(PD_r(0))$ associated to the polydisc of radius r.

Then, as above, for each $p \in \mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)$ we have a map

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

This map is smooth, multilinear, and compatible with the differential. Further, this map varies smoothly with p.

The fact that $\mathcal{F}$ is a holomorphically translation invariant prefactorization algebra means that these maps are equipped with extra structure. We have derivations η_j of $\mathcal{F}$ which make the derivations $\frac{\mathrm{d}}{\mathrm{d}\overline{z}_j}$ homotopically trivial.

For $i = 1, \dots, k$ and $j = 1, \dots, n$, let $z_{ij}, \overline{z}_{ij}$ refer to coordinates on $(\mathbb{C}^n)^k$, and so on the open subset

$$\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{C}^n)^k.$$

Thus, we have operations

$$\frac{\mathrm{d}p}{\mathrm{d}\overline{z}_{ij}} m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s.$$

Let $m[p] \circ_i \eta_j$ denote the operation

$$\begin{aligned} m[p] \circ_i \eta_j : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} &\to \mathcal{F}_s \\ \alpha_1 \times \cdots \times \alpha_k &\mapsto \pm m[p](\alpha_1, \dots, \eta_j \alpha_i, \dots, \alpha_k) \end{aligned}$$

(where $\pm$ indicates the usual Koszul rule of signs).

Then, the identity

$$\left[\mathrm{d}, m[p] \circ_i \eta_j\right] = \frac{\mathrm{d}p}{\mathrm{d}\overline{z}_{ij}} m[p]$$

holds, describing the fact that the product map $m[p]$ is holomorphic in p, up to a homotopy given by η_i.

2.7.8. In the smooth case, we saw that we could describe the structure as that of an algebra over a $\mathbb{R}_{>0}$-colored co-operad built from smooth functions on the spaces $\mathrm{Discs}_n(r_1, \ldots, r_k \mid s)$. In this section we will see that there is an analogous story in the complex world, where we use the Dolbeault complex of the spaces $\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)$.

Let us first introduce some notation. For any complex manifold X, and any collection $V_1, \ldots, V_k, W$ of differentiable cochain complexes over $\mathbb{C}$, let

$$\Omega^{0,*}(X, \mathrm{Hom}(V_1, \ldots, V_k \mid W_i))$$

denote the cochain complex of smooth multilinear maps

$$V_1 \times \cdots \times V_k \to \Omega^{0,*}(X, W).$$

Recall that

$$\Omega^{0,*}(X, W) = C^\infty(X, W) \otimes_{C^\infty(X)} \Omega^{0,*}(W);$$

as we see in the appendix, a differentiable vector space W has enough structure to define the $\bar{\partial}$ operator on $\Omega^{0,*}(X, W)$. The differential on $\Omega^{0,*}(X, \mathrm{Hom}(V_1, \ldots, V_k \mid W_i))$ is a combination of the Dolbeault differential on X with the differentials on the differentiable cochain complexes V_i, W.

Let

$$\circ_i : \mathrm{PDiscs}_n(r_1, \ldots, r_k \mid t_i) \times \mathrm{PDiscs}_n(t_1, \ldots, t_m \mid s) \to \mathrm{PDiscs}_n(t_1, \ldots, t_{i-1}, r_1, \ldots, r_k, t_{i+1}, \ldots, t_m \mid s).$$

be the composition map, which is a holomorphic map of complex manifolds. We let

$$\circ_i^* : \Omega^{0,*}(\mathrm{PDiscs}_n(t_1, \ldots, t_{i-1}, r_1, \ldots, r_k, t_{i+1}, \ldots, t_m \mid s)) \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid t_i) \times \mathrm{PDiscs}_n(t_1, \ldots, t_m \mid s)$$

be the corresponding pullback map on Dolbeault complexes.

2.7.8.1 Proposition. *Let $\mathcal{F}$ be a holomorphically translation invariant factorization algebra on $\mathbb{C}^n$. Then, the product maps*

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \to \mathcal{F}_s$$

for $p \in \mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s)$ lift to closed elements

$$\mu^{\bar{\partial}}(r_1, \ldots, r_k \mid s) \in \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid \mathcal{F}_s))$$

satisfying the following properties.

(1) $\mu^{\overline{\partial}}(r_1, \dots, r_k \mid s)$ *is closed under the natural differential on*

$$\Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s))$$

which incorporates the Dolbeault differential as well as the internal differentials on the the complexes $\mathcal{F}_{r_i}, \mathcal{F}_s$.

(2) *Let* $\sigma \in S_k$. *Then, as in the smooth case,*

$$\sigma_* \mu^{\overline{\partial}}(r_1, \dots, r_k \mid s) = \mu^{\overline{\partial}}(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s)$$

where

$$\sigma_* : \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid \mathcal{F}_s)) \\ \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_{\sigma(1)}, \dots, r_{\sigma(k)} \mid s), \mathrm{Hom}(\mathcal{F}_{r_{\sigma(1)}}, \dots, \mathcal{F}_{r_{\sigma(k)}} \mid \mathcal{F}_s))$$

is the natural isomorphism.

(3) *For all* $1 \leq i \leq m$,

$$\circ_i^* \mu^{\overline{\partial}}(t_1, \dots, t_{i-1}, r_1, \dots, r_k, t_{i+1}, \dots, t_m \mid s) = \mu^{\overline{\partial}}(t_1, \dots, t_m \mid s) \circ_i \mu^{\overline{\partial}}(r_1, \dots, r_k \mid t_i) \\ \in \Omega^{0,*}\left(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid t_i) \times \mathrm{PDiscs}_n(t_1, \dots, t_m \mid s), \right. \\ \left. \mathrm{Hom}(\mathcal{F}_{t_1} \times \dots \times \mathcal{F}_{t_{i-1}} \times \mathcal{F}_{r_1} \times \dots \mathcal{F}_{r_k} \times \mathcal{F}_{t_{i+1}} \times \dots \times \mathcal{F}_{t_m} \mid \mathcal{F}_s)\right).$$

Essentially, this proposition asserts that we can construct a co-operad from the complexes $\Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s))$ and that this co-operad acts on the differentiable cochain complexes $\mathcal{F}_{r_i}$.

PROOF. We will produce this action starting from the operations

$$\mu^0(r_1, \dots, r_k \mid s) \in C^\infty(\mathrm{PDiscs}(r_1, \dots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \dots, \mathcal{F}_{r_k} \mid s))$$

that we already have because $\mathcal{F}$ is a smoothly translation-invariant factorization algebra.

First, we need to introduce some notation. Recall that

$$\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{C}^n)^k$$

is an open (possibly empty) subset. Thus,

$$\Omega^{0,*}\left(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)\right) = \Omega^{0,0}\left(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s)\right) \otimes \mathbb{C}[\mathrm{d}\bar{z}_{ij}]$$

where the $\mathrm{d}\bar{z}_{ij}$ are commuting variables of cohomological degree 1, with $i = 1, \ldots, k$ and $j = 1, \ldots, n$. We let $\frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}\bar{z}_{ij})}$ denote the graded derivation which removes $\mathrm{d}\bar{z}_{ij}$.

As before, let $\eta_j : \mathcal{F}_r \to \mathcal{F}_r$ denote the derivation which cobounds the derivation $\frac{\mathrm{d}}{\mathrm{d}\bar{z}_j}$. We can compose any element

$$\alpha \in \Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s))$$

with η_j acting on $\mathcal{F}_{r_i}$, to get

$$\alpha \circ_i \eta_j \in \Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s)).$$

Then, the cochains $\mu^{\bar{\partial}}(r_1, \ldots, r_k \mid s)$ are characterized by the following properties.

(1) When restricted to $\Omega^{0,0}(\mathrm{PDiscs}_n(r_1, \ldots, r_k \mid s))$ they are the elements constructed from the fact that $\mathcal{F}$ is a smoothly translation-invariant factorization algebra.
(2) The identity

$$\frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}\bar{z}_{ij})} \mu^{\bar{\partial}}(r_1, \ldots, r_k \mid s) = \mu^{\bar{\partial}}(r_1, \ldots, r_k \mid s) \circ_i \eta_j$$

holds for all i and j.

It is easy to verify that there is a unique $\mu^{\bar{\partial}}$ satisfying these properties.

Next, we need to verify that $\mathrm{d}\mu^{\bar{\partial}}(r_1, \ldots, r_k \mid s) = 0$, where d is the differential on

$$\Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s)).$$

To see this, note that, for all

$$\alpha \in \Omega^{0,*}(\mathrm{PDisc}_n(r_1, \ldots, r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1}, \ldots, \mathcal{F}_{r_k} \mid s)),$$

$$\begin{aligned} \mathrm{d}\frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}\bar{z}_{ij})}\alpha + \frac{\mathrm{d}}{\mathrm{d}(\mathrm{d}\bar{z}_{ij})}\mathrm{d}\alpha &= \frac{\mathrm{d}}{\mathrm{d}\bar{z}_{ij}}\alpha \\ \mathrm{d}(\alpha \circ_i \eta_j) - (\mathrm{d}\alpha) \circ_i \eta_j &= (-1)^{|\alpha|} \alpha \circ_i \frac{\mathrm{d}}{\mathrm{d}\bar{z}_j}, \end{aligned}$$

where on the right hand side of the second equation $\frac{\mathrm{d}}{\mathrm{d}\bar{z}_j}$ indicates the action of this vector field on $\mathcal{F}_{r_i}$.

Now, we know that

$$\mu^0(r_1,\dots,r_k \mid s) \in \Omega^{0,0}(\mathrm{PDisc}_n(r_1,\dots,r_k \mid s), \mathrm{Hom}(\mathcal{F}_{r_1},\dots,\mathcal{F}_{r_k} \mid s))$$

satisfies

$$\mathrm{d}\mu^0(r_1,\dots,r_k \mid s) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}_{ij}}\mu^0(r_1,\dots,r_k \mid s) = \mu^0(r_1,\dots,r_k \mid s) \circ_i \frac{\mathrm{d}}{\mathrm{d}\bar{z}_j}.$$

It follows from these identities that $\mu^{\bar\partial}(r_1,\dots,r_k \mid s)$ is closed.

It is straightforward to verify that the elements $\mu^{\bar\partial}$ are compatible with composition and with the symmetric group actions. □

CHAPTER 3

Factorization algebras: definition and formal aspects

Our definition of a prefactorization algebra is closely related to that of a precosheaf or of a presheaf. Mathematicians have found it useful to refine the axioms of a presheaf to that of a sheaf: a sheaf is a presheaf whose value on a large open set is determined, in a precise way, by values on arbitrarily small subsets. In this chapter we describe a similar "descent" axiom for prefactorization algebras. We call a prefactorization algebra satisfying this axiom a factorization algebra.

After defining this axiom, our next task is to verify that the examples we have constructed so far (such as the free field) satisfy it. This we do in section 3.2.

The rest of this chapter is devoted to exploring formal aspects of the theory of factorization algebras: we show how factorization algebras form a (simplicially-enriched) multicategory, and construct push-forward and pull-back functors in certain cases.

Philosophically, our descent axiom for factorization algebras is important: a prefactorization algebra satisfying descent (i.e. a factorization algebra) is built from local data, in a way that a general prefactorization algebra is not. However, for practical purposes, this axiom is often not essential.

Thus, a reader with little taste for formal mathematics could skip this chapter and still be able to follow the rest of this book.

3.1. Factorization algebras

A factorization algebra is a prefactorization algebra that satisfies the *local-to-global* axiom. This axiom is the analog of the gluing axiom for sheaves; it expresses how the values on big open sets are determined by the values on small open sets. For sheaves, the gluing axiom says that for any open set U and any cover of that open set, we can determine the value of the sheaf on U from the values on the open cover. For factorization algebras, we require our covers to be fine enough that they capture all the "multiplicative structure."

We will describe the local-to-global axiom for factorization algebras taking values in vector spaces or chain complexes, but the generalization to an arbitrary symmetric monoidal category is straightforward.

3.1.0.2 Definition. *Let U be an open set and $\mathfrak{U} := \{U_i \mid i \in I\}$ a cover of U by open sets. The cover $\mathfrak{U}$ is* factorizing *if for any finite collection of points $\{x_1, \ldots, x_k\}$ in U, there is a finite collection of pairwise disjoint opens $\{U_{i_1}, \ldots, U_{i_n}\}$ from the cover such that $\{x_1, \ldots, x_k\} \subset U_{i_1} \cup \cdots \cup U_{i_n}$.*

Remark: Every Hausdorff space admits a nontrivial factorizing cover (i.e., a cover not containing the whole space as an element).

Remark: For a smooth n-manifold M, there is a simple way to construct a factorizing basis for M. Namely, fix a Riemannian metric on M, and consider

$$\{B_r(x) \,:\, \forall x \in M, \text{ with } 0 < r < InjRad(x)\},$$

the collection of open balls, running over each point $x \in M$, whose radii are less than the injectivity radius at x. Another construction is simply to take the collection of open sets in M diffeomorphic to the open n-ball.

3.1.1. Strict factorization algebras. The value of a factorization algebra on U is determined by its behavior on a factorizing cover, just as the value of a cosheaf on an open set U is determined by its value on any cover of U.

In order to motivate our definition of factorization algebra, let us write briefly recall the cosheaf axiom. A precosheaf Φ on M is a cosheaf if, for every open cover $\{U_i \mid i \in I\}$ of an open set $U \subset M$, the sequence

$$\oplus_{i,j}\Phi(U_i \cap U_j) \to \oplus_k \Phi(U_k) \to \Phi(U)$$

is exact on the right. (Alternatively, one can say the map $\oplus\Phi(U_k) \to \Phi(U)$ coequalizes the pair of maps $\oplus\Phi(U_i \cap U_j) \rightrightarrows \oplus\Phi(U_k)$.)

We will define the notion of factorization algebra in a similar way, except that instead of considering elements U_i of the cover, one considers finite collections of disjoint elements of the cover.

In order to make this precise, we need to introduce some notation. Let PI denote the set of finite subsets $\alpha \subset I$, with the property that if $j, j' \in \alpha$, $U_j \cap U_{j'} = \emptyset$. These are the tuples of open subsets that appear in the structure maps of a prefactorization algebra.

If $\alpha \in PI$, let us define $\mathcal{F}(\alpha)$ by

$$\mathcal{F}(\alpha) = \mathcal{F}(\amalg_{j \in \alpha} U_j).$$

Similarly, if $\alpha_1, \ldots, \alpha_k \in PI$, we will let

$$\mathcal{F}(\alpha_1, \ldots, \alpha_k) = \mathcal{F}\left(\amalg_{j_1 \in \alpha_1, \ldots, j_k \in \alpha_k} U_{j_1} \cap \cdots \cap U_{j_k}\right).$$

Note that there are natural maps

$$p_i : \mathcal{F}(\alpha_1, \ldots, \alpha_k) \to \mathcal{F}(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k)$$

for each $1 \leq i \leq k$.

3.1.1.1 Definition. *A prefactorization algebra is a* lax factorization algebra *if it has the property: For every open subset $U \subset M$ and every factorizing cover $\{U_i \mid i \in I\}$ of U, the sequence*

$$\oplus_{\alpha_1, \alpha_2 \in PI} \mathcal{F}(\alpha_1, \alpha_2) \xrightarrow{p_1 - p_2} \oplus_{\beta \in PI} \mathcal{F}(\beta) \to \mathcal{F}(U)$$

is exact on the right.

A lax factorization algebra is a strict factorization algebra *if, in addition, for every pair of disjoint open sets $U, V \in M$, the natural map*

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \amalg V)$$

is an isomorphism.

3.1.2. The Čech complex and homotopy factorization algebras. Now suppose we have a prefactorization algebra $\mathcal{F}$, taking values in complexes. We will define what it means for $\mathcal{F}$ to be a *homotopy factorization algebra*. This will happen when $\mathcal{F}(U)$ is quasi-isomorphic to a certain Čech complex constructed from any factorizing cover.

To motivate the definition, let us first recall the definition of a homotopy cosheaf. Let Φ be a pre-cosheaf on M, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a cover of some open subset U of M. The Čech complex of $\mathfrak{U}$ with coefficients in Φ is is defined in the usual way, as

$$\oplus_k \oplus_{j_1, \ldots, j_k \in I} \Phi(U_{j_1} \cap \cdots \cap U_{j_k})[k-1]$$

where the differential is defined in the usual way. We say that Φ is a homotopy cosheaf if the natural map from the Čech complex to $\Phi(U)$ is a quasi-isomorphism, for every open $U \subset M$ and every open cover of U.

Now let $\mathcal{F}$ be a prefactorization algebra on M, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be a factorizing cover of an open subset $U \subset M$. The Čech complex of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ is defined by

$$\check{C}(\mathfrak{U}, \mathcal{F}) = \oplus_{k \geq 0} \oplus_{\alpha_1, \dots, \alpha_k \in PI} \mathcal{F}(\alpha_1, \dots, \alpha_k)[k-1]$$

with differential defined as the alternating sum of the restriction maps $\mathcal{F}(\alpha_1, \dots, \alpha_k) \to \mathcal{F}(\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_k)$, as in the Čech complex for a homotopy cosheaf.

3.1.2.1 Definition. *A lax homotopy factorization algebra on X is a prefactorization algebra $\mathcal{F}$ valued in cochain complexes, with the property that for every open set $U \subset X$, factorizing cover $\mathfrak{U}$ of U, the natural map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a quasi-isomorphism.

A lax homotopy factorization algebra is a homotopy factorization algebra if, in addition, for every pair U, V of disjoint open subsets of X, the natural map

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \amalg V)$$

is a quasi-isomorphism.

Remark: The notion of strict factorization algebra is not appropriate for the world of cochain complexes. Whenever we refer to a factorization algebra in cochain complexes, we will mean a homotopy factorization algebra.

3.1.3. Factorization algebras valued in a multicategory. The factorization algebras of ultimate interest to us take values, not in the symmetric monoidal category of cochain complexes, but in the multicategory of differentiable cochain complexes (section A).

Thus, we need to define what it means to be a factorization algebra in a multicategory $\mathcal{C}$. We will assume that $\mathcal{C}$ is equipped with a realization functor from the category $\mathcal{C}^{\triangle}$ of simplicial objects of $\mathcal{C}$ to the original category $\mathcal{C}$. This will allow us to define the Čech complex of an object of $\mathcal{C}$. (In the category of differentiable cochain complexes, the geometric realization of a simplicial object is defined in the same way as it is in the category of ordinary cochain complexes).

We will also assume that $\mathcal{C}$ is equipped with some notion of weak equivalence (weak equivalences in differentiable cochain complexes are defined in section A).

As before, let $\mathrm{Op}(X)$ be the multicategory whose objects are open sets in X and whose multi-morphisms are defined by

$$\mathrm{Hom}(U_1, \dots, U_n; V) = \begin{cases} * & \text{if } U_1 \amalg \dots U_n \subset V \\ \emptyset & \text{otherwise.} \end{cases}$$

3.1.3.1 Definition. *Let $\mathcal{C}$ be a multicategory with the structures listed above. A factorization algebra $\mathcal{F}$ with values in $\mathcal{C}$ is a functor $\mathrm{Op}(X) \to \mathcal{C}$ with the property that, for all open subsets $U \subset X$, and all factorizing open covers $\mathfrak{U}$ of U, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a weak equivalence.

Remark: Suppose that $\mathcal{C}$ is the multicategory underlying a symmetric monoidal category $\mathcal{C}^{\otimes}$. Then a factorization algebra valued in $\mathcal{C}$, considered as a multicategory, is the same as a lax factorization algebra valued in $\mathcal{C}^{\otimes}$, considered as a symmetric monoidal category.

3.1.3.2 Definition. *A differentiable factorization algebra is a factorization algebra valued in the multicategory of differentiable cochain complexes.*

3.1.4. Factorization algebras in quantum field theory. We have seen (section 1.4) how prefactorization algebras appear naturally when one thinks about the structure of observables of a quantum field theory. It is natural to ask whether the local-to-global axiom which distinguishes factorization algebras from prefactorization algebras also has a quantum-field theoretic interpretation.

The local-to-global axiom we posit states, roughly speaking, that all observables on an open set $U \subset M$ can be built up as sums of observables supported on arbitrarily small open subsets of M. To be concrete, let us consider a factorizing cover $\mathfrak{U}_\varepsilon$ of M, consisting of all balls in M of radius $< \varepsilon$. Applied to this factorizing cover, our local-to-global axiom states that any observable $O \in \mathrm{Obs}(U)$ can be written as a sum of observables of the form $O_1 O_2 \cdots O_k$, where $O_i \in \mathrm{Obs}(B_{\delta_i}(x_i))$ and $x_1, \dots, x_k \in M$.

By taking ε to be very small, we see that our local-to-global axiom implies that all observables can be written as sums of products of observables which are supported as close as we like to points in U.

This is a physically reasonable assumption: most of the observables (or operators) which are considered in quantum field theory textbooks are supported at points, so it might make sense to restrict attention to observables built from these.

However, more global observables are also considered in the physics literature. For example, in a gauge theory, one might consider the observable which measures the monodromy of a connection around some loop in the space-time manifold. How would such observables fit into the factorization algebra picture?

The answer reveals a key limitation of our axioms: *the concept of factorization algebra is only appropriate for perturbative quantum field theories*. Indeed, in a perturbative gauge theory, the gauge field (i.e., the connection) is taken to be an infinitesimally small perturbation $A_0 + \delta A$ of a fixed connection A_0, which is a solution to the equations of motion. There is a well-known formula (the time-ordered exponential) expressing the holonomy of $A_0 + \delta A$ as a power series in δA, where the coefficients of the power series are given as integrals over L^k, where L is the loop which we are considering.

This expression shows that the holonomy of $A_0 + \delta A$ can be built up from observables supported at points (which happen to lie on the loop L). This, the holonomy observable will form part of our factorization algebra.

However, if we are not working in a perturbative setting, this formula does not apply, and we would not expect (in general) that the prefactorization algebra of observables satisfies the local-to-global axiom.

3.2. Factorization algebras from cosheaves

The goal of this section is to describe a natural class of factorization algebras. The factorization algebras which we construct from classical and quantum field theory will be closely related to the factorization algebras discussed here.

The main result of this section is that, given a nice cosheaf of vector spaces or cochain complexes F on a manifold M, the functor $\mathrm{Sym}^* F : U \mapsto \mathrm{Sym}(F(U))$ is a factorization algebra. It is clear how this functor is a prefactorization algebra; the hard part is verifying when it satisfies the local-to-global axiom. The examples we are ultimately interested in arise from cosheaves F which are compactly supported sections of a vector bundle, so we will focus on cosheaves like this.

We begin by providing the definitions necessary to state the main result of this section. We then state the main result and explain its role for the rest of the book. Finally, we prove the lemmas that culminate in the proof of the main result.

3.2.1. Preliminary definitions.

3.2.1.1 Definition. *A* local cochain complex *on M is a graded vector bundle E on M, whose sections will be denoted by $\mathscr{E}$, equipped with a differential operator $\mathrm{d} : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 satisfying $\mathrm{d}^2 = 0$.*

Let E be a local cochain complex on M, and let U be an open subset of M. We let $\mathscr{E}(U)$ denote the cochain complex of smooth sections of E on U, and $\mathscr{E}_c(U)$ denote the cochain complex of compactly supported sections of E on U. Similarly, let $\overline{\mathscr{E}}(U)$ denote the distributional sections on U, defined by

$$\overline{\mathscr{E}}(U) = \mathscr{E}(U) \otimes_{C^\infty(U)} \mathcal{D}(U)$$

where $\mathcal{D}(U)$ is the space of distributions on U. Let $\overline{\mathscr{E}}_c(U)$ denote the compactly supported distributional sections of E on U. In the appendix (A) it is shown that these four cochain complexes are differentiable cochain complexes in a natural way.

If, as above, E is a graded vector bundle on M, let $E^! = E \otimes \mathrm{Dens}_M$. Let us give $E^!$ a differential which is the formal adjoint to that on E. Let $\mathscr{E}^!(U)$, $\mathscr{E}_c^!(U)$ denote the cochain complexes of smooth and compactly supported sections of $E^!$, and let $\overline{\mathscr{E}}^!(U)$

and $\overline{\mathscr{E}}^!_c(U)$ denote the cochain complexes of distributional and compactly-supported distributional sections of $E^!$.

Note that $\overline{\mathscr{E}}_c(U)$ is the continuous dual to $\mathscr{E}^!(U)$, and that $\mathscr{E}_c(U)$ is the continuous dual to $\overline{\mathscr{E}}^!(U)$.

The factorization algebras we will discuss are constructed from the symmetric algebra on the vector spaces $\mathscr{E}_c(U)$ and $\overline{\mathscr{E}}_c(U)$. This symmetric algebra can be defined in two ways: either using the completed projective tensor product of topological vector spaces, or in terms of sections of bundles on U^n. For concreteness, we will discuss the latter construction.

Thus, let us define $(\mathscr{E}_c(U))^{\boxtimes n}$ to be the tensor power defined using the completed projective tensor product on the topological vector space $\mathscr{E}_c(U)$. Concretely, if $E^{\boxtimes n}$ denotes the vector bundle on M^n obtained as the external tensor product, then

$$(\mathscr{E}_c(U))^{\boxtimes n} = \Gamma_c(U^n, E^{\boxtimes n})$$

is the compactly supported smooth sections of $E^{\boxtimes n}$ on U^n. Similarly, we have

$$(\overline{\mathscr{E}}_c(U))^{\boxtimes n} = \overline{\Gamma}_c(U^n, E^{\boxtimes n})$$

is the compactly supported distributional sections of $E^{\boxtimes n}$.

The symmetric powers $\operatorname{Sym}^n \mathscr{E}_c(U)$ and $\operatorname{Sym}^n \overline{\mathscr{E}}_c(U)$ are defined as the coinvariants of the symmetric group action on the tensor powers. Finally, we can define the symmetric and completed symmetric algebra of $\mathscr{E}_c(U)$ and $\overline{\mathscr{E}}_c(U)$ as

$$\begin{aligned}
\operatorname{Sym}^* \mathscr{E}_c(U) &= \oplus_n \operatorname{Sym}^n \mathscr{E}_c(U) \\
\operatorname{Sym}^* \overline{\mathscr{E}}_c(U) &= \oplus_n \operatorname{Sym}^n \overline{\mathscr{E}}_c(U) \\
\widehat{\operatorname{Sym}}^* \mathscr{E}_c(U) &= \prod_n \operatorname{Sym}^n \mathscr{E}_c(U) \\
\widehat{\operatorname{Sym}}^* \overline{\mathscr{E}}_c(U) &= \prod_n \operatorname{Sym}^n \overline{\mathscr{E}}_c(U).
\end{aligned}$$

Note that these are all commutative algebras in the multi-category of differentiable cochain complexes (section A), and the completed symmetric algebras are commutative algebras in the multi-category of differentiable pro-cochain complexes (section A).

Note that, since $\overline{\mathscr{E}}^!_c(U)$ is dual to $\mathscr{E}(U)$, we can view $\widehat{\mathrm{Sym}}\overline{\mathscr{E}}^!_c(U)$ as the algebra of formal power series on $\mathscr{E}(U)$. Thus, we often write

$$\widehat{\mathrm{Sym}}\overline{\mathscr{E}}^!_c(U) = \mathscr{O}(\mathscr{E}(U)).$$

3.2.2. Note that if $U \to V$ is an inclusion of open sets in M, then there are natural maps of commutative dg algebras

$$\begin{aligned} \mathrm{Sym}^* \mathscr{E}_c(U) &\to \mathrm{Sym}^* \mathscr{E}_c(V) \\ \widehat{\mathrm{Sym}}^* \mathscr{E}_c(U) &\to \widehat{\mathrm{Sym}}^* \mathscr{E}_c(V) \\ \mathrm{Sym}^* \overline{\mathscr{E}}_c(U) &\to \mathrm{Sym}^* \overline{\mathscr{E}}_c(V) \\ \widehat{\mathrm{Sym}}^* \overline{\mathscr{E}}_c(U) &\to \widehat{\mathrm{Sym}}^* br\mathscr{E}_c(V). \end{aligned}$$

Thus, each of these symmetric algebras forms a precosheaf of commutative algebras, and thus a prefactorization algebra. We denote these prefactorization algebras by $\mathrm{Sym}^* \mathscr{E}_c$, etc.

3.2.3. The main result of this section is the following.

3.2.3.1 Theorem. *(1) Let E be a vector bundle on M. Then,*

- *(a)* $\mathrm{Sym}^* \mathscr{E}_c$ *and* $\mathrm{Sym}^* \overline{\mathscr{E}}_c$ *are strict (non-homotopical) factorization algebras valued in the category of differentiable vector spaces.*
- *(b)* $\widehat{\mathrm{Sym}}^* \mathscr{E}_c$ *and* $\widehat{\mathrm{Sym}}^* \overline{\mathscr{E}}_c$ *are strict (non-homotopical) factorization algebras valued in the category of differentiable pro-vector spaces.*

(2) Let E be a local cochain complex on M. Then,

- *(a)* $\mathrm{Sym}^* \mathscr{E}_c$ *and* $\mathrm{Sym}^* \mathscr{E}_c$ *are homotopy factorization algebras valued in the category of differentiable cochain complexes.*
- *(b)* $\widehat{\mathrm{Sym}}^* \mathscr{E}_c$ *and* $\widehat{\mathrm{Sym}}^* \mathscr{E}_c$ *are homotopy factorization algebras valued in the category of differentiable pro-cochain complexes.*

PROOF. Let us first prove the strict (non-homotopy) version of the result. To start with, consider the case of $\mathrm{Sym}^* \mathscr{E}_c$. We need to verify the local-to-global axiom (section 3.1).

Let U be an open set in M and $\mathfrak{U} = \{U_i \mid i \in I\}$ be a factorizing cover of U. As before, let PI denote the set of finite subsets of I, where for each $\alpha \in PI$, and every

$i, j \in \alpha$, $U_i \cap U_j = \emptyset$. If $\alpha \in PI$ let $U_\alpha = \amalg_{i \in \alpha} U_i$. More generally, let

$$U_{\alpha_1, \dots, \alpha_n} = \amalg_{i_1 \in \alpha_1, \dots, i_n \in \alpha_n} \left(U_{i_1} \cap \dots \cap U_{i_n} \right).$$

We need to prove that $\operatorname{Sym}^* \mathscr{E}_c(U)$ is the cokernel of the map

$$\oplus_{\alpha, \beta \in PI} \operatorname{Sym}^*(\mathscr{E}_c(U_{\alpha,\beta})) \to \oplus_{\gamma \in PI} \operatorname{Sym}^a st(\mathscr{E}_c(U_\gamma))$$

This map compatible with the decomposition of $\operatorname{Sym}^* \mathscr{E}_c(U)$ into symmetric powers. Thus, it suffices to show that, for all m,

$$\operatorname{Sym}^m \mathscr{E}_c(U) = \operatorname{coker} \left(\oplus_{\alpha, \beta \in PI} \operatorname{Sym}^m (\mathscr{E}_c(U_\alpha \cap U_\beta) \to \oplus_{\gamma \in PI} \operatorname{Sym}^m \mathscr{E}_c(U_\gamma) \right).$$

Now, observe that

$$\mathscr{E}_c(U)^{\otimes m} = \mathscr{E}_c^{\boxtimes m}(U^m)$$

where $\mathscr{E}_c^{\boxtimes m}$ is the cosheaf on U^m obtained as the external product of $\mathscr{E}_c$ with itself m times.

Thus it is enough to show that

$$\mathscr{E}_c^{\boxtimes m}(U^m) = \operatorname{coker} \left(\oplus_{\alpha,\beta \in PI} \mathscr{E}_c^{\boxtimes m} \left((U_\alpha \cap U_\beta)^m \right) \to \oplus_{\gamma \in PI} \mathscr{E}_c^{\boxtimes m} \left(U_\gamma^m \right) \right).$$

Our cover $\mathfrak{U}$ is a factorizing cover. This means that, for every finite set of points $x_1, \dots, x_k \in M$ we can find disjoint open subsets $U_{i_1}, \dots, U_{i_k}$ in the cover $\mathfrak{U}$ with $x_i \in U_{i_i}$. This implies that the subsets of U^m of the form $(U_\alpha)^m$, where $\alpha \in PI$, cover U^m. Further,

$$(U_\alpha)^m \cap (U_\beta)^m = (U_\alpha \cap U_\beta)^m.$$

The desired isomorphism now follows from the fact that $\mathscr{E}_c^{\boxtimes m}$ is a cosheaf on M^m.

The same argument applies to show that $\operatorname{Sym}^* \mathscr{E}_c^!$ is a factorization algebra. In the completed case, essentially the same argument applies, with the subtlety (A) that when working with pro-cochain complexes the direct sum is completed.

For the homotopy case, the argument is similar. Let $\mathfrak{U}$ be a factorizing cover of an open subset V of M, indexed by a set I. Let PI denote the set of finite subsets $\alpha \subset I$ with the property that, for $i, j \in \alpha$, $U_i \cap U_j = \emptyset$. Let $U_\alpha = \amalg_{i \in \alpha} U_i$. Let $U_{\alpha_1, \dots, \alpha_n}$ denote the intersection $U_{\alpha_1} \cap \dots \cap U_{\alpha_n}$.

Let $\mathcal{F} = \operatorname{Sym}^* \mathscr{E}_c$ denote the prefactorization algebras we are considering (the argument will below will apply when we use the completed symmetric product or use $\overline{\mathscr{E}}_c$ instead of $\mathscr{E}_c$). We need to show that map

$$\check{C}(\mathfrak{U}, \mathcal{F}) = \oplus_{\alpha_1,\dots,\alpha_n} \mathcal{F}(U_{\alpha_1,\dots,\alpha_n})[n-1] \to \mathcal{F}(U)$$

is an equivalence (where the left hand side is equipped with the standard Čech differential).

Let $\mathcal{F}^m(U) = \operatorname{Sym}^m \mathscr{E}_c$. Both sides of the displayed equation above split as a direct sum over m, and the map is compatible with this splitting. (If we use the completed symmetric product, this decomposition is of course as a product).

We thus need to show that the map

$$\oplus_{\alpha_1,\dots,\alpha_n} \operatorname{Sym}^m \left(\mathscr{E}_c(U_{\alpha_1,\dots,\alpha_n})\right)[n-1] \to \operatorname{Sym}^m \left(\mathscr{E}_c(U)\right)$$

is a weak equivalence.

For $\alpha \in PI$, we get an open subset $U^m_\alpha \subset U^m$. Since $\mathfrak{U}$ is a factorizing cover of V, these open subsets form a cover of V^m. Note that

$$(U_{\alpha_1})^m \cap \dots \cap (U_{\alpha_n})^m = (U_{\alpha_1,\dots,\alpha_n})^m.$$

Note that $\mathscr{E}_c(U)^{\otimes m}$ can be naturally identified with $\Gamma_c(U^m, E^{\boxtimes m})$ (where the tensor product is the completed projective tensor product).

Thus, to show that the Čech descent axiom holds, we need to verify that the map

$$\oplus_{\alpha_1,\dots,\alpha_n \in PI} \Gamma_c \left(U^m_{\alpha_1} \cap \dots \cap U^m_{\alpha_n}, E^{\boxtimes m}\right)[n-1] \to \Gamma_c(V^m, E^{\boxtimes m})$$

is a quasi-isomorphism. The left hand side above is the Čech complex for the cosheaf of compactly supported sections of $E^{\boxtimes m}$ on V^m. Standard partition-of-unity arguments show that this map is a weak equivalence.

□

3.3. Locally constant factorization algebras

If M is an n-dimensional manifold, then prefactorization algebras locally bear a resemblance to E_n algebras. After all, a prefactorization algebra prescribes a way to combine the elements associated to k distinct balls into an element associated to a big ball containing all k balls. In fact, E_n algebras form a full subcategory of factorization algebras on $\mathbb{R}^n$.

3.3.0.2 Definition. *A factorization algebra $\mathcal{F}$ on an n-manifold M is* locally constant *if whenever U, U' are open subsets with $U \subset U'$, such that U is a deformation retraction of U', then the map $\mathcal{F}(U) \to \mathcal{F}(U')$ is a quasi-isomorphism of cochain complexes.*

Lurie [] has shown the following.

3.3.0.3 Theorem. *There is an equivalence of $(\infty, 1)$-categories between locally constant factorization algebras on $\mathbb{R}^n$, and E_n algebras.*

3.3.1. Many examples of E_n algebras arise naturally from topology, such as labelled configuration spaces (as discussed in the work of Segal, McDuff, Bodigheimer, Salvatore, and Lurie). We will discuss an important example, that of mapping spaces.

Recall that (in the appropriate category of spaces) $\mathrm{Maps}(U \sqcup V, X) \cong \mathrm{Maps}(U, X) \times \mathrm{Maps}(V, X)$. This fact suggests that we might fix a target space X and define a factorization algebra by sending an open set U to $\mathrm{Maps}(U, X)$. This construction almost works, but it is not clear how to "extend" a map $f : U \to X$ from U to a larger open set $V \ni U$. By working with "compactly-supported" maps, we solve this issue.

Fix (X, p) a pointed space. Let F denote the prefactorization algebra on M sending an open set U to the space of compactly-supported maps from U to (X, p). (Here, "f is compactly-supported" means that the closure of $f^{-1}(X - p)$ is compact.) Then F is a prefactorization algebra in the category of pointed spaces. (Composing with the singular chains functor gives a prefactorization algebra in abelian groups, but we will work at the level of spaces.)

Note that this prefactorization algebra is locally constant: if $U \hookrightarrow U'$ is an inclusion of open subsets where U is a deformation retraction of U', then the map $F(U) \to F(U')$ is a weak homotopy equivalence.

Note also that there is a natural isomorphism

$$F(U_1 \amalg U_2) = F(U_1) \times F(U_2)$$

if U_1, U_2 are disjoint.

Let's consider for a moment the case when $M = \mathbb{R}^n$. Then, if $D \subset \mathbb{R}^n$ is a ball, there is a equivalence

$$F(D) \simeq \Omega^n X$$

between the space of compactly supported maps $D \to X$ and the n-fold based loop space of X. (We are using the topologists notation $\Omega^n X$ for the n-fold loop space: we apologize with the conflict of notation with the space of n-forms).

To see this, note that a compactly supported map $D \to X$ extends uniquely to a map from the closed ball $\overline{D}$, sending the boundary $\partial\overline{D}$ to the base point p of X. Since $\Omega^n(X)$ is defined to be the space of maps of pairs

$$(\overline{D}, \partial\overline{D}) \to (X, p)$$

we have constructed the desired map from $F(D)$ to $\Omega^n(X)$. It is easily verified that this map is a homotopy equivalence.

If D_1, D_2 are disjoint discs contained in a disc D_3, the prefactorization structure gives us a map

$$F(D_1) \times F(D_2) \to F(D_3).$$

These maps correspond to the standard E_n structure on the n-fold loop space.

For a particularly nice example, let $M = \mathbb{R}$. Then, these maps describe the standard product on the space ΩX of based loops in X. This is the product which, on components, is the standard product on $\pi_0 \Omega X = \pi_1(X, x)$.

This prefactorization algebra does not always satisfy the gluing axiom. However, Salvatore [Sal99] and Lurie [Lur09a] has shown that if X is sufficiently connected, this prefactorization algebra is in fact a factorization algebra. (Technically, Salvatore and Lurie use a slightly different descent axiom than we do here, but in the locally constant case we believe our descent axioms our equivalent).

3.4. The category of factorization algebras

In this section, we explain how prefactorization algebras and factorization algebras form categories. In fact, they naturally form multicategories (or colored operads). We also explain how these multicategories are enriched in simplicial sets when the (pre)factorization algebras take values in cochain complexes.

3.4.1. Morphisms and the category structure.

3.4.1.1 Definition. *A morphism of prefactorization algebras $\phi : F \to G$ consists of a map $\phi_U : F(U) \to G(U)$ for each open $U \subset M$, compatible with the structure maps. That is, for any open V and any finite collection $U_1, \ldots, U_k$ of pairwise disjoint open sets, each contained in V, the following diagram commutes:*

$$\begin{array}{ccc} F(U_1) \otimes \cdots \otimes F(U_k) & \xrightarrow{\phi_{U_1} \otimes \cdots \otimes \phi_{U_k}} & G(U_1) \otimes \cdots \otimes G(U_k) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\phi_V} & G(V) \end{array}$$

Likewise, all the obvious associativity relations are respected.

Remark: When our prefactorization algebras take values in cochain complexes, we require the ϕ_U to be cochain maps, i.e., they each have degree 0 and commute with the differentials. When our prefactorization algebras take values in differentiable cochain complexes, we require in addition that the maps ϕ_U are smooth.

3.4.1.2 Definition. *On a space X, we denote the category of prefactorization algebras on X taking values in the multicategory $\mathcal{C}$ by $PreFA(X, \mathcal{C})$. The category of factorization algebras, $FA(X, \mathcal{C})$, is the full subcategory whose objects are the factorization algebras.*

In practise, $\mathcal{C}$ will normally be the multicategory of differentiable cochain complexes.

3.4.2. The multicategory structure. Let $S\mathcal{C}$ denote the universal symmetric monoidal category containing the multicategory $\mathcal{C}$. Any prefactorization algebra valued in $\mathcal{C}$ gives rise to one valued in $S\mathcal{C}$.

There is a natural tensor product on $PreFA(X, S\mathcal{C})$, as follows. Let F, G be prefactorization algebras. We define $F \otimes G$ by

$$F \otimes G(U) := F(U) \otimes G(U),$$

and we simply define the structure maps as the tensor product of the structure maps. For instance, if $U \subset V$, then the structure map is

$$F(U \subset V) \otimes G(U \subset V) : F \otimes G(U) = F(U) \otimes G(U) \to F(V) \otimes G(V) = F \otimes G(V).$$

3.4.2.1 Definition. *Let $PreFA_{mc}(X, \mathcal{C})$ denote the multicategory arising from the symmetric monoidal product on $PreFA(X, S\mathcal{C})$. That is, if F_i, G are prefactorization algebras valued in $\mathcal{C}$,* *we define the set of multi-morphisms by*

$$PreFA_{mc}(F_1, \cdots, F_n \mid G)$$

to be the set of maps of $\mathcal{SC}$-valued prefactorization algebras

$$F_1 \otimes \cdots \otimes F_n \to G.$$

Factorization algebras inherit this multicategory structure.

3.4.3. Enrichment over simplicial sets. In this subsection we will explain how the multicategory of factorization algebras with values in an appropriate multicategory is simplicially enriched, in a natural way. In order to define this simplicial enrichment, we need to introduce some notation.

The first thing to define is the algebra $\Omega^*(X)$ of smooth forms on a simplicial set X. We define an element $\omega \in \Omega^i(X)$ is, for every n-simplex $f : \triangle^n \to X$, an i-form $f^*\Omega^i(\triangle^n)$. If $\sigma : \triangle^m \to \triangle^n$ is a face or degeneracy map, we require that $\sigma^* f^* \omega = (f \circ \sigma)^* \omega$.

If V is a differentiable cochain complex, we can define a complex

$$\Omega^*(X, V) = \Omega^*(X) \otimes_{C^\infty(X)} C^\infty(X, V)$$

where $C^\infty(X, V)$ refers to the cochain complex of smooth maps from X to V. In this way, we see that the multicategory differentiable cochain complexes is tensored over the opposite category SSet^{op} to the category of simplicial sets.

This allows us to lift the multicategory DVS of differentiable cochain complexes to a simplicially enriched multicategory, where we define the n-simplices in the simplicial set of multimorphisms by

$$\mathrm{Hom}(V_1, \ldots, V_n \mid W)[n] = \mathrm{Hom}(V_1, \ldots, V_n \mid \Omega^*(\triangle^n, W))$$

where on the right hand side, Hom denotes smooth multilinear cochain maps

$$V_1 \times \cdots \times V_n \to \Omega^*(\triangle^n, W)$$

which are compatible with differentials.

Now, in general, suppose we have a multicategory $\mathcal{C}$ which is tensored over SSet^{op}. Then the multicategory $\mathrm{PreFA}(X, \mathcal{C})$ is also tensored over SSet^{op}: the tensor product

$F \boxtimes X$ of a prefactorization algebra F with a simplicial set X is defined by

$$(F \boxtimes X)(U) = F(U) \boxtimes X.$$

Then, we can lift our multicategory of $\mathcal{C}$-valued prefactorization algebras to a simplicially enriched multicategory, by defining

$$PreFA^{\triangle}_{mc}(F_1, \ldots, F_n \mid G) = PreFA^{\triangle}_{mc}(F_1, \ldots, F_n \mid G \boxtimes X).$$

In particular, we see that the multicategory of prefactorization algebras valued in differentiable cochain complexes is simplicially enriched.

3.4.4. Equivalences. In the appendix A, we define a notion of *weak equivalence* or *quasi-isomorphism* of differentiable cochain complexes.

3.4.4.1 Definition. *Let F, G be factorization algebras valued in cochain complexes. Let $\phi : F \to G$ be a map. We say that ϕ is a weak equivalence if, for all open subsets $U \subset M$, the map $F(U) \to G(U)$ is a quasi-isomorphism of cochain complexes.*

Similarly, let F, G be factorization algebras valued in differentiable cochain complexes. We say a map $\phi : F \to G$ is a weak equivalence if, for all U, the map $F(U) \to G(U)$ is a weak equivalence of differentiable cochain complexes.

3.4.4.2 Lemma. *A map $F \to G$ between differentiable factorization algebras is a weak equivalence if and only if, for all factorizing bases $\mathfrak{U}$ of X, and all $U_1, \ldots, U_k$ disjoint elements of $\mathcal{U}$, the maps*

$$F(U_1 \amalg \cdots \amalg U_k) \to G(U_1 \amalg \cdots \amalg U_k)$$

are weak equivalences.

PROOF. For any open subset $V \subset X$, let $\mathfrak{U}_V$ denote the factorizing cover of V consisting of open subsets in $\mathfrak{U}$ which lie in V. By the descent axiom the map

$$\check{C}(\mathfrak{U}_V, F) \to F(V)$$

is a weak equivalence, and similarly for G. Thus, it suffices to check that the map

$$\check{C}(\mathfrak{U}_V, F) \to \check{C}(\mathfrak{U}_V, G)$$

is a weak equivalence. This follows from a spectral sequence argument and the fact that the maps

$$F(U_1 \amalg \cdots \amalg U_k) \to G(U_1 \amalg \cdots \amalg U_k)$$

are weak equivalences. □

3.5. Pushing forward factorization algebras

A crucial feature of factorization algebras is that they push forward nicely. Let M and N be topological spaces admitting factorizing covers and let $f : M \to N$ be a continuous map. Given a factorizing cover $\mathfrak{U} = \{U_\alpha\}$ of an open $U \subset N$, let $f^{-1}\mathfrak{U} = \{f^{-1}U_\alpha\}$ denote the preimage cover of $f^{-1}U \subset M$. Observe that $f^{-1}\mathfrak{U}$ is factorizing: given a finite collection of points $\{x_1, \ldots, x_n\}$ in $f^{-1}U$, the image points $\{f(x_1), \ldots, f(x_n)\}$ can be covered by a disjoint collection of opens $U_{\alpha_1}, \ldots, U_{\alpha_k}$ in $\mathfrak{U}$ and hence $f^{-1}U_{\alpha_1}, \ldots, f^{-1}U_{\alpha_k}$ is a disjoint collection of opens in $f^{-1}\mathfrak{U}$ covering the x_j.

3.5.0.3 Definition. *Given a factorization algebra $\mathcal{F}$ on a space M and a continuous map $f : M \to N$, the* pushforward factorization algebra *$f_*\mathcal{F}$ on N is defined by*

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Note that for the map to a point $f : M \to pt$, the pushforward factorization algebra $f_*\mathcal{F}$ is simply the global sections of $\mathcal{F}$. We also call this the *factorization homology* of $\mathcal{F}$ on M. We sometimes denote this $\mathrm{FH}(M, \mathcal{F})$.

3.6. Extension from a basis

3.6.1. Factorization algebras defined on a factorizing basis. Let X be a topological space, and let $\mathfrak{U}$ be a basis for X, which is closed under taking finite intersections. It is well-known that there is an equivalence of categories between sheaves on X and sheaves which are only defined for open sets in the basis $\mathfrak{U}$. In this section we will prove a similar statement for factorization algebras. This will allow us to perform several useful formal constructions with factorization algebras, such as gluing.

3.6.1.1 Definition. *A* factorizing basis *for X is a basis $\mathfrak{U}$ of open sets of X which is closed under finite intersections, and which is also a factorizing cover.*

Let $\mathfrak{U}$ be a factorizing basis.

3.6.1.2 Definition. *A $\mathfrak{U}$-prefactorization algebra $\mathcal{F}$ is like a factorization algebra, except that $\mathcal{F}(U)$ is only defined for sets U of the form $U = U_1 \amalg \ldots U_k$ with U_i disjoint sets in $\mathfrak{U}$. A $\mathfrak{U}$-factorization algebra is a $\mathfrak{U}$-prefactorization algebra with the property that, for all U which* *is a disjoint union of sets in* $\mathfrak{U}$ *and all factorizing covers* $\mathfrak{V}$ *of* U *consisting of open sets in* $\mathfrak{U}$*,*

$$\check{C}(\mathfrak{V}, \mathcal{F}) \simeq \mathcal{F}(U),$$

where $\check{C}(\mathfrak{V}, \mathcal{F})$ *denotes the Čech complex described earlier (section 3.1).*

In this section we will show that any $\mathfrak{U}$-factorization algebra on X extends to a factorization algebra on X. This extension is unique up to quasi-isomorphism.

Let $\mathcal{F}$ be a $\mathfrak{U}$-factorization algebra. Let us define a prefactorization algebra $i_*^{\mathfrak{U}}\mathcal{F}$ on X by

$$i_*^{\mathfrak{U}}(\mathcal{F})(V) = \check{C}(\mathfrak{U}_V, \mathcal{F}),$$

- for each $V \subset X$ open. Here $\mathfrak{U}_V$ is the cover of V consisting of those open subsets in the cover $\mathfrak{U}$ which are contained in V.

3.6.1.3 Proposition. *With this definition,* $i_*^{\mathfrak{U}}(\mathcal{F})$ *is a factorization algebra whose restriction to open sets in the cover* $\mathfrak{U}$ *is quasi-isomorphic to* $\mathcal{F}$*.*

The proof is rather technical, so we put it in the appendix B. The statement holds if $\mathcal{F}$ is a factorization algebra valued in differentiable cochain complexes or in ordinary cochain complexes.

3.7. Pulling back factorization algebras

Let $\mathcal{F}$ be a factorization algebra on M. Let $U \subset M$ be an open subset. Then we can restrict $\mathcal{F}$ to a factorization algebra $\mathcal{F} \mid_U$ on U, whose value on an open subset $V \subset U$ is simply $\mathcal{F}(V)$.

In this section we will discuss a generalization of this construction. We will not try to define pull-backs for arbitrary maps, but only for open immersions.

Let $f : N \to M$ be an open immersion. Let $\mathfrak{U}_f$ be the cover of N consisting of those open subsets $U \subset N$ with the property that

$$f \mid_U : U \to f(U)$$

is a homeomorphism. (To say that f is an open immersion means that sets of this form cover N).

Now, $\mathfrak{U}_f$ is a factorizing basis for N. Let us define a $\mathfrak{U}_f$-prefactorization algebra $f^*\mathcal{F}$ by

$$f^*\mathcal{F}(U) = \mathcal{F}(f(U))$$

if $U \in \mathfrak{U}_f$.

3.7.0.4 Lemma. *$f^*\mathcal{F}$ is a $\mathfrak{U}_f$-factorization algebra.*

PROOF. We need to verify that if $U \in \mathfrak{U}_f$, and $\mathfrak{V}$ is a factorizing cover of U by elements of $\mathfrak{U}_f$, that

$$\check{C}(\mathfrak{V}, f^*\mathcal{F}) \simeq f^*\mathcal{F}(U) = \mathcal{F}(f(U)).$$

Now, $f(\mathfrak{V})$ is a factorizing cover of $f(U)$, and

$$\check{C}(\mathfrak{V}, f^*\mathcal{F}) = \check{C}(f(\mathfrak{V}, \mathcal{F}).$$

The result follows from the fact that $\mathcal{F}$ is a factorization algebra on M.

□

So far we have defined $f^*\mathcal{F}$ as a $\mathfrak{U}_f$-factorization algebra. We can extend (section 3.6) $f^*\mathcal{F}$ to an actual factorization algebra, which we will continue to call $f^*\mathcal{F}$.

3.8. Descent and gluing

3.8.1. Gluing. Suppose we have open cover $\{U_i\}$ of X and a factorization algebra $\mathcal{F}_i$ on each U_i, with gluing data on double intersections, coherence data on triple intersections, and so on. In this section we will show that we can construct a unique factorization algebra $\mathcal{F}$ on X from this data. In fact, this is an easy consequence of the fact that we can extend factorization algebras from a factorizing basis. This construction only works for factorization algebras valued in a symmetric monoidal category.

3.8.2. Covers by two open sets. Let us start with the simple case when we cover X by two open sets U, V. Let $\mathcal{F}_U$ be a factorization algebra on U and $\mathcal{F}_V$ be a factorization algebra on V, both valued in a symmetric monoidal category $\mathcal{C}$.

Let $\mathcal{F}_{U\cap V}$ be a factorization algebra on $U \cap V$ equipped with weak equivalences of factorization algebras on $U \cap V$,

$$\mathcal{F}_{U\cap V} \to i_U^* \mathcal{F}_U$$
$$\mathcal{F}_{U\cap V} \to i_V^* \mathcal{F}_V.$$

This is one natural way of saying what it means to have "gluing data".

Let us define a factorizing basis $\mathfrak{W}$ of X by saying that a set is in $\mathfrak{W}$ if it lies in U or in V.

We want to define an $\mathfrak{W}$ prefactorization algebra; to do this, we need to define $\mathcal{F}(W)$ for all sets W which are disjoint unions of sets in $\mathfrak{W}$. Thus, given such a W, let us write it as

$$W = U_1 \amalg \cdots \amalg U_k \amalg V_1 \amalg \cdots \amalg V_l$$

where the $U_i \subset U$ and $V_i \subset V$ are open subsets. We then define

$$\mathcal{F}(W) = \otimes \mathcal{F}_U(U_i) \otimes \mathcal{F}_V(V_i).$$

It is clear that $\mathcal{F}$ is a $\mathfrak{W}$-prefactorization algebra (the structure maps use the weak equivalences $\mathcal{F}_{U\cap V}(W) \to \mathcal{F}_U(W)$ if $W \subset U \cap V$).

3.8.2.1 Lemma. *$\mathcal{F}$ is a $\mathfrak{W}$-factorization algebra.*

PROOF. Let $W \in \mathfrak{W}$, and let $\mathfrak{W}'$ be a factorizing cover of W consisting of sets in $\mathfrak{W}$. We need to check that

$$\check{C}(\mathfrak{W}', \mathcal{F}) \simeq \mathcal{F}(W).$$

There are three cases: either $W \subset U \cap V$, or $W \subset U$ and $W \not\subset V$, or $W \subset V$ and $W \not\subset U$. The first case is immediate from the fact that $\mathcal{F}_{U\cap V}$ is a factorization algebra on $U \cap V$.

The second two cases are the same; so let us assume that $W \subset U$ and $W \not\subset V$.

We need to verify that

$$\check{C}(\mathfrak{W}', \mathcal{F}) \simeq \mathcal{F}(W).$$

Now, there are two prefactorization algebras on U, $\mathcal{F}$ and $\mathcal{F}_U$. $\mathcal{F}_U$ is given to us, and $\mathcal{F}$ is defined by

$$\mathcal{F}(W') = \begin{cases} \mathcal{F}_U(W') & \text{if } W' \not\subset V \\ \mathcal{F}_{U\cap V}(W') & \text{if } W' \subset V. \end{cases}$$

What we're checking is that $\mathcal{F}$ is a factorization algebra on U.

If $W' \in \mathfrak{W}'$, and $W' \subset U \cap V$, then there is a weak equivalence

$$\mathcal{F}_{U \cap V}(W') \to \mathcal{F}_U(W').$$

This induces a weak equivalence map of prefactorization algebras on U,

$$\mathcal{F} \to \mathcal{F}_U.$$

Since $\mathcal{F}_U$ is a factorization algebra, and this map is a weak equivalence, $\mathcal{F}$ is also a factorization algebra. □

3.8.3. The general case. Now let us consider a general open cover $\mathfrak{U}$ of X. We want to show how to construct, from a factorization algebra on every set in $\mathfrak{U}$ with certain gluing data, a factorization algebra on X.

The way we will encode the factorization algebra on the sets in the cover $\mathfrak{U}$, together with gluing data, is as follows. We have, for each finite subset $I \subset \mathfrak{U}$, a factorization algebra $\mathcal{F}_I$ on

$$U_I = \cap_{i \in I} U_i.$$

Further, we have weak equivalences

$$\mathcal{F}_I \to r_i^* \mathcal{F}_{I \setminus \{i\}}$$

of factorization algebras on U_I, for each $i \in I$.

$$r_i : U_I \to U_{I \setminus \{i\}}$$

is the natural inclusion.

Finally, we require that, for every I and every $i, j \in I$, the following diagram of commutes:

$$\begin{array}{ccc} \mathcal{F}_I & \to & \mathcal{F}_{I \setminus \{i\}} \\ \downarrow & & \downarrow \\ \mathcal{F}_{I \setminus \{j\}} & \to & \mathcal{F}_{I \setminus \{i,j\}} \end{array}$$

We can think of this data as defining a factorization algebra $\mathcal{F}_i$ for each $i \in \mathfrak{U}$, together with weak equivalences on double intersections, provided by $\mathcal{F}_{\{i,j\}}$; and with coherences provided by the factorization algebras $\mathcal{F}_I$ where $\#I \geq 3$.

In this situation, we can construct a factorization algebra on X, as follows. We will let $\mathfrak{W}$ be the factorizing basis of X consisting of open sets subordinate to the cover $\mathfrak{U}$. We will define a $\mathfrak{W}$-factorization algebra $\mathcal{F}$ by saying that, if $W \in \mathfrak{W}$,

$$\mathcal{F}(W) = \mathcal{F}_I(W)$$

where I is the largest subset of $\mathfrak{U}$ such that

$$W \subset U_I.$$

If $W = W_1 \amalg \cdots \amalg W_k$ where $W_i \in \mathfrak{W}$, we set

$$\mathcal{F}(W) = \otimes_{i=1}^{k} \mathcal{F}(W_i).$$

It is straightforward to check that $\mathcal{F}$ is a $\mathfrak{W}$-factorization algebra, and so extends to factorization algebra on X.

3.8.4. Descent. Let G be a discrete group acting on a space X.

3.8.4.1 Definition. *A G-equivariant factorization algebra on X is a factorization algebra $\mathcal{F}$ on X together with isomorphisms*

$$\rho_g : g^* \mathcal{F} \cong \mathcal{F},$$

for each $g \in G$, such that

$$\begin{aligned} \rho_{\mathrm{Id}} &= \mathrm{Id} \\ \rho_{gh} &= \rho_h \circ h^*(\rho_g) : h^* g^* \mathcal{F} \to \mathcal{F}. \end{aligned}$$

3.8.4.2 Proposition. *Let G be a discrete group acting properly discontinuously on X, so that $X \to X/G$ is a principal G-bundle. Then, there is an equivalence of categories between G-equivariant factorization algebras on X and factorization algebras on X/G.*

This result holds for factorization algebras valued in multicategories as well as symmetric monoidal categories.

PROOF. If $\mathcal{F}$ is a factorization algebra on X/G, then $f^* \mathcal{F}$ is a G-equivariant factorization algebra on X.

Conversely, let $\mathcal{F}$ be a G-equivariant factorization algebra on $\mathcal{F}$. Let $\mathfrak{U}$ be the open cover of X/G consisting of those connected sets where the G-bundle $X \to X/G$ admits a section. Note that $\mathfrak{U}$ is a factorizing basis for X/G, and that $\mathfrak{U}$ is closed under taking both finite intersection and disjoint union. We will define a $\mathfrak{U}$-factorization algebra $\mathcal{F}^G$ by defining

$$\mathcal{F}^G(U) = \mathcal{F}(\sigma(U))$$

where σ is any section of the G-bundle $\pi^{-1}(U) \to U$.

Because $\mathcal{F}$ is G-equivariant, $\mathcal{F}(\sigma(U))$ is independent of the section σ chosen. Since $\mathfrak{U}$ is a factorizing basis, $\mathcal{F}^G$ extends canonically to a factorization algebra on X/G.

□

CHAPTER 4

Classical field theory

4.1. Introduction

Our goal here is to describe how the observables of a classical field theory naturally form a factorization algebra (section 3.1). More accurately, we are interested in what might be called classical perturbative field theory. "Classical" means that the main object of interest is the sheaf of solutions to the Euler-Lagrange equations for some local action functional. "Perturbative" means that we will only consider those solutions which are infinitesimally close to a given solution. Much of the chapter is devoted to providing a precise mathematical definition of these ideas, with inspiration taken from deformation theory and derived geometry.

4.1.1. The Euler-Lagrange equations. The fundamental objects of a physical theory are the observables of a theory, that is, the measurements one can make in that theory. In a classical field theory, our fields are constrained to be solutions to the Euler-Lagrange equations. Thus, the measurements one can make are the functions on the space of solutions to the Euler-Lagrange equations.

However, it is essential that we do not take the naive moduli space of solutions. Instead, we consider the *derived* moduli space of solutions. Since we are working perturbatively – that is, infinitesimally close to a given solution – this derived moduli space will be a "formal moduli problem" [**Lur10**, **Lur11**]. In the physics literature, the procedure of taking the derived critical locus is implemented by the BV formalism. Thus, the first step (section 4.3.3) in our treatment of classical field theory is to develop a language to treat formal moduli problems cut out by systems of partial differential equations on a manifold M. Since it is essential that the differential equations we consider are elliptic, we call such an object a *formal elliptic moduli problem*.

Since one can consider the solutions to a differential equation on any open subset $U \subset M$, a formal elliptic moduli problem $\mathcal{F}$ yields, in particular, a sheaf of formal moduli problems on M; which sends U to the formal moduli space $\mathcal{F}(U)$ of solutions on U.

We will use the notation $\mathcal{EL}$ to denote the formal elliptic moduli problem of solutions to the Euler-Lagrange equation on M; so that $\mathcal{EL}(U)$ will denote the space of solutions on an open subset $U \subset M$.

4.1.2. Observables. In a field theory, we tend to focus on measurements that are localized in spacetime. Hence, we want a method that associates a set of observables to each region in M. If $U \subset M$ is an open subset, the observables on U are

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathcal{EL}(U)),$$

our notation for the algebra of functions on the formal moduli space $\mathcal{EL}(U)$ of solutions to the Euler-Lagrange equations on U. (We will be more precise about which class of functions we are using later). As we are working in the derived world, $\mathrm{Obs}^{cl}(U)$ is a differential-graded commutative algebra. Using these functions, we can answer any question we might ask about the behavior of our system in the region U.

The factorization algebra structure arises naturally on the observables in a classical field theory. Let U be an open set in M, and $V_1, \ldots, V_k$ a disjoint collection of open subsets of U. Then restriction of solutions from U to each V_i induces a natural map

$$\mathcal{EL}(U) \to \mathcal{EL}(V_1) \times \cdots \times \mathcal{EL}(V_k).$$

Since functions pullback under maps of spaces, we get a natural map

$$\mathrm{Obs}^{cl}(V_1) \otimes \cdots \otimes \mathrm{Obs}^{cl}(V_k) \to \mathrm{Obs}^{cl}(U)$$

so that Obs^{cl} forms a *pre*factorization algebra. To see that Obs^{cl} is indeed a factorization algebra, it suffices to observe that the functor $\mathcal{EL}$ is a sheaf.

Since the space $\mathrm{Obs}^{cl}(U)$ of observables on a subset $U \subset M$ is a commutative algebra, and not just a vector space, we see that the observables of a classical field theory form a commutative factorization algebra (section 2.2).

4.1.3. The symplectic structure. Above, we outlined a way to construct, from the elliptic moduli problem associated to the Euler-Lagrange equations, a commutative factorization algebra. However, this construction would apply equally well to any system of differential equations. The Euler-Lagrange equations, of course, have the special property that they arise as the critical points of a functional.

In finite dimensions, a formal moduli problem which arises as the derived critical locus (section 4.8) of a function is equipped with an extra structure: a symplectic form of cohomological degree -1. For us, this symplectic form is an intrinsic way of indicating that a formal moduli problem arises as the critical locus of a functional. Indeed, any formal moduli problem with such a symplectic form can be expressed (non-uniquely) in this way.

We give (section 4.8.2) a definition of symplectic form on an elliptic moduli problem. We then simply *define* a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

Given a local action functional satisfying certain non-degeneracy properties, we construct (section 4.9.1) an elliptic moduli problem describing the corresponding Euler-Lagrange equations, and show that this elliptic moduli problem has a symplectic form of degree -1.

In ordinary symplectic geometry, the simplest construction of a symplectic manifold is as a cotangent bundle. In our setting, there is a similar construction: given any elliptic moduli problem $\mathcal{F}$, we construct (section 4.12) a new elliptic moduli problem $T^*[-1]\mathcal{F}$ which has a symplectic form of degree -1. It turns out that many examples of field theories of interest in mathematics and physics arise in this way.

4.1.4. The P_0 structure. In finite dimensions, if X is a formal moduli problem with a symplectic form of degree -1, then the dga $\mathscr{O}(X)$ of functions on X is equipped with a Poisson bracket of degree 1. In other words, $\mathscr{O}(X)$ is a P_0 algebra (section 2.3).

In infinite dimensions, we show that something similar happens. If $\mathcal{F}$ is a classical field theory, then we show that the commutative algebra $\mathscr{O}(\mathcal{F}(U)) = \mathrm{Obs}^{cl}(U)$ has a P_0 structure; and that the commutative factorization algebra Obs^{cl} forms a P_0 factorization algebra. This is not quite trivial; it is at this point that we need the assumption that our Euler-Lagrange equations are elliptic.

4.2. Elliptic moduli problems and local Lie algebras

The essential data of a classical field theory is the moduli space of solutions to the equations of motion of the field theory. For us, it is essential that we take not the naive moduli space of solutions, but rather the *derived* moduli space of solutions. In the physics literature, the procedure of taking the derived moduli of solutions to the Euler-Lagrange equations is known as the Batalin-Vilkovisky formalism.

The derived moduli space of solutions to the equations of motion of a field theory on X is a sheaf on X. In this section we will introduce a general language for discussing sheaves of "derived spaces" on X which are cut out by differential equations.

4.3. Formal moduli problems and Lie algebras

Before we discuss the concepts specific to classical field theory, we will explain some general techniques from deformation theory.

In ordinary algebraic geometry, the fundamental objects are commutative rings. In derived algebraic geometry, commutative rings are replaced by commutative differential graded rings concentrated in non-positive degrees (or, if one prefers, simplicial commutative rings; over $\mathbb{Q}$, there is no difference).

We are interested in formal derived geometry, which is described by nilpotent commutative dgas.

4.3.0.1 Definition. *An Artinian dga over a field K of characteristic zero is a differential graded K-algebra R, concentrated in degrees ≤ 0, such that*

(1) Each graded component R^i is finite dimensional, and $R^i = 0$ for $i << 0$.
(2) R has a unique maximal differential ideal m such that $R/m = K$, and such that $m^N = 0$ for $N >> 0$.

Given the first condition, the second condition is equivalent to the statement that $H^0(R)$ is Artinian in the classical sense.

The category of Artinian dg rings is a simplicially enriched category. A map $R \to S$ is simply a map of dg rings taking the maximal ideal m_R to that of m_S. Equivalently, such a map is a map of non-unital dg rings $m_R \to m_S$. An n-simplex in the space $\mathrm{Hom}(R, S)$ of maps from R to S is defined to be a map of non-unital dg rings

$$m_R \to m_S \otimes \Omega^*(\triangle^n)$$

where $\Omega^*(\triangle^n)$ is some commutative algebra model for the cochains on the n-simplex. (Normally, we will work over $\mathbb{R}$, and $\Omega^*(\triangle^n)$ will be the usual de Rham complex).

We will (temporarily) let Art denote the simplicially enriched category of Artinian dg rings over K.

4.3.0.2 Definition. *A formal moduli problem over a field k is a functor (of simplicially enriched categories)*

$$F : \mathrm{Art}_k \to \mathrm{sSets}$$

from Art_k *to the category* sSets *of simplicial sets, with the following additional properties.*

1. *$F(k)$ is contractible.*
2. *F takes surjective maps of dg Artinian rings to fibrations of simplicial sets.*
3. *Suppose that A, B, C are dg Artinian rings, and that $B \to A$, $C \to A$ are surjective maps. Then we can form the fiber product $B \times_A C$. We require that the natural map*

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

Note that, in light of the second property, the fiber product $F(B) \times_{F(A)} F(C)$ coincides with the homotopy fiber product.

The category of formal moduli problems is itself simplicially enriched, in an evident way. If F, G are formal moduli problems, and $\phi : F \to G$ is a map, we say that ϕ is a weak equivalence if for all dg Artinian rings R, the map

$$\phi(R) : F(R) \to G(R)$$

is a weak homotopy equivalence of simplicial sets.

4.3.1. Formal moduli problems and L_∞ algebras. One very important way in which formal moduli problems arise is as the solutions to the Maurer-Cartan equation in an L_∞ algebra. As we will see later, all formal moduli problems are equivalent to formal moduli problems of this form.

If $\mathfrak{g}$ is an L_∞ algebra, and (R, m) is a dg Artinian ring, we will let

$$\mathrm{MC}(\mathfrak{g} \otimes m)$$

denote the simplicial set of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes m$. Thus, an n-simplex in this simplicial set is an element

$$\alpha \in \mathfrak{g} \otimes m \otimes \Omega^*(\triangle^n)$$

of cohomological degree 1, which satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \sum_{n \geq 2} \tfrac{1}{n!} l_n(\alpha, \ldots, \alpha) = 0.$$

It is a standard lemma that sending R to $\mathrm{MC}(\mathfrak{g} \otimes m)$ defines a formal moduli problem. We will often use the notation $B\mathfrak{g}$ to denote this formal moduli problem.

If $\mathfrak{g}$ is finite dimensional, then a Maurer-Cartan element of $\mathfrak{g} \otimes m$ is the same thing as a map of commutative dgas

$$C^*(\mathfrak{g}) \to R$$

which takes the maximal ideal of $C^*(\mathfrak{g})$ to that of R.

Thus, we can think of the Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ as the algebra of functions on $B\mathfrak{g}$.

Under the dictionary between formal moduli problems and L_∞ algebras, a dg vector bundle on $B\mathfrak{g}$ is the same thing as a dg module over $\mathfrak{g}$. The cotangent complex to $B\mathfrak{g}$ corresponds to the $\mathfrak{g}$-module $\mathfrak{g}^\vee[-1]$. The tangent complex corresponds to the $\mathfrak{g}$-module $\mathfrak{g}[1]$.

If M is a $\mathfrak{g}$-module, then sections of the corresponding vector bundle on $B\mathfrak{g}$ is the Chevalley cochains with coefficients in M. Thus, we can define $\Omega^1(B\mathfrak{g})$ to be

$$\Omega^1(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]).$$

Similarly, the complex of vector fields on $B\mathfrak{g}$ is

$$\mathrm{Vect}(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1]).$$

Note that, if $\mathfrak{g}$ is finite dimensional, this is the same as the cochain complex of derivations of $C^*(\mathfrak{g})$. Even if $\mathfrak{g}$ is not finite dimensional, the complex $\mathrm{Vect}(B\mathfrak{g})$ is, up to a shift of one, the Lie algebra controlling deformations of the L_∞ structure on $\mathfrak{g}$.

4.3.2. The fundamental theorem of deformation theory. The following statement is at the heart of the philosophy of deformation theory:

> There is an equivalence of $(\infty, 1)$ categories between the category of differential graded Lie algebras, and the category of formal pointed derived moduli problems.

In a different guise, this statement goes back to Quillen's work [Qui69] on rational homotopy theory. A precise formulation of this theorem has been proved by Hinich [Hin01]; more general theorems of this nature are considered in [Lur11] and in [Lur10], which is also an excellent survey of these ideas.

It would take us too far afield to describe the language in which this statement can be made precise. We will simply use this statement as motivation: we will only consider formal moduli problems described by L_∞ algebras, and this statement asserts that we lose no information in doing so.

4.3.3. Elliptic moduli problems. We are interested in formal moduli problems which describe solutions to differential equations on a manifold M. Since we can discuss solutions to a differential equation on any open subset of M, such an object will give a (homotopy) sheaf of derived moduli problems on M. Let us give a formal definition of such a sheaf.

4.3.3.1 Definition. *Let M be a manifold. A simplicial presheaf on M is a homotopy sheaf if it satisfies Čech descent. A homotopy sheaf of formal moduli problems on M is a presheaf F of formal moduli problems, with the property that for all Artinian dgas R, the simplicial presheaf $F(R)$ is a homotopy sheaf.*

We will often refer to a homotopy sheaf as just a sheaf.

4.3.4. Elliptic moduli problems. We are interested in elliptic derived moduli problems: that is, derived moduli problems described by a system of elliptic partial differential equations on a manifold M. We will define a formal pointed elliptic moduli problem on a manifold M to be a sheaf of formal moduli problems represented by a sheaf of L_∞ algebras on M of a certain kind.

4.3.4.1 Definition. *Let M be a manifold. A local L_∞ algebra on M consists of the following data.*

(1) *A graded vector bundle L on M, whose space of smooth sections will be denoted $\mathcal{L}$.*
(2) *A differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$, of cohomological degree 1 and square 0.*
(3) *A collection of poly-differential operators*

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

which are alternating, of cohomological degree $2 - n$, and which endow $\mathcal{L}$ with the structure of L_∞ algebra.

4.3.4.2 Definition. *An elliptic L_∞ algebra is a local L_∞ algebra $\mathcal{L}$ as above with the property that $(\mathcal{L}, \mathrm{d})$ is an elliptic complex.*

If $\mathcal{L}$ is a local L_∞ algebra on a manifold M, then it yields a presheaf $B\mathcal{L}$ of formal moduli problems on M. This presheaf sends a dg Artinian ring (R, m), and an open subset $U \subset M$, to the simplicial set then we can consider the simplicial set

$$B\mathcal{L}(U)(R) = \mathrm{MC}(\mathcal{L}(U) \otimes m)$$

of Maurer-Cartan elements of the L_∞ algebra $\mathcal{L}(U) \otimes m$ (where $\mathcal{L}(U)$ refers to the sections of L on U). We will think of this as the R-points of the formal pointed moduli problem associated to $\mathcal{L}(U)$.

4.3.4.3 Definition. *A formal pointed elliptic moduli problem (or simply elliptic moduli problem) is a sheaf of formal moduli problems on M, which is represented by an elliptic L_∞ algebra.*

4.4. Examples of elliptic moduli problems related to gauge theories

4.4.1. Flat bundles.

Next, let us discuss a more geometric example of an elliptic moduli problem: that describing flat bundles on a manifold M. In this case, because flat bundles have automorphisms, it is more difficult to give a direct definition of the formal moduli problem.

Thus, let G be a Lie group, and let $P \to M$ be a principal G-bundle equipped with a flat connection. Let $\mathfrak{g}_P$ be the adjoint bundle (associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$). Thus, $\mathfrak{g}_P$ is a bundle of Lie algebras on M, with a flat connection.

Let R be an Artinian dg ring. We want to define the simplicial set $\mathrm{Def}_P(R)$ of R-families of flat G-bundles on M which deform P.

As the underlying topological bundle of P is rigid, we can only deform the flat connection on P. A deformation of the connection on P is given by an element

$$A \in \Omega^1(M, \mathfrak{g}_P) \otimes m$$

of cohomological degree 0.

We would like to ask that A is flat up to homotopy. The curvature $F(A)$ is

$$F(A) = \mathrm{d}A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g}_P) \otimes m.$$

Note that, by the Bianchi identity, $\mathrm{d}F(A) + [A, F(A)] = 0$.

For A to be flat up to homotopy, we should ask that $F(A)$ is exact in the cochain complex $\Omega^2(M, \mathfrak{g}_P) \otimes m$ of two-forms on M. However, we should also ask that $F(A)$ be made exact in a way compatible with the Bianchi identity.

Thus, as a first approximation, we will define the zero-simplices of the deformation functor by

$$\mathrm{Def}_P^{prelim}(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m, B \in \Omega^2(M, \mathfrak{g}_P) \otimes m \mid F(A) = \mathrm{d}_R B,\ \mathrm{d}_{dR} B + [A, B] = 0\}.$$

Here, A is of cohomological degree 0 and B is of cohomological degree -1.

Note that if m is of square zero, then the Bianchi constraint on B just says that $\mathrm{d}_{dR} B = 0$. This leads to a problem: the sheaf of closed 2-forms on M is *not* fine: it has higher cohomology groups. Thus, we cannot hope to construct a deformation functor with values in homotopy sheaves of simplicial sets on M in this way.

Instead, we will ask that B satisfy the Bianchi constraint up a sequence of higher homotopies. Thus, the zero simplices of our simplicial set of deformations are defined by

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m, B \in \Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m \mid F(A) + \mathrm{d}_{dR} B + [A, B] + \tfrac{1}{2}[B, B] = 0.\}.$$

Here, d refers to the total differential on the tensor product cochain complex $\Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m$. As before, A is of cohomological degree 0 and B is of cohomological degree -1.

If we let $B_i \in \Omega^i(M, \mathfrak{g}_P) \otimes m$, then the first few constraints on the B_i can be written as

$$\begin{aligned} d_{dR} B_2 + [A, B_2] + d_R B_3 &= 0 \\ d_{dR} B_3 + [A, B_3] + \tfrac{1}{2}[B_2, B_2] + d_R B_4 &= 0. \end{aligned}$$

Thus, B_2 satisfies the Bianchi constraint up to a homotopy defined by B_3, and so on.

The higher simplices of this simplicial set must relate gauge-equivalent flat connections. If the dg ring R is concentrated in degree 0 (and so has zero differential), then we can define the simplicial set $\mathrm{Def}_P(R)$ to be the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$, which acts on $\mathrm{Def}_P(R)[0]$ in a standard way.

If R is not concentrated in degree 0, however, then the higher simplices of $\mathrm{Def}_P(R)$ must also involve elements of R of negative cohomological degree. Indeed, degree -1 elements of R should be thought of as homotopies between degree 0 elements of R, and so should contribute 1-simplices to our simplicial set.

A slick way to define a simplicial set with both desiderata is to set

$$\mathrm{Def}_P(R)[n] = \{A \in \Omega^*(M, \mathfrak{g}_P) \otimes m \otimes \Omega^*(\triangle^n) \mid d_{dR} A + d_R A + \tfrac{1}{2}[A, A] = 0\}.$$

Suppose that R is concentrated in degree 0 (so that the differential on R is zero). Then, the higher forms on M don't play any role, and

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^!(M, \mathfrak{g}_P) \otimes m \mid d_{dR} A + \tfrac{1}{2}[A, A] = 0\}.$$

One can show [Get04] that the simplicial set $\mathrm{Def}_P(R)$ is weakly homotopy equivalent to the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$. Indeed, a one-simplex in the simplicial set $\mathrm{Def}_P(R)$ is given by a family of the form $A_0(t) + A_1(t)\mathrm{d}t$, where $A_0(t)$ is a smooth family of elements of $\Omega^1(M, \mathfrak{g}_P) \otimes m$ depending on $t \in [0, 1]$, and $A_1(t)$ is a smooth family of elements of $\Omega^0(M, \mathfrak{g}_P) \otimes m$. The Maurer-Cartan equation in this context says that

$$\begin{aligned} d_{dR} A_0(t) + \tfrac{1}{2}[A_0(t), A_0(t)] &= 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} A_0(t) + [A_1(t), A_0(t)] &= 0. \end{aligned}$$

The first equation says that $A_0(t)$ defines a family of flat connections. The second equation says that the gauge equivalence class of $A_0(t)$ is independent of t. In this way, gauge equivalences are represented by one-simplices in $\mathrm{Def}_P(R)$.

It is immediate that the formal moduli problem $\mathrm{Def}_P(R)$ is represented by the elliptic dg Lie algebra

$$\mathcal{L} = \Omega^*(M, \mathfrak{g}).$$

The differential on $\mathcal{L}$ is the de Rham differential on M coupled to the flat connection on $\mathfrak{g}$.

4.4.2. Self-dual bundles. Next, we will discuss the formal moduli problem associated to the self-duality equations on a 4-manifold. We won't go into as much detail as we did for flat connections; instead, we will simply write down the elliptic L_∞ algebra representing this formal moduli problem.

Let M be an oriented 4-manifold. Let G be a Lie group, and let $P \to M$ be a principal G-bundle, and let $\mathfrak{g}_P$ be the adjoint bundle of Lie algebras. Suppose we have a connection A on P with anti-self-dual curvature:

$$F(A)_+ = 0 \in \Omega^2_+(M, \mathfrak{g}_P)$$

(here $\Omega^2_+(M)$ denotes the space of self-dual two-forms).

Then, the elliptic Lie algebra controlling deformations of (P, A) is described by the diagram

$$\Omega^0(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}} \Omega^1(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_+} \Omega^2_+(M, \mathfrak{g}_P).$$

Here d_+ is the composition of the de Rham differential (coupled to the connection on $\mathfrak{g}_P$) with the projection onto $\Omega^2_+(M, \mathfrak{g}_P)$.

Note that this elliptic Lie algebra is a quotient of that describing the moduli of flat G-bundles on M.

4.4.3. Holomorphic bundles. In a similar way, if M is a complex manifold and if $P \to M$ is a holomorphic principal G-bundle, then the elliptic dg Lie algebra $\Omega^{0,*}(M, \mathfrak{g}_P)$, with differential $\overline{\partial}$, describes the formal moduli space of holomorphic G-bundles on M.

4.5. Examples of elliptic moduli problems related to scalar field theories

4.5.1. The free scalar field theory. Let us start with the most basic example of an elliptic moduli problem, that of harmonic functions. Let M be a Riemannian manifold.

We want to consider the formal moduli problem describing functions ϕ on M which are harmonic, that is, satisfy $\mathrm{D}\,\phi = 0$ where D is the Laplacian. The base point of this formal moduli problem is the zero function.

The elliptic L_∞ algebra describing this formal moduli problem is defined by

$$\mathscr{L} = C^\infty(M) \xrightarrow{\mathrm{D}} C^\infty(M).$$

This is situated in degrees 1 and 2. The products l_n in this L_∞ algebra are all zero for $n \geq 2$.

In order to justify this definition, let us analyze the Maurer-Cartan functor of this L_∞ algebra. Let R be an ordinary (not dg) Artinian ring, and let m be the maximal ideal of R. The set of 0-simplices of the simplicial set $\mathrm{MC}_{\mathscr{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m \mid \mathrm{D}\,\phi = 0.\}$$

Indeed, because the L_∞ algebra $\mathscr{L}$ is Abelian, the set of solutions to the Maurer-Cartan equation is simply the set of closed degree 1 elements of the cochain complex $\mathscr{L} \otimes m$. All higher simplices in the simplicial set $\mathrm{MC}_{\mathscr{L}}(R)$ are constant. Indeed, if $\phi \in \mathscr{L} \otimes m \otimes \Omega^*(\triangle^n)$ is a closed element in degree 1, then ϕ must be in $C^\infty(M) \otimes m \otimes \Omega^0(\triangle^n)$. The fact that ϕ is closed amounts to the statement that $\mathrm{D}\,\phi = 0$ and that $\mathrm{d}_{dR}\phi = 0$, where d_{dR} is the de Rham differential on $\Omega^*(\triangle^n)$.

Let us now consider the Maurer-Cartan simplicial set associated to a differential graded Artinian ring (R, m) with differential d_R. The the set of 0-simplices of $\mathrm{MC}_{\mathscr{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m^0,\ \ \psi \in C^\infty(M) \otimes m^{-1} \mid \mathrm{D}\,\phi = \mathrm{d}_R\psi.\}$$

(The superscripts on m indicate the cohomological degree). Thus, the zero-simplices of our simplicial set can be identified with the set R-valued smooth functions ϕ on M which are harmonic up to a homotopy given by ψ, and which vanish modulo the maximal ideal m.

Next, let us identify the set of 1-simplices of the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathscr{L}}(R)$. This is the set of closed degree 0 elements of $\mathscr{L} \otimes m \otimes \Omega^*([0,1])$. Such a closed degree 0 element has four terms:

$$\begin{aligned}\phi_0(t) &\in C^\infty(M) \otimes m^0 \otimes \Omega^0([0,1]) \\ \phi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^1([0,1]) \\ \psi_0(t) &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^0([0,1]) \\ \psi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-2} \otimes \Omega^1([0,1]).\end{aligned}$$

Being closed amounts to the sequence of equations

$$\begin{aligned}\mathrm{D}\,\phi_0(t) &= \mathrm{d}_R\psi_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_0(t) &= \mathrm{d}_R\phi_1(t) \\ \mathrm{D}\,\phi_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\psi_0(t) &= \mathrm{d}_R\psi_1(t).\end{aligned}$$

These equations can be interpreted as follows. We think of $\phi_0(t)$ as providing a family of R-valued smooth functions on M, which are harmonic up to a homotopy specified by $\psi_0(t)$. Further, $\phi_0(t)$ is independent of t, up to a homotopy specified by $\phi_1(t)$. Finally, we have a coherence condition among our two homotopies.

The higher simplices of the simplicial set have a similar interpretation.

4.5.2. Interacting scalar field theories. Next, we will consider an elliptic moduli problem which arises as the Euler-Lagrange equation for an interacting scalar field theory. The Euler-Lagrange equation for the action functional $\frac{1}{2}\int \phi\,\mathrm{D}\,\phi + \frac{1}{4!}\phi^4$ (where $\phi \in C^\infty(M)$ is a smooth function) is the equation

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = 0.$$

The formal moduli problem of solutions to this equation can be described as the solutions to the Maurer-Cartan equation in a certain elliptic L_∞ algebra which (as always) we call $\mathcal{L}$. As a cochain complex, $\mathcal{L}$ is

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

Thus, $C^\infty(M)$ is situated in degrees 1 and 2, and the differential is the Laplacian.

The L_∞ brackets l_n are all zero except for l_3. The cubic bracket l_3 is the map

$$\begin{aligned} l_3 : C^\infty(M)^{\otimes 3} &\to C^\infty(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3. \end{aligned}$$

Here, the copy of $C^\infty(M)$ appearing in the source of l_3 is the one situated in degree 1, whereas that appearing in the target is the one situated in degree 2.

If R is an ordinary (not dg) Artinian ring, then the Maurer-Cartan simplicial set $MC_{\mathscr{L}}(R)$ associated to R has for 0-simplices the set $\phi \in C^\infty(M) \otimes m$ such that $D\phi + \frac{1}{3!}\phi^3 = 0$. The higher simplices of this simplicial set are constant.

If R is a dg Artinian ring, then the simplicial set $MC_{\mathscr{L}}(R)$ has for zero simplices the set of pairs $\phi \in C^\infty(M) \otimes m^0$ and $\psi \in C^\infty(M) \otimes m^{-1}$ such that

$$D\phi + \tfrac{1}{3!}\phi^3 = d_R \psi.$$

We should interpret this as saying that ϕ satisfies the Euler-Lagrange equations up to a homotopy given by ψ.

The higher simplices of this simplicial set have an interpretation similar to that described for the free theory.

4.6. Cochains of a local Lie algebra

Let L be a local L_∞ algebra on M. If $U \subset M$ is an open subset, then $\mathscr{L}(U)$ denotes the L_∞ algebra of sections of L on U. Let $\mathcal{L}_c(U) \subset \mathcal{L}(U)$ denote the sub-L_∞ algebra of compactly supported sections.

In the appendix (section A.7) we defined the algebra of functions on the space of sections on a vector bundle on a manifold. We are interested in the algebra

$$\mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq 0} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R} \right)_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps.

This space is naturally a graded differentiable vector space. However, it is important that we treat this object as a differentiable pro-vector space. Basic facts about differentiable pro-vector spaces are developed in the Appendix A. The pro-structure comes the filtration

$$F^i \mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq i} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R}\right)_{S_n}.$$

The L_∞ algebra structure on $\mathcal{L}(U)$ gives, as usual, a differential on $\mathscr{O}(\mathcal{L}(U)[1])$, making $\mathscr{O}(\mathcal{L}(U)[1])$ into a differentiable pro-cochain complex.

4.6.0.1 Definition. *Define the Lie algebra cochain complex $C^*(\mathcal{L}(U))$ to be*

$$C^*(\mathcal{L}(U)) = \mathscr{O}(\mathcal{L}(U)[1])$$

equipped with the usual Chevalley-Eilenberg differential. Similarly, define

$$C^*_{red}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

to be the reduced Chevalley-Eilenberg complex, that is, the kernel of the natural augmentation map $C^*(\mathcal{L}(U) \to R$. These are both differentiable pro-cochain complexes.

Of course, one can define $C^*(\mathcal{L}_c(U)$ in the same way.

We will think of $C^*(\mathcal{L}(U))$ as the algebra of functions on the formal moduli problem $B\mathcal{L}(U)$ associated to the L_∞ algebra $\mathcal{L}(U)$.

4.6.1. Cochains with coefficients in a module. Let L be a local L_∞ algebra on M, and let E be a graded vector bundle on M, equipped with a differential which is a differential operator. As usual, we will let $\mathscr{L}$ and $\mathscr{E}$ denote the global sections of L and E, respectively.

4.6.1.1 Definition. *A local action of L on E is an action of $\mathscr{L}$ on $\mathscr{E}$ with the property that the structure maps*

$$\mathscr{L}^{\otimes n} \otimes \mathscr{E} \to \mathscr{E}$$

are all polydifferential operators.

Note that $L^! = L^\vee \otimes_{C^\infty_M} \mathrm{Dens}_M$ has the structure of a local module over $\mathscr{L}$.

If E is a local module over L, then, for each $U \subset M$, we can define the Chevalley-Eilenberg cochains

$$C^*(\mathcal{L}(U), \mathscr{E}(U))$$

of $\mathcal{L}(U)$ with coefficients in $\mathscr{E}(U)$. As above, one needs to take account of the topologies on the vector spaces $\mathcal{L}(U)$ and $\mathscr{E}(U)$ when defining this Chevalley-Eilenberg cochain complex. Thus, as a graded vector space,

$$C^*(\mathcal{L}(U), \mathscr{E}(U)) = \prod_{n \geq 0} \mathrm{Hom}((\mathcal{L}(U)[1])^{\otimes n}, \mathscr{E}(U))_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps. Again, we treat this object as a differentiable pro-cochain complex.

As explained in the section on formal moduli problems (section 4.3), we should think of a local module E over L as providing, on each open subset $U \subset M$, a vector bundle on the formal moduli problem $B\mathcal{L}(U)$ associated to $\mathcal{L}(U)$. Then the Chevalley-Eilenberg cochain complex $C^*(\mathcal{L}(U), \mathscr{E}(U))$ should be thought of as the space of sections of this vector bundle.

4.7. D-modules and local Lie algebras

Our definition of local L_∞ algebra is designed to encode derived moduli spaces of solutions to non-linear differential equations. An alternative language for describing differential equations is the theory of D-modules. In this section we will show how our local L_∞ algebras can also be viewed as L_∞ algebras in the symmetric monoidal category of D-modules.

The main motivation for this extra layer of formalism is that local action functionals – which play a central role in classical field theory – are elegantly described using the language of D-modules.

Let C^∞_M denote the sheaf of smooth functions on the manifold M, Dens_M the sheaf of densities, and D_M the sheaf of differential operators. The infinite jet bundle $Jet(E)$ of our vector bundle E is the vector bundle whose fiber at a point $x \in M$ is the space of formal germs at x of sections of E. The sheaf of sections of $Jet(E)$, denoted $J(E)$, is equipped with a canonical D_M-module structure, i.e., the natural flat connection sometimes known as the Cartan distribution. (For motivation, observe that a field ϕ (a section of E) gives a section of $Jet(E)$ that encodes all the *local* information about ϕ.)

The category of D_M modules has a symmetric monoidal structure, given by tensoring over C^∞_M. The following lemma allows us to translate our definition of local L_∞ algebra into the world of D-modules.

4.7.0.2 Lemma. *Let $E_1, \dots, E_n, F$ be vector bundles on M, and let $\mathscr{E}_i, \mathscr{F}$ denote their spaces of global sections. Then, there is a natural bijection*

$$\mathrm{PolyDiff}(\mathscr{E}_1 \times \cdots \times \mathscr{E}_n, \mathscr{F}) \cong \mathrm{Hom}_{D_M}(J(E_1) \otimes \cdots \otimes J(E_n), J(F))$$

where PolyDiff *refers to the space of polydifferential operators. Further, this bijection is compatible with composition.*

A more formal statement of this lemma is that the multi-category of vector bundles on M, with morphisms given by polydifferential operators, is a full subcategory of the symmetric monoidal category of D_M modules; with the embedding given by taking jets. The proof of this lemma (which is straightforward) is presented in [Cos11c], Chapter 5.

4.7.0.3 Corollary. *Let L be a local L_∞ algebra on M. Then, $J(L)$ has the structure of L_∞ algebra in the category of D_M modules.*

Indeed, the lemma implies that to give a local L_∞ algebra on M is the same as to give a graded vector bundle L on M together with an L_∞ structure on the D_M module $J(L)$.

We are interested in the Chevalley cochains of $J(L)$, taken in the category of D_M modules. Because $J(L)$ is an inverse limit of the sheaves of finite-order jets, some care needs to be taken when defining this Chevalley cochain complex.

In general, if E is a vector bundle, let $J(E)^\vee$ denote the sheaf $\mathrm{Hom}_{C^\infty_M}(J(E), C^\infty_M)$, where $\mathrm{Hom}_{C^\infty_M}$ denotes continuous linear maps of C^∞_M-modules. This sheaf is naturally a D_M-module. We can form the completed symmetric algebra

$$\mathscr{O}_{red}(J(E)) = \prod_{n>0} \mathrm{Sym}^n\ J(E)^\vee,$$

which is a D_M-algebra.

We should think of an element of $\mathscr{O}_{red}(J(E))$ as a Lagrangian on the space $\mathscr{E}$ of sections of E. Indeed, a section

$$F_n \in \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)$$

is something which takes n elements $\phi_1, \dots, \phi_n \in \mathscr{E}$ and yields a smooth function $F_n(\phi_1, \dots, \phi_n) \in C^\infty(M)$, with the property that $F_n(\phi_1, \dots, \phi_n)(x)$ only depends on the jet of ϕ_i at x.

Let F be a section of $\mathscr{O}_{red}(E)$, and let us write F as a sum $F = \sum F_n$, where

$$F_n \in \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)_{S_n}.$$

Then, we can interpret F as something which takes a section $\phi \in \mathscr{E}$ and yields a smooth function

$$\sum F_n(\phi, \dots, \phi) \in C^\infty(M),$$

with the property that $F(\phi)(x)$ only depends on the jet of ϕ at x.

Of course, the functional F is a formal power series in the variable ϕ. A formal way to say what such a power series is is to use the functor of points: if R is an auxiliary graded Artin ring with maximal ideal m, and if $\phi \in \mathscr{E} \otimes m$, then $F(\phi)$ is an element of $C^\infty(M) \otimes m$. This assignment is functorial with respect to maps of graded Artin rings.

4.7.1. Local functionals. We have seen that we can interpret $\mathscr{O}_{red}(J(E))$ as the sheaf of Lagrangians on a graded vector bundle E on M. Thus, the sheaf

$$\mathrm{Dens}_M \otimes_{C^\infty_M} \mathscr{O}_{red}(J(E))$$

is the sheaf of Lagrangian densities on M. A section F of this sheaf is something which takes a section $\phi \in \mathscr{E}$ of $\mathscr{E}$, and yields a density $F(\phi)$ on M, in such a way that $F(\phi)(x)$ only depends on the jet of ϕ at x. (As before, F is of course a formal power series in the variable ϕ).

The sheaf of local action functionals is the sheaf of Lagrangians, modulo total derivatives. The formal definition is as follows.

4.7.1.1 Definition. *Let E be a graded vector bundle on M, whose space of global sections is $\mathscr{E}$. Then, the space of* local action functionals *on $\mathscr{E}$ is*

$$\mathscr{O}_{loc}(\mathscr{E}) = \mathrm{Dens}_M \otimes_{D_M} \mathscr{O}_{red}(J(E)).$$

Here, Dens_M is the right D_M-module of densities on M.

Let $\mathscr{O}_{red}(\mathscr{E}_c)$ denote the algebra of functionals modulo constants on the space $\mathscr{E}_c$ of compactly supported sections of E. Integration over M provides a natural inclusion

$$\mathscr{O}_{loc}(\mathscr{E}) \to \mathscr{O}_{red}(\mathscr{E}_c).$$

4.7.2. Local Chevalley complex of a local Lie algebra. Let L be a local Lie algebra. Then, we can form, as above, the reduced Chevalley cochain complex $C^*_{red}(J(L))$ of L. This is the D_M-algebra $\mathscr{O}_{red}(J(L)[1])$ equipped with a differential encoding the L_∞ structure on L.

In general, if $\mathfrak{g}$ is an L_∞ algebra, we will think of the Lie algebra cochain complex $C^*(\mathfrak{g})$ as being the algebra of functions on $B\mathfrak{g}$. Thus, we will let

$$\mathscr{O}_{loc}(B\mathscr{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L))$$

denote the space of local action functionals on $J(L)[1]$, equipped with the Chevalley-Eilenberg differential. This is the local Chevalley cochain complex. We could also use the notation $C^*_{red,loc}(\mathcal{L})$ for this complex.

Note that there's a natural inclusion of cochain complexes

$$\mathscr{O}_{loc}(B\mathscr{L}) \to C^*_{red}(\mathcal{L}_c)$$

where $\mathcal{L}_c$ denotes the L_∞ algebra of compactly supported sections of L.

4.7.3. Modules over local L_∞ algebras. Let L be a local L_∞ algebra on M, and let E be a local module for L. Then, $J(E)$ has an action of the L_∞ algebra $J(L)$, in a way compatible with the D_M-module on both $J(E)$ and $J(L)$.

4.7.3.1 Definition. *Suppose that E has a local action of L. Then the local cochains $C^*_{loc}(\mathscr{L}, \mathscr{E})$ of $\mathscr{L}$ with coefficients in $\mathscr{E}$ is defined to be the flat sections of the D_M-module of cochains of $J(L)$ with coefficients in $J(E)$.*

More explicitly, the D_M-module $C^*(J(L), J(E)$ is

$$\prod_{n \geq 0} \mathrm{Hom}_{C^\infty_M} \left((J(L)[1])^{\otimes n}, J(E) \right)_{S_n}$$

equipped with the usual Chevalley-Eilenberg differential. The sheaf of flat sections of this D_M module is the subsheaf

$$\prod_{n\geq 0} \mathrm{Hom}_{D_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n}$$

where the maps must be D_M-linear. In light of the fact that

$$\mathrm{Hom}_{D_M}\left(J(L)^{\otimes n}, J(E)\right) = \mathrm{PolyDiff}(\mathscr{L}^{\otimes n}, \mathscr{E})$$

we see that $C^*_{loc}(\mathscr{L}, \mathscr{E})$ is precisely the subcomplex of the Chevalley-Eilenberg cochain complex

$$C^*(\mathscr{L}, \mathscr{E}) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}((\mathscr{L}[1])^{\otimes n}, \mathscr{E})_{S_n}$$

consisting of those cochains built up from polydifferential operators.

4.8. The classical BV formalism in finite dimensions

Before we discuss the Batalin-Vilkovisky formalism for classical field theory, we will discuss a finite-dimensional toy model (which we can think of as a 0-dimensional classical field theory). Our model for the space of fields is a finite-dimensional smooth manifold manifold M. The "action functional" is given by a smooth function $S \in C^\infty(M)$. Classical field theory is concerned with solutions to the equations of motion. In our setting, the equations of motion are given by the subspace $\mathrm{Crit}(S) \subset M$. Our toy model will not change if M is a smooth algebraic variety or a complex manifold, or indeed a smooth formal scheme. Thus we will write $\mathscr{O}(M)$ to indicate whatever class of functions (smooth, polynomial, holomorphic, power series) we are considering on M.

If S is not a nice function, then this critical set can by highly singular. The classical Batalin-Vilkovisky formalism tells us to take, instead the *derived* critical locus of S. (Of course, this is exactly what a derived algebraic geometer ([**Lur09b**], [**Toë06**]) would tell us to do as well).

The critical locus of S is the intersection of the graph

$$\Gamma(\mathrm{d}S) \subset T^*M$$

with the zero-section of the cotangent bundle of M. Algebraically, this means that we can write the algebra $\mathscr{O}(\mathrm{Crit}(S))$ of functions on $\mathrm{Crit}(S)$ as a tensor product

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

Derived algebraic geometry tells us that the derived critical locus is obtained by replacing this tensor product with a derived tensor product. Thus, the derived critical locus of S (which we denote $\mathrm{Crit}^h(S)$ is an object such that

$$\mathscr{O}(\mathrm{Crit}^h(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

In derived algebraic geometry, as in ordinary geometry, spaces are determined by their algebras of functions. In derived geometry, however, one allows differential-graded algebras as algebras of functions (normally one restricts attention to differential-graded algebras concentrated in non-positive cohomological degrees).

We will take this derived tensor product as a definition of $\mathscr{O}(\mathrm{Crit}^h(S))$.

4.8.1. An explicit model. It is convenient to consider an explicit model for the derived tensor product. By taking a standard Koszul resolution of $\mathscr{O}(M)$ as a module over $\mathscr{O}(T^*M)$, one sees that $\mathscr{O}(\mathrm{Crit}^h(S))$ can be realized as the complex

$$\mathscr{O}(\mathrm{Crit}^h(S)) \simeq \ldots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} \mathscr{O}(M).$$

In other words, we can identify $\mathscr{O}(\mathrm{Crit}^h(S))$ with functions on the graded manifold $T^*[-1]M$, equipped with the differential given by contracting with $\mathrm{d}S$.

Note that

$$\mathscr{O}(T^*[-1]M) = \Gamma(M, \wedge^* TM)$$

has a Poisson bracket of cohomological degree 1, called the Schouten-Nijenhuis bracket. This Poisson bracket is characterized by the fact that if $f, g \in \mathscr{O}(M)$ and $X, Y \in \Gamma(M, TM)$, then

$$\begin{aligned} \{X, Y\} &= [X, Y] \\ \{X, f\} &= Xf \\ \{f, g\} &= 0 \end{aligned}$$

(the Poisson bracket between other elements of $\mathscr{O}(T^*[-1]M)$ is inferred from the Leibniz rule).

The differential on $\mathscr{O}(T^*[-1]M)$ corresponding to that on $\mathscr{O}(\mathrm{Crit}^h(S))$ is given by

$$\mathrm{d}\phi = \{S, \phi\}$$

for $\phi \in \mathscr{O}(T^*[-1]M)$.

The derived critical locus of any function is a dg manifold equipped with a symplectic form of cohomological degree -1. In the Batalin-Vilkovisky formalism, the space of fields always has such a symplectic structure. However, one does not require that the space of fields arises as the derived critical locus of a function.

4.8.2. The classical BV formalism in infinite dimensions. We would like to consider classical field theories in the BV formalism. For us, such a classical field theory will be specified by an elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

We defined the notion of formal elliptic moduli problem on a manifold M using the language of L_∞ algebras. Thus, in order to give the definition of a classical field theory, we need to understand the following question: what extra structure on an L_∞ algebra $\mathfrak{g}$ endows the corresponding formal moduli problem with a symplectic form?

The answer to this question was given by Kontsevich [**Kon93**]. Given a pointed formal moduli problem $\mathcal{M}$, the associated L_∞ algebra $\mathfrak{g}_{\mathcal{M}}$ has the property that

$$\mathfrak{g}_{\mathcal{M}} = T_p\mathcal{M}[-1].$$

Further, we can identify geometric objects on $\mathcal{M}$ in terms of $\mathfrak{g}_{\mathcal{M}}$ as follows.

$C^*(\mathfrak{g}_{\mathcal{M}})$	The algebra $\mathscr{O}(\mathcal{M})$ of functions on $\mathcal{M}$
$\mathfrak{g}_{\mathcal{M}}$-modules V	$\mathscr{O}_{\mathcal{M}}$-modules
$C^*(\mathfrak{g}_{\mathcal{M}}, V)$	the $\mathscr{O}_{\mathcal{M}}$ module corresponding to V
The $\mathfrak{g}_{\mathcal{M}}$-module $g_{\mathcal{M}}[1]$	$T\mathcal{M}$

Following this logic, we see that the complex of two-forms on $\mathcal{M}$ can be identified with $C^*(\mathfrak{g}_{\mathcal{M}}, \wedge^2(\mathfrak{g}_{\mathcal{M}}^\vee[-1]))$.

However, on a symplectic formal manifold, one can always choose Darboux coordinates. Changes of coordinates on $\mathcal{M}$ correspond to L_∞ isomorphisms on $\mathfrak{g}_{\mathcal{M}}$. In Darboux coordinates, the symplectic form has constant coefficients, and thus can be viewed as a $\mathfrak{g}_{\mathcal{M}}$-invariant element of $\wedge^2(g_{\mathcal{M}}^\vee[-1])$.

Note that the usual Koszul rules of signs imply that

$$\wedge^2(g_{\mathcal{M}}^\vee[-1]) = \mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}}^\vee)[-2].$$

To give a $\mathfrak{g}_{\mathcal{M}}$-invariant element of $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}}^\vee)$ is the same as to give an invariant symmetric bilinear form on $\mathfrak{g}_{\mathcal{M}}$.

Thus, we arrive at the following principle:

> To give a formal pointed derived moduli problem with a symplectic form of cohomological degree k is the same as to give an L_∞ algebra with an invariant and non-degenerate pairing of cohomological degree $k-2$.

We will define a classical field theory to be an elliptic L_∞ algebra equipped with a non-degenerate invariant pairing of cohomological degree -3. Let us first define what it means to have an invariant pairing on an elliptic L_∞ algebra.

4.8.2.1 Definition. *Let M be a manifold, and let E be an elliptic L_∞ algebra on M. An invariant pairing on E of cohomological degree k is a symmetric vector bundle map*

$$\langle -,- \rangle_E : E \otimes E\, \mathrm{Dens}(M)[k]$$

satisfying some additional conditions:

(1) Non-degeneracy: we require that this pairing induces a vector bundle isomorphism

$$E \to E^\vee \otimes \mathrm{Dens}(M)[-3].$$

(2) Invariance: let $\mathscr{E}_c$ denotes the space of compactly supported sections of E. The pairing on E induces an inner product on $\mathscr{E}_c$, defined by

$$\begin{aligned} \langle -,- \rangle : \mathscr{E}_c \otimes \mathscr{E}_c &\to \mathbb{R} \\ \alpha \otimes \beta &\to \int_M \langle \alpha, \beta \rangle . \end{aligned}$$

We require that this is an invariant pairing on the L_∞ algebra $\mathscr{E}_c$.

Recall that a symmetric pairing on an L_∞ algebra $\mathfrak{g}$ is called invariant if, for all n, the linear map

$$\mathfrak{g}^{\otimes n+1} \to \mathbb{R}$$
$$\alpha_1 \otimes \cdots \otimes \alpha_{n+1} \mapsto \langle l_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle$$

is graded anti-symmetric in the α_i.

4.8.2.2 Definition. *A formal pointed elliptic moduli problem on with a symplectic form of cohomological degree k on a manifold M is an elliptic L_∞ algebra on M with an invariant pairing of cohomological degree $k-2$.*

4.8.2.3 Definition. *A (perturbative) classical field theory on M in the BV formalism is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.*

4.9. The exterior derivative of a local action functional

The critical locus of a function f is, of course, the zero locus of the one-form $\mathrm{d}f$. We are interested in constructing the derived critical locus of a local functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$ on the formal moduli problem associated to a local L_∞ algebra on a manifold M. Thus, we need to understand what kind of object the exterior derivative $\mathrm{d}S$ of such an S is.

If $\mathfrak{g}$ is an L_∞ algebra, then we should think of $C^*_{red}(\mathfrak{g})$ as the algebra of functions on the formal moduli problem $B\mathfrak{g}$ associated to $\mathfrak{g}$, which vanish at the base point. Similarly, $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$ should be the thought of as the space of one-forms on $B\mathfrak{g}$. The exterior derivative is thus a map

$$C^*_{red}(\mathfrak{g}) \to C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]).$$

We will define a similar exterior derivative for a local Lie algebra L on M. The analog of $\mathfrak{g}^\vee$ is the L-module $L^!$. Thus, our exterior derivative will be a map

$$\mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Recall that $\mathscr{O}_{loc}(B\mathcal{L})$ is the subcomplex of the complex $C^*_{red}(\mathcal{L}_c(M))$ consisting of local functionals. The exterior derivative for the L_∞ algebra $\mathcal{L}_c(M)$ is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \mathcal{L}_c(M)^\vee[-1]).$$

Note that the dual $\mathcal{L}_c(M)^\vee$ of $\mathcal{L}_c(M)$ is the space $\overline{\mathcal{L}}^!(M)$ of distributional sections of the bundle $L^!$ on M. Thus, the exterior derivative can be viewed as a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)[-1]).$$

Note that

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \subset C^*(\mathcal{L}_c(M), \mathcal{L}^!(M)) \subset C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)).$$

4.9.0.4 Lemma. *The exterior derivative takes the subspace $\mathscr{O}_{loc}(B\mathcal{L})$ of $C^*_{red}(\mathcal{L}_c(M))$ to the subspace $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ of $C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M))$.*

PROOF. The content of this lemma is the familiar statement that the Euler-Lagrange equations associated to a local action functional are differential equations. We will give a formal proof, but the reader will see that all that is used is integration by parts.

Any functional

$$F \in \mathscr{O}_{loc}(B\mathcal{L})$$

can be written as a sum $F = \sum F_n$ where

$$F_n \in \mathrm{Dens}_M \otimes_{D_M} \mathrm{Hom}_{C^\infty_M}\left(J(L)^{\otimes n}, C^\infty_M\right)_{S_n}.$$

Any such F_n can be written as a finite sum

$$F_n = \sum_i \omega D^i_1 \dots D^i_n$$

where ω is a section of Dens_M and D^i_j are differential operators from $\mathcal{L} \to C^\infty_M$.

If we view $F \in \mathscr{O}(\mathcal{L}_c(M))$, then the n^{th} Taylor component of F is the linear map

$$\mathcal{L}_c(M)^{\otimes n} \to \mathbb{R}$$

defined by

$$\phi_1 \otimes \cdots \otimes \phi_n \to \sum_i \int_M \omega (D^i_1\phi_1) \dots (D^i_n\phi_n).$$

Thus, the $(n-1)^{th}$ Taylor component of $\mathrm{d}F$ is given by the linear map

$$\mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} \to \overline{L}^!(M) = \mathcal{L}_c(M)^\vee$$
$$\phi_1 \otimes \cdots \otimes \phi_{n-1} \sum_i \mapsto \omega(D_1^i \phi_1)\dots(D_{n-1}^i\phi_{n-1})D_n^i(-) + \text{symmetric terms}$$

where the right hand side is viewed as a linear map from $\mathcal{L}_c(M)$ to $\mathbb{R}$. Now, by integration by parts, we see that

$$(\mathrm{d}F_n)(\phi_1, \dots, \phi_{n-1})$$

is in the subspace $\mathcal{L}^!(M) \subset \overline{L}^!(M)$ of smooth sections of the bundle $L^!(M)$, inside the space of distributional sections.

It is clear from the explicit expressions that the map

$$\mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} \to \mathcal{L}^!(M)$$

is a polydifferential operator, and so defines an element of $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ as desired.

□

4.9.1. Field theories from action functionals. Physicists normally think of a classical field theory as being associated to an action functional. In this section we will show how to construct a classical field theory in our sense from an action functional.

We will work in a quite general setting. Recall (section 4.3.3) that we defined a local L_∞ algebra on a manifold M to be a sheaf of L_∞ algebras where the structure maps are given by differential operators. We will think of a local L_∞ algebra $\mathscr{L}$ on M as defining a formal moduli problem cut out by some elliptic equations. We will use the notation $B\mathscr{L}$ to denote this formal moduli problem.

We want to take the derived critical locus of a local action functional

$$S \in \mathscr{O}_{loc}(B\mathscr{L})$$

of cohomological degree 0. (We also need to assume that S is at least quadratic: this means that the base-point of our formal moduli problem $B\mathscr{L}$ is a critical point of S). We have seen (section 4.9) how to apply the exterior derivative to a local action functional S yields an element

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$

which we think of as being a local one-form on $B\mathscr{L}$.

The critical locus of S is the zero locus of dS. We thus need to explain how to construct a new local L_∞ algebra which we interpret as being the zero locus of dS.

4.9.2. Finite dimensional model. We will first describe the analogous construction in finite dimensions. Let $\mathfrak{g}$ be an L_∞ algebra, M be a $\mathfrak{g}$-module of finite total dimension, and α be a closed degree zero element of $C^*_{red}(\mathfrak{g}, M)$. The subscript *red* indicates that we are taking the reduced cochain complex, so that α is in the kernel of the augmentation map $C^*(\mathfrak{g}, M) \to M$.

We should think of M as a dg vector bundle on the formal derived moduli problem $B\mathfrak{g}$, and α as a section of this vector bundle. The condition that α is in the reduced cochain complex means translates into the statement that α vanishes at the basepoint of $B\mathfrak{g}$. We are interested in constructing the L_∞ algebra representing the zero locus of α.

The commutative dga representing this zero locus is given by the total complex of the double complex

$$\cdots \to C^*(\mathfrak{g}, \wedge^2 M^\vee) \xrightarrow{\alpha^\vee} C^*(\mathfrak{g}, M^\vee) \xrightarrow{\alpha^\vee} C^*(\mathfrak{g}).$$

This commutative dga is the symmetric algebra on the dual of $\mathfrak{g}[1] \oplus M[-1]$. It follows that this commutative dga is the Chevalley-Eilenberg cochain complex of $\mathfrak{g} \oplus M[-2]$, equipped with an L_∞ structure arising from the differential on this complex.

Note that $\mathfrak{g} \oplus M[-2]$ has a natural semi-direct product L_∞ structure, arising from the L_∞ structure on $\mathfrak{g}$ and the $\mathfrak{g}$ action on $M[-2]$. This L_∞ structure corresponds to the case $\alpha = 0$.

4.9.2.1 Lemma. *The L_∞ structure on $\mathfrak{g} \oplus M[-2]$ describing the zero locus of α is a deformation of the semidirect product L_∞ structure, obtained by adding to the structure maps l_n the maps*

$$\begin{aligned} D_n\alpha : \mathfrak{g}^{\otimes n} &\to M \\ X_1 \otimes \cdots \otimes X_n &\mapsto \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n} \alpha. \end{aligned}$$

PROOF. The proof is a straightforward computation. □

Note that the maps $D_n\alpha$ in the statement of the lemma are simply the homogeneous components of the cochain α.

We will let $Z(\alpha)$ denote $\mathfrak{g} \oplus M[-2]$, equipped with this L_∞ structure.

Recall that the formal moduli problem $B\mathfrak{g}$ is the functor from dg Artin rings (R, m) to simplicial sets, sending (R, m) to the simplicial set of Maurer-Cartan elements of $\mathfrak{g} \otimes m$. In order to check that we have constructed the correct derived zero locus for α, we should describe the formal moduli problem associated $Z(\alpha)$.

Thus, let (R, m) be a dg Artin ring, and $x \in \mathfrak{g} \otimes m$ be an element of degree 1, and $y \in M \oplus m$ be an element of degree -1. Then, (x, y) satisfies the Maurer-Cartan equation in $Z(\alpha)$ if and only if:

(1) x satisfies the Maurer-Cartan equation in $\mathfrak{g} \otimes m$.
(2) $\alpha(x) = \mathrm{d}_x y \in M$, where $\mathrm{d}_x : M \to M$ is the differential obtained by deforming the original differential by that arising from the Maurer-Cartan element x.

In other words, we see that an R-point of $BZ(\alpha)$ is an R-point x of $B\mathfrak{g}$, and a homotopy between $\alpha(x)$ and 0, in the fiber M_x of the bundle M at $x \in B\mathfrak{g}$.

4.9.3. The derived critical locus of a local functional. Let us now return to the situation where $\mathcal{L}$ is a local L_∞ algebra on a manifold M, and $S \in \mathscr{O}(B\mathcal{L})$ is a local functional which is at least quadratic. We let

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$

denote the exterior derivative of S. Note that $\mathrm{d}S$ is in the reduced cochain complex (that is, the kernel of the augmentation map $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \to L^![-1]$).

Let

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

be the n^{th} Taylor component of $\mathrm{d}S$. The fact that $\mathrm{d}S$ is a local cochain means that $\mathrm{d}_n S$ is a polydifferential operator.

4.9.3.1 Definition. *The derived critical locus of S is the local L_∞ algebra obtained by adding to the structure maps l_n of the semi-direct product L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![-3]$ the maps*

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!.$$

Let us denote this local L_∞ algebra by $\mathrm{Crit}(S)$. If (R, m) is an auxiliary Artinian dg ring, then a solution to the Maurer-Cartan equation in $\mathrm{Crit}(S) \otimes m$ consists of the following data.

(1) A Maurer-Cartan element $x \in \mathcal{L} \otimes m$.
(2) An element $y \in \mathcal{L}^! \otimes m$, such that

$$(\mathrm{d}S)(x) = \mathrm{d}_x y.$$

Here, $\mathrm{d}_x y$ is the differential on $L^! \otimes m$ induced by the Maurer-Cartan element x. These two equations say that x is an R-point of $B\mathcal{L}$ which satisfies the Euler-Lagrange equations up to a homotopy specified by y.

4.9.4. Symplectic structure on the derived critical locus. Recall that a classical field theory is given by a local L_∞ algebra which is elliptic, and which also has an invariant pairing of degree -3. The pairing on the local L_∞ algebra $\mathrm{Crit}(S)$ constructed above is evident: it is given by the natural bundle isomorphism

$$(L \oplus L^![-3])^![-3] \cong L \oplus L^![-3].$$

In other words, the pairing arises from the bundle map

$$L \otimes L^! \to \mathrm{Dens}_M .$$

4.9.4.1 Lemma. *This pairing is invariant under the L_∞ structure constructed from* $\mathrm{d}S$.

PROOF. The original L_∞ structure on $\mathcal{L} \oplus \mathcal{L}^![-3]$ (that is, the L_∞ structure not involving S) is easily seen to be invariant. We will verify that the deformation of this structure coming from S is also invariant.

We need to show that, if

$$\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{L}_c \oplus \mathcal{L}_c^![-3]$$

are compactly supported sections of $L \oplus L^1[-3]$, that

$$\langle l_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle$$

is totally antisymmetric in the variables α_i. Now, the part of this expression which comes from S is just

$$\left(\frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_{n+1}} \right) S(0).$$

The fact that partial derivatives commute, and the shift in grading coming from the fact that $C^*(\mathcal{L}_c) = \mathscr{O}(\mathcal{L}_c[1])$, immediately implies that this is totally antisymmetric. □

Note that, although the local L_∞ algebra $\mathrm{Crit}(S)$ has a symplectic form, it does not always define a classical field theory. It only does so under the additional assumption that the local L_∞ algebra $\mathrm{Crit}(S)$ is elliptic.

4.10. A succinct definition of a classical field theory

We defined a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In this section we will rewrite this definition in a more concise (but less conceptual) way. This is included largely for consistency with [Cos11c], and for ease of reference when we discuss the quantum theory.

4.10.0.2 Definition. *Let E be a graded vector bundle on a manifold M. Then, a degree -1 symplectic structure on E is an isomorphism of graded vector bundles*

$$\phi : E \cong E^![-1]$$

which is anti-symmetric, in the sense that $\phi^* = -\phi$ where ϕ^* is the formal adjoint of ϕ.

Note that if L is an elliptic L_∞ algebra on M with an invariant pairing of degree -3, then the graded vector bundle $L[1]$ on M has a -1 symplectic form. Indeed, by definition, L is equipped with a symmetric isomorphism $L \cong L^![-3]$, which becomes an antisymmetric isomorphism $L[1] \cong (L[1])^![-1]$.

Note also that the tangent space at the basepoint to the formal moduli problem $B\mathcal{L}$ associated to $\mathcal{L}$ is $\mathcal{L}[1]$ (equipped with the differential induced from that on $\mathcal{L}$). Thus, the algebra $C^*(\mathcal{L})$ of cochains of $\mathcal{L}$ is isomorphic, as a graded algebra without the differential, to the algebra $\mathscr{O}(\mathcal{L}[1])$ of functionals on $\mathcal{L}[1]$.

Now suppose that E is a graded vector bundle equipped with a -1 symplectic form. Let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$.

4.10.0.3 Proposition. *(1) The symplectic form on $\mathscr{E}$ induces a Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$, of degree $+1$.*

(2) *To give a local L_∞ algebra structure on $E[1]$, compatible with the given pairing on $E[1]$, is the same as to give an element $S \in \mathscr{O}_{loc}(\mathscr{E})$ which is of cohomological degree 0, at least quadratic, and satisfies the classical master equation*

$$\{S, S\} = 0.$$

PROOF. Let $L = E[-1]$. Note that L is a local L_∞ algebra, with 0 differential and bracket. We have seen that the exterior derivative (section 4.9) gives a map

$$\mathrm{d} : \mathscr{O}_{loc}(\mathscr{E}) = \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(L, L^![-1]).$$

Note that the isomorphism

$$L \cong L^![-3]$$

gives an isomorphism

$$C^*_{loc}(L, L^![-1]) \cong C^*_{loc}(L, L[2]).$$

Finally, $C^*_{loc}(L, L[2])$ is the Lie algebra controlling the deformations of L as a local L_∞ algebra. It thus remains to verify that $\mathscr{O}_{loc}(B\mathcal{L}) \subset C^*_{loc}(L, L[2])$ is a sub-Lie algebra; but this is straightforward. □

Note that the finite-dimensional analog of this statement is simply the fact that, on a formal symplectic manifold, all symplectic derivations (which correspond, after a shift, to deformations of the formal symplectic manifold) are given by Hamiltonian functions, defined up to the addition of an additive constant. The additive constant is not mentioned in our formulation because $\mathscr{O}_{loc}(\mathscr{E})$ by definition consists of functionals without a constant term.

Thus, we can make a concise definition of a field theory.

4.10.0.4 Definition. *A pre-classical field theory on a manifold M consists of a graded vector bundle E on M, equipped with a symplectic pairing of degree -1, and a local functional*

$$S \in \mathscr{O}_{loc}(\mathscr{E}_c(M))$$

of cohomological degree 0, satisfying the following properties.

(1) *S satisfies the classical master equation $\{S, S\} = 0$.*
(2) *S is at least quadratic (so that $0 \in \mathscr{E}_c(M)$ is a critical point of S).*

In this situation, we can write S as a sum (in a unique way)

$$S(e) = \langle e, Qe \rangle + I(e)$$

where $Q : \mathscr{E} \to \mathscr{E}$ is a skew self-adjoint differential operator of cohomological degree 1 and square zero.

4.10.0.5 Definition. *A pre-classical field is a classical field theory if the complex $(\mathscr{E}, Q)$ is elliptic.*

There is one more property we need of a classical field theories in order to be apply the quantization machinery of [**Cos11c**].

4.10.0.6 Definition. *A* gauge fixing operator *is a map*

$$Q^{GF} : \mathscr{E}(M) \to \mathscr{E}(M)$$

which is a differential operator of cohomological degree -1 and square zero, such that

$$[Q, Q^{GF}] : \mathscr{E}(M) \to \mathscr{E}(M)$$

is a generalized Laplacian in the sense of [**BGV92**].

The only classical field theories we can try to quantize are those which admit a gauge fixing operator. We will only consider classical field theories which have a gauge fixing operator.

4.11. Examples of field theories from action functionals

Let us now give some basic examples of field theories arising as the derived critical locus of an action functional. We will only discuss scalar field theories in this section.

Let M be a Riemannian manifold. Let $E = \underline{\mathbb{R}}$ be the trivial vector bundle on M, and let

$$S(\phi) = \frac{1}{2} \int_M \phi \, \mathrm{D} \, \phi$$

denote the action functional for the free massless field theory on M. Here D is the Laplacian on M, viewed as a differential operator from $C^\infty(M)$ to $\mathrm{Dens}(M)$.

The derived critical locus of S is described by the elliptic L_∞ algebra

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2]$$

where Dens(M) is the global sections of the bundle of densities on M. Thus, $C^\infty(M)$ is situated in degree 1, and the space Dens(M) is situated in degree 2. The pairing between Dens(M) and $C^\infty(M)$ gives the invariant pairing on $\mathcal{L}$, which is symmetric of degree -3 as desired.

4.11.1. Interacting scalar field theories. Next, let us write down the derived critical locus for a basic interacting scalar field theory, given by the action functional

$$S(\phi) = \tfrac{1}{2}\int_M \phi \, \mathrm{D}\, \phi + \tfrac{1}{4!}\int_M \phi^4.$$

The cochain complex underlying our elliptic L_∞ algebra is, as before,

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2].$$

The interacting term $\frac{1}{4!}\int_M \phi^4$ gives rise to a higher bracket l_3 on $\mathcal{L}$, defined by the map

$$C^\infty(M)^{\otimes 3} \to \mathrm{Dens}(M)$$
$$\phi_1 \otimes \phi_2 \otimes \phi_3 \mapsto \phi_1\phi_2\phi_3 \mathrm{d}Vol.$$

Let (R, m) be a nilpotent Artinian ring, concentrated in degree 0. Then, a section of $\phi \in C^\infty(M) \otimes m$ satisfies the Maurer-Cartan equation in this L_∞ algebra if and only if

$$\mathrm{D}\, \phi + \tfrac{1}{3!}\phi^3 \mathrm{d}Vol = 0.$$

Note that this is precisely the Euler-Lagrange equation for S. Thus, the formal moduli problem associated to $\mathcal{L}$ is, as desired, the derived version of the moduli of solutions to the Euler-Lagrange equations for S.

4.12. Cotangent field theories

We have defined a field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. The basic way symplectic manifolds arise in geometry is, of course, as cotangent bundles. We can apply this in our setting: given any elliptic moduli problem, we will construct a new elliptic moduli problem – its shifted cotangent bundle – which has a symplectic form of degree -1. We will call field theories which arise by this construction *cotangent field theories*. It turns out that a surprising number of field theories of interested in mathematics and physics (including, for example, both the A- and the B-models of mirror symmetry, as well as their half-twisted versions) arise as cotangent theories.

We should regard cotangent field theories as the simplest and most basic class of non-linear field theories, just as cotangent bundles are the simplest class of symplectic vector spaces. One can show, for example, that the phase space of a cotangent field theory is always an (infinite-dimensional) cotangent bundle, whose classical Hamiltonian function is linear on the cotangent fibers.

4.12.1. The cotangent bundle to an elliptic moduli problem. Let L be an elliptic L_∞ algebra on a manifold X; and let $\mathcal{M}_L$ be the associated elliptic moduli problem.

Let $L^!$ be the bundle $L^\vee \otimes \mathrm{Dens}(X)$. Note that there is a natural pairing between compactly supported sections of L and compactly supported sections of $L^!$.

Recall that we use the notation $\mathscr{L}$ to denote the space of sections of L; we will let $\mathscr{L}^!$ denote the space of sections of $L^!$.

4.12.1.1 Definition. *Let us define $T^*[k]\mathcal{M}_L$ to be the elliptic moduli problem associated to the elliptic L_∞ algebra $L \oplus L^![k-2]$.*

This elliptic L_∞ algebra has a pairing of cohomological degree $k-2$.

The L_∞ structure on the space $\mathscr{L} \oplus \mathscr{L}^![k-2]$ of sections of the direct sum bundle $L \oplus L^![k-2]$ arises from the natural $\mathscr{L}$-module structure on $\mathscr{L}^!$.

4.12.1.2 Definition. *Let $\mathcal{M}$ be an elliptic moduli problem. Then, the* cotangent field theory *associated to $\mathcal{M}$ is the -1-symplectic elliptic moduli problem $T^*[-1]\mathcal{M}$.*

4.12.2. Examples. In this section we will list some basic examples of cotangent theories.

In order to make the discussion more transparent, we will normally not explicitly describe the elliptic L_∞ algebra related to an elliptic moduli problem; instead, we will simply define the elliptic moduli problem in terms of the geometric objects it classifies. In all examples, it is straightforward using the techniques we have discussed so far to right down the elliptic L_∞ algebra describing the formal neighborhood of a point in any of the elliptic moduli problems we will consider.

4.12.3. Self-dual Yang-Mills theory. Let X be an oriented 4-manifold equipped with a conformal class of a metric. Let G be a compact Lie group. Let $\mathcal{M}(X,G)$ denote the elliptic moduli problem parametrizing principal G-bundles on X with a connection whose curvature is self-dual.

Then, we can consider the cotangent theory $T^*[-1]\mathcal{M}(X,G)$. This theory is known in the physics literature as *self-dual Yang-Mills theory*.

Let us describe the L_∞ algebra of this theory explicitly. Observe that the elliptic L_∞ algebra describing the completion of $\mathcal{M}(X,G)$ near a point (P,∇) is

$$\Omega^0(X,\mathfrak{g}_P) \xrightarrow{d} \Omega^1(X,\mathfrak{g}_P) \xrightarrow{d_-} \Omega^2_-(X,\mathfrak{g}_P)$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to the principal G-bundle P.

Thus, the elliptic L_∞ algebra describing $T^*[-1]\mathcal{M}$ is given by the diagram

$$\begin{array}{cccc} \Omega^0(X,\mathfrak{g}_P) \xrightarrow{d} & \Omega^1(X,\mathfrak{g}_P) \xrightarrow{d_-} & \Omega^2_-(X,\mathfrak{g}_P) & \\ & \oplus & \oplus & \\ & \Omega^2_-(X,\mathfrak{g}_P) \xrightarrow{d} & \Omega^3(X,\mathfrak{g}_P) \xrightarrow{d} & \Omega^4(X,\mathfrak{g}_P) \end{array}$$

This is a standard presentation of the fields of self-dual Yang-Mills theory in the BV formalism.

Ordinary Yang-Mills theory arises as a deformation of the self-dual theory. The deformation is given by simply deforming the differential in the dg Lie algebra presented in the diagram above by including a term in the differential which is the identity mapping $\Omega^2_-(X,\mathfrak{g}_P)$ in degree 1 to the copy of $\Omega^2_-(X,\mathfrak{g}_P)$ situated in degree 2.

4.12.4. The holomorphic σ-model. Let E be an elliptic curve and let X be a complex manifold. Let $\mathcal{M}(E,X)$ denote the elliptic moduli problem parametrizing holomorphic maps from $E \to X$. As before, there is an associated cotangent field theory $T^*[-1]\mathcal{M}(E,X)$. (In [**Cos11a**] it is explained how to describe the formal neighborhood of any point in this mapping space in terms of an elliptic L_∞ algebra on E).

In [**Cos10**], this field theory was called a holomorphic Chern-Simons theory. In the physics literature ([**Wit05**], [**Kap05**]) this theory is known as the $(0,2)$ supersymmetric sigma model.

This theory has an interesting role in both mathematics and physics. For instance, it was shown in [Cos10, Cos11a] that the partition function of this theory (at least, the part which discards the contributions of non-constant maps to X) is the Witten genus of X.

4.12.5. Twisted supersymmetric gauge theories. Of course, there are a great many more examples of cotangent theories, as there are very many elliptic moduli problems. In [Cos11b], it is shown how twisted versions of supersymmetric gauge theories can be written as cotangent theories.

The most basic example is the twisted $\mathscr{N} = 1$ field theory. If X is a complex surface, and G is a complex Lie group, then the $\mathscr{N} = 1$ twisted theory is simply the cotangent theory to the elliptic moduli problem of holomorphic principal G-bundles on X. If we fix one such principal G-bundle $P \to X$, then the elliptic L_∞ algebra describing this formal moduli problem near P is

$$\Omega^{0,*}(X, \mathfrak{g}_P)$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to P.

The cotangent theory to this elliptic moduli problem is thus described by the elliptic L_∞ algebra

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus \mathfrak{g}_P^\vee \otimes K_X[-1].).$$

4.12.6. The twisted $\mathscr{N} = 2$ theory. Twisted versions of gauge theories with more supersymmetry have similar descriptions, as is explained in [Cos11b]. The $\mathscr{N} = 2$ theory is the cotangent theory to the elliptic moduli problem for holomorphic G-bundles $P \to X$ together with a holomorphic section of the adjoint bundle $\mathfrak{g}_P$. The elliptic L_∞ algebra describing this moduli problem is

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1]).$$

Thus, the elliptic L_∞ algebra for the cotangent theory is

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1] \oplus \mathfrak{g}_P^\vee \otimes K_X \oplus \mathfrak{g}_P^\vee \otimes K_X[-1]).$$

4.12.7. The twisted $\mathscr{N} = 4$ theory. Finally we will describe the twisted $\mathscr{N} = 4$ theory. There are two versions of this twisted theory: one used in the work of Vafa-Witten [VW94] on S-duality, and another considered more recently by Kapustin-Witten [KW06] in their work on geometric Langlands. Here we will describe only the latter.

Let X again be a complex surface, and G a complex Lie group. Then, the twisted $\mathscr{N} = 4$ theory is the cotangent theory to the elliptic moduli problem describing principal G-bundles $P \to X$, together with a holomorphic section $\phi \in H^0(X, T^*X \otimes \mathfrak{g}_P)$, satisfying

$$[\phi, \phi] = 0 \in H^0(X, K_X \otimes g_P).$$

Here T^*X is the holomorphic cotangent bundle of X.

The elliptic L_∞ algebra describing this is

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus T^*X \otimes \mathfrak{g}_P[-1] \oplus K_X \otimes \mathfrak{g}_P[-2]).$$

Of course, this elliptic L_∞ algebra can be rewritten as

$$(\Omega^{*,*}(X, \mathfrak{g}_P), \overline{\partial})$$

(so that the differential is just $\overline{\partial}$ and does not involve ∂). The Lie bracket arises from the commutative algebra structure on the algebra $\Omega^{*,*}(X)$ of forms on X, and the Lie bracket on $\mathfrak{g}_P$.

Thus, the elliptic Lie algebra describing the corresponding cotangent theory is

$$\Omega^{*,*}(X, \mathfrak{g}_P) \oplus \Omega^{*,*}(X, \mathfrak{g}_P)[1].$$

4.13. The graded Poisson structure on classical observables

Recall the following definition.

4.13.0.1 Definition. *A P_0 algebra (in the category of cochain complexes) is a commutative differential graded algebra together with a Poisson bracket $\{-,-\}$ of cohomological degree 1, which satisfies the Jacobi identity and the Leibniz rule.*

The main result of this section is the following.

4.13.0.2 Theorem. *For any classical field theory (section 4.10) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a weak equivalence of commutative factorization algebras.*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

Concretely, $\widetilde{\mathrm{Obs}}^{cl}(U)$ is built from functionals on the space of solutions to the Euler-Lagrange equations which have more regularity than the functionals in $\mathrm{Obs}^{cl}(U)$.

The idea of the definition of the P_0 structure is very simple. Let us start with a finite dimensional model. Let $\mathfrak{g}$ be an L_∞ algebra equipped with an invariant antisymmetric element $P \in \mathfrak{g} \otimes \mathfrak{g}$, of cohomological degree 3. This can be viewed (according to the correspondence between formal moduli problems and Lie algebras (section 4.3)) as a bivector on $B\mathfrak{g}$, and so defines a Poisson bracket on $\mathscr{O}(B\mathfrak{g}) = C^*(\mathfrak{g})$. Concretely, this Poisson bracket is defined, on the generators $\mathfrak{g}^\vee[-1]$ of $C^*(\mathfrak{g})$, to be the tensor P viewed as a map

$$\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{R}.$$

Now, let $\mathcal{L}$ be an elliptic L_∞ algebra describing a classical field theory. Then, the kernel for the isomorphism $\mathcal{L}(U) \cong \mathcal{L}^!(U)[-3]$ is an element $P \in \overline{\mathcal{L}}(U) \otimes \overline{\mathcal{L}}(U)$, which is symmetric, invariant, and of degree 3.

We would like to use this idea to define the Poisson bracket on

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

As in the finite dimensional case, in order to define such a Poisson bracket we would need an invariant tensor in $\mathcal{L}(U)^{\otimes 2}$. The tensor representing our pairing is instead in $\overline{\mathcal{L}}(U)^{\otimes 2}$, which contains $\mathcal{L}(U)^{\otimes 2}$ as a dense subspace.

We solve this problem by finding a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

on which the Poisson bracket is well-defined and such that the inclusion is a weak equivalence.

4.13.1. Overview.

Here is an overview of this section.

(1) The Poisson structure for free field theories (section 4.14) In this section, we give a simple direct construction of the P_0 structure on the observables for a

free field theory. For this class of field theories, we can find a simple model for the classical observables which has a P_0 structure.

(2) The Poisson structure for a general classical field theory (section 4.15). In this section we construct the Poisson structure in general: this construction is a little more involved.

4.14. The Poisson structure for free field theories

In this section, we will construct a P_0 structure on the factorization algebra of observables of a free field theory. More precisely, in each case, we will construct a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

of the complex of classical observables for each open subset U, such that

(1) $\widetilde{\mathrm{Obs}}^{cl}$ forms a sub-commutative factorization algebra of Obs^{cl};
(2) The inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$ is a weak equivalence of differentiable pro-cochain complexes, for each U;
(3) $\widetilde{\mathrm{Obs}}^{cl}$ has the structure of P_0 factorization algebra.

Recall that a free field theory is a classical field theory associated to an elliptic L_∞ algebra $\mathcal{L}$ which is abelian, that is, where all the brackets $\{l_n \mid n \geq 2\}$ vanish.

Thus, let L be the graded vector bundle associated to an abelian elliptic L_∞ algebra, and let $\mathcal{L}(U)$ be the elliptic complex of sections of L on U. To say that L defines a field theory means we have a symmetric isomorphism $\mathcal{L} \cong \mathcal{L}^![-3]$.

Recall (section A.2) that we use the notation $\overline{\mathcal{L}}(U)$ to denote the space of distributional sections of L on U. A lemma of Atiyah-Bott (section A.9) shows that the inclusion

$$\mathcal{L}(U) \hookrightarrow \overline{\mathcal{L}}(U)$$

is a homotopy equivalence of topological cochain complexes.

It follows that the natural map

$$C^*(\overline{L}(U)) \hookrightarrow C^*(\mathcal{L}(U))$$

is a cochain homotopy equivalence. Indeed, because we are dealing with an abelian L_∞ algebra,

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}(U)^\vee[-1])$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}(U)^\vee[-1]),$$

where, as always, the symmetric algebra is defined using the completed tensor product.

Note that
$$\mathcal{L}(U)^\vee = \overline{L}_c^!(U) = \overline{L}_c(U)[3].$$
Thus,

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}_c(U)[2])$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}_c(U)[2]).$$

We can define a Poisson bracket of degree 1 on $C^*(\overline{\mathcal{L}}(U)$ as follows. On the generators $\mathcal{L}_c(U)[2]$, it is defined to be the given pairing

$$\langle -, - \rangle : \mathcal{L}_c(U) \times \mathcal{L}_c(U) \to \mathbb{R}.$$

This extends uniquely (by the Leibniz rule) to continuous bilinear map

$$C^*(\overline{\mathcal{L}}(U) \times C^*(\overline{\mathcal{L}}(U) \to C^*(\overline{\mathcal{L}}(U).$$

In particular, we see that $C^*(\overline{\mathcal{L}}(U))$ has the structure of a P_0 algebra in the multicategory of differentiable cochain complexes.

Let us define the modified observables in this theory by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*(\overline{\mathcal{L}}(U)).$$

We have seen that $\widetilde{\mathrm{Obs}}^{cl}(U)$ is homotopy equivalent to $\mathrm{Obs}^{cl}(U)$, and that $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a P_0 structure.

4.14.0.1 Lemma. $\mathrm{Obs}^{cl}(U)$ *has the structure of a P_0 factorization algebra.*

PROOF. It remains to verify that, if $U_1, \dots, U_n$ are disjoint open subsets of M, each contained in an open subset W, then the map

$$\widetilde{\mathrm{Obs}}^{cl}(U_1) \times \cdots \times \widetilde{\mathrm{Obs}}^{cl}(U_n) \to \widetilde{\mathrm{Obs}}^{cl}(W)$$

is compatible with the P_0 structures. This map is automatically respects the commutative structure; so it suffices to verify that $\alpha \in \widetilde{\mathrm{Obs}}^{cl}(U_i)$, and $\beta \in \widetilde{\mathrm{Obs}}^{cl}(U_j)$, where $i \neq j$, then

$$\{\alpha, \beta\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

This follows immediately from the fact that if $\phi, \psi \in \mathcal{L}_c(W)$ have disjoint support, then

$$\langle \phi, \psi \rangle = 0.$$

□

4.15. The Poisson structure for a general classical field theory

In this section we will prove the following.

4.15.0.2 Theorem. *For any classical field theory (section 4.10) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a quasi-isomorphism of factorization algebras*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

4.15.1. Functionals with smooth first derivative. As in our discussion of the P_0 algebra for free field theories (section 4.14), we will define a subalgebra $\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$ consisting of functionals which have some additional smoothness properties.

Let $\mathcal{L}$ be an elliptic L_∞ algebra on M, which defines a classical field theory. Recall that the cochain complex of observables is

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)),$$

where $\mathcal{L}(U)$ is the L_∞ algebra of sections of L on U.

Recall that, as a graded vector space, $C^*(\mathcal{L}(U))$ is the algebra of functionals $\mathscr{O}(\mathcal{L}(U)[1])$ on the graded vector space $\mathcal{L}(U)[1]$. In the appendix (section A.7), given any graded vector bundle E on M, we define a subspace

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}(\mathscr{E}(U))$$

of functionals which have "smooth first derivative". Precisely, a function $\Phi \in \mathscr{O}(\mathscr{E}(U))$ is in $\mathscr{O}^{sm}(\mathscr{E}(U))$ if

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U).$$

(The exterior derivative of a general function in $\mathscr{O}(\mathscr{E}(U))$ will lie in the larger space $\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}_c^!(U)$). The space $\mathscr{O}^{sm}(\mathscr{E}(U))$ is a differentiable pro-vector space.

Recall that, if $\mathfrak{g}$ is an L_∞ algebra, the exterior derivative maps $C^*(\mathfrak{g})$ to $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$. The complex $C^*_{sm}(\mathcal{L}(U))$ of cochains with smooth first derivative is thus defined to be the subcomplex of $C^*(\mathcal{L}(U))$ consisting of this cochains whose first derivative lies in $C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1])$, which is a subcomplex of $C^*(\mathcal{L}(U), \otimes\mathcal{L}(U)^\vee[-1])$.

In other words, $C^*_{sm}(\mathcal{L}(U))$ is defined by the fiber diagram

$$\begin{array}{ccc} C^*_{sm}(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1]) \\ \downarrow & & \downarrow \\ C^*(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \overline{\mathcal{L}_c}^!(U)[-1]). \end{array}$$

(Note that differentiable pro-cochain complexes are closed under taking limits, so that this fiber product is again a differentiable pro-cochain complex; more details are provided in the appendix A.7).

Note that

$$C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

is a sub-commutative dga, and further, as U varies, $C^*_{sm}(\mathcal{L}(U))$ defines a sub-commutative prefactorization algebra of that defined by $C^*(\mathcal{L}(U))$.

We will let

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U)) = \mathrm{Obs}^{cl}(U).$$

4.15.2. The Poisson bracket. Because the elliptic L_∞ algebra L defines a classical field theory, it is equipped with an isomorphism $L \cong L^![-3]$. Thus, we have an isomorphism

$$\Phi : C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1]) \cong C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]).$$

In the appendix (section A.8) we show that $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ (which we think of as vector fields on the formal manifold $B\mathcal{L}(U)$) has a natural structure of dg Lie algebra, in the multicategory category of differentiable pro-cochain complexes. Further, $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ acts on $C^*(\mathcal{L}(U))$ by derivations. This action is in the multicategory of differentiable pro-cochain complexes: the map

$$C^*(\mathcal{L}(U), \mathcal{L}(U)[1]) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is a smooth bilinear cochain map. We will write $\mathrm{Der}(C^*(\mathcal{L}(U))$ for the Lie algebra $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$.

Thus, composing the map Φ above with the exterior derivative d, and with the inclusion $\mathcal{L}_c(U) \hookrightarrow \mathcal{L}(U)$, we find a cochain map

$$C^*_{sm}(\mathcal{L}(U)) \to C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]) \to \mathrm{Der}(C^*(\mathcal{L}(U))[1].$$

If $f \in C^*_{sm}(\mathcal{L}(U))$ we will let $X_f \in \mathrm{Der}(C^*(\mathcal{L}(U))$ be the corresponding derivation. If f has cohomological degree k, then X_f has cohomological degree $k+1$.

If $f, g \in C^*_{sm}(\mathcal{L}(U)) = \widetilde{\mathrm{Obs}}^{cl}(U)$, we define

$$\{f, g\} = X_f g \in \widetilde{\mathrm{Obs}}^{cl}(U).$$

This bracket defines a bilinear map

$$\widetilde{\mathrm{Obs}}^{cl}(U) \times \widetilde{\mathrm{Obs}}^{cl}(U) \to \widetilde{\mathrm{Obs}}^{cl}(U).$$

4.15.2.1 Lemma. *This map is smooth, that is, is a bilinear map in the multicategory of differentiable pro-cochain complexes.*

PROOF. This follows from the fact that the map

$$\mathrm{d} : \widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Der}(C^*(\mathcal{L}(U))[1]$$

is smooth (which is immediate from the definitions), and from the fact that the map

$$\mathrm{Der}(C^*(\mathcal{L}(U)) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is smooth (which is proved in the appendix A.8). □

4.15.2.2 Lemma. *This bracket satisfies the Jacobi rule and the Leibniz rule. Further, if U, V are disjoint subsets of M, both contained in W, and if $f \in \widetilde{\mathrm{Obs}}^{cl}(U)$, $g \in \widetilde{\mathrm{Obs}}^{cl}(V)$, then*

$$\{f, g\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

PROOF. The proof is straightforward. □

4.15.2.3 Corollary. $\widetilde{\mathrm{Obs}}^{cl}$ *defines a* P_0 *factorization algebra in the valued in the multicategory of differentiable pro-cochain complexes.*

The final thing we need to verify is the following.

4.15.2.4 Proposition. *For all open subset* $U \subset M$*, the map*

$$\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

is a weak equivalence.

PROOF. It suffices to show that it is a weak equivalence on the associated graded for the natural filtration on both sides. Now, $\mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U)$ fits into a fiber diagram

$$\begin{array}{ccc} \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U) & \longrightarrow & \mathrm{Sym}^n(\overline{\mathcal{L}}_c^!(U)[-1]) \otimes \mathcal{L}_c^!(U) \\ \downarrow & & \downarrow \\ \mathrm{Gr}^n \mathrm{Obs}^{cl}(U) & \longrightarrow & \mathrm{Sym}^n(\overline{\mathcal{L}}_c^!(U)[-1]) \otimes \overline{\mathcal{L}}_c^!(U). \end{array}$$

Note also that

$$\mathrm{Gr}^n \mathrm{Obs}^{cl}(U) = \mathrm{Sym}^n \overline{\mathcal{L}}_c^!(U).$$

The Atiyah-Bott lemma A.9 shows that the inclusion

$$\mathcal{L}_c^!(U) \hookrightarrow \overline{\mathcal{L}}_c^!(U)$$

is a continuous cochain homotopy equivalence. We can thus choose a homotopy inverse $P\overline{\mathcal{L}}_c^!(U) \to \mathcal{L}_c^!(U)$, and a homotopy $H : \overline{\mathcal{L}}_c^!(U) \to \overline{\mathcal{L}}_c^!(U)$ with $[\mathrm{d}, H] = P - \mathrm{Id}$; with the property that H preserves the subspace $\mathcal{L}_c^!(U)$.

Now,

$$\mathrm{Sym}^n \mathcal{L}_c^!(U) \subset \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Sym}^n \overline{\mathcal{L}}_c^!(U).$$

From the projector P and the homotopy H one can construct a projector

$$P_n = P^{\otimes n} : \overline{\mathcal{L}}_c^!(U)^{\otimes n} \to \mathcal{L}_c^!(U)^{\otimes n}$$

and a homotopy

$$H_n : \overline{L}_c^!(U)^{\otimes n} \to \mathcal{L}_c^!(U)^{\otimes n}.$$

The homotopy H_n is defined inductively by the formula

$$H_n = H \otimes P_{n-1} + 1 \otimes H_{n-1}.$$

This defines a homotopy, because

$$[\mathrm{d}, H_n] = P \otimes P_{n-1} - 1 \otimes P_{n-1} + 1 \otimes P_{n-1} - 1 \otimes 1.$$

Notice that the homotopy H_n preserves all the subspaces of the form

$$\overline{L}_c^!(U)^{\otimes k} \otimes \mathcal{L}_c^!(U) \otimes \overline{L}_c^!(U)^{\otimes n-k-1}.$$

This will be important momentarily. Next, let

$$\pi : \overline{L}_c^!(U)^{\otimes n}[-n] \to \mathrm{Sym}^n(\overline{L}_c^!(U)[-1])$$

be the projection, and let

$$\Gamma_n = \pi^{-1} \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U).$$

Then, Γ_n is acted on by the symmetric group S_n, and the S_n invariants are $\widetilde{\mathrm{Obs}^{cl}}(U)$.

Thus, it suffices to show that the inclusion

$$\Gamma_n \hookrightarrow \overline{L}_c(U)^{\otimes n}$$

is a weak equivalence of differentiable spaces; we will show that it is continuous homotopy equivalence.

The definition of $\widetilde{\mathrm{Obs}}^{cl}(U)$ allows one can identify

$$\Gamma_n = \cap_{k=0}^{n-1} \overline{L}_c^!(U)^{\otimes k} \otimes \mathcal{L}_c^!(U) \otimes \overline{L}_c^!(U)^{\otimes n-k-1}.$$

The homotopy H_n preserves Γ_n, and the projector P_n maps

$$\overline{L}_c^!(U)^{\otimes n} \to \mathcal{L}_c(U)^{\otimes n} \subset \Gamma_n.$$

Thus, P_n and H_n provide a continuous homotopy equivalence between $\overline{L}_c^!(U)^{\otimes n}$ and Γ_n, as desired. $\square$

4.16. Symmetries and deformations of a classical field theory

In this section we will give the definition of an action of an L_∞ algebra on a classical field theory. We will start by saying what it means for an L_∞ algebra to act on an elliptic moduli problem.

Recall that in homotopy theory, to give an action of a group G on an object is the same as to give a family of objects over the classifying space BG. There is a similar picture in homotopical algebra: to given an action of an L_∞ algebra $\mathfrak{g}$ on some object is the same as to give a family of such objects over $C^*(\mathfrak{g})$. We will take this as our definition of action of an L_∞ algebra $\mathfrak{g}$ on an elliptic L_∞ algebra. Thus, we need to define what it means to have a family of elliptic L_∞ algebras over some differential graded base ring.

Suppose that R is a differential graded algebra. Let $R^\sharp$ refer to R without the differential.

4.16.0.5 Definition. *An R-family of elliptic L_∞ algebras on X consists of graded bundle L of $R^\sharp$-modules on X, whose sheaf of sections will be denoted $\mathcal{L}$; together with an $R^\sharp$-linear differential operator*

$$\mathrm{d} : \mathcal{L} \to \mathcal{L}$$

which makes $\mathcal{L}$ into a sheaf of dg R-modules; and, collection of R-linear polydifferential operators

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

making $\mathcal{L}$ into a sheaf of L_∞ algebras on X over R.

Remark: Note that in this definition, R can be a nuclear Frchet dg algebra. In that case, the tensor products should be the completed projective tensor product.

4.16.0.6 Definition. *If $\mathfrak{g}$ is an L_∞ algebra, and $\mathscr{L}$ is an elliptic L_∞ algebra on a space X, a $\mathfrak{g}$-action on $\mathscr{L}$ is a family of elliptic moduli problems $\mathscr{L}^{\mathfrak{g}}$ on X, over the base ring $C^*(\mathfrak{g})$, which specialize to $\mathscr{L}$ modulo the maximal ideal $C^{>0}(\mathfrak{g})$ of $C^*(\mathfrak{g})$.*

Remark: The Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ is the completed pro-nilpotent dg algebra, which is an inverse limit

$$C^*(\mathfrak{g}) = \varprojlim C^*(\mathfrak{g}) / I^n$$

where I is the maximal ideal $C^{>0}(\mathfrak{g})$.

Let R be a differential graded algebra, and let $\mathscr{L}$ be an R-family of elliptic L_∞ algebras. Recall that this means that we have a graded bundle L of $R^\sharp$-modules on X, whose sheaf $\mathscr{L}$ of sections is equipped with a differential making it into a sheaf of dg R-modules, and with an R-linear L_∞ structure. We will let

$$L^! = L^\vee \otimes \mathrm{Dens}_X$$

where $L^\vee$ is the $R^\sharp$-linear dual of L. We will let $\mathscr{L}^!$ denote the sheaf of sections of $L^!$. This has a natural structure of sheaf of dg modules over R, with an L_∞ action of $\mathscr{L}$.

4.16.0.7 Definition. *An invariant pairing of degree k on an R-family of elliptic L_∞ algebras $\mathscr{L}$ is an R-linear isomorphism*

$$\mathscr{L} \cong \mathscr{L}^![k]$$

of sheaves of $\mathscr{L}$-modules, which is symmetric as before.

4.16.0.8 Definition. *Let $\mathfrak{g}$ be an L_∞ algebra, and let $\mathscr{L}$ be a classical field theory on a space X. Thus $\mathscr{L}$ is an elliptic L_∞ algebra on X with an invariant pairing $\mathscr{L} \cong \mathscr{L}^![-3]$ of degree -3. Then a $\mathfrak{g}$-action on $\mathscr{L}$ is a family of elliptic moduli problems $\mathscr{L}^{\mathfrak{g}}$ on X, flat over the base ring $C^*(\mathfrak{g})$, equipped with an invariant pairing of degree -3, which specializes to $\mathscr{L}$ modulo the maximal ideal $C^{>0}(\mathfrak{g})$ of $C^*(\mathfrak{g})$.*

If $\mathscr{L}$ is an elliptic L_∞ algebra on X with an action of $\mathfrak{g}$, then the cotangent field theory $T^*[-1]\mathscr{L}$ also has a natural action of $\mathfrak{g}$, compatible with the invariant pairing.

4.16.1. Symmetries and local functionals.

Let L be a classical field theory, so that L is a local L_∞ algebra on M equipped with a non-degenerate symmetric isomorphism $L \cong L^![-3]$. Recall (section 4.10) that the L_∞ structure on L can be described by a local functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$, satisfying the classical master equation $\{S, S\} = 0$.

We will use this presentation to show that the dg Lie algebra controlling symmetries and deformations of the classical field theory L is $\mathscr{O}_{loc}(B\mathcal{L})[-1]$, with the bracket $\{-,-\}$ and differential $\{S,-\}$.

More precisely, we will verify the following.

4.16.1.1 Proposition.

//Let $\mathfrak{g}$ be an L_∞ algebra. Then every $\mathfrak{g}$ action on L (in the sense described above) arises from an L_∞ map $\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{L})[-1]$. This L_∞ map is unique up to homotopy. //

PROOF. An L_∞ map $\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{L})[-1]$ is the same as an element

$$\alpha \in C^*(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{L})[-1]$$

which

Satisfies the Maurer-Cartan equation.

Vanishes modulo the maximal ideal $C^{>0}(\mathfrak{g})$ of $C^*(\mathfrak{g})$.

The dg Lie algebra structure on $C^*(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{L})[-1]$ arises from the dg commutative algebra structure on $C^*(\mathfrak{g})$ and the dg Lie algebra structure on $\mathscr{O}_{loc}(B\mathcal{L})[-1]$.

The Maurer-Cartan equation is, of course,

$$\mathrm{d}_{\mathfrak{g}}\alpha + \{S, \alpha\} + \tfrac{1}{2}\alpha, \alpha = 0$$

where $\mathrm{d}_{\mathfrak{g}}$ refers to the differential on $C^*_{red}(\mathfrak{g})$.

Given such an α, it is immediate that $\alpha + S$ gives $C^*(\mathfrak{g}) \otimes L$ the structure of a family of classical field theories over $C^*(\mathfrak{g})$, as desired.

Conversely, suppose that we have a family of classical field theories over $C^*(\mathfrak{g})$, whose space of fields is the sections of some bundle L' of $C^*(\mathfrak{g})$-modules L' on M. Let us suppose that this family reduces modulo the maximal ideal $C^{>0}(\mathfrak{g})$ of $C^*(\mathfrak{g})$ to the given classical field theory, with underlying vector bundle L on M and action functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$. Then, it is easy to verify that we can choose an isomorphism of graded $C^\sharp(\mathfrak{g})$-modules $L' \cong C^\sharp(\mathfrak{g}) \otimes L$. The structure of family of classical field theories on sections of L' is encoded in some $\phi \in C^*(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{L})[-1]$. Modulo the maximal ideal of $C^*(\mathfrak{g})$, ϕ must reduce to the given action functional S, so that $\phi = S + \alpha$ where α is as above.

The final thing we need to verify is the statement that the L_∞ map $\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{L})[-1]$ corresponding to a given family of classical field theories over $C^*(\mathfrak{g})$ is unique up to homotopy. The L_∞ map is associated to the family of classical field theories together with the isomorphism of graded $C^\sharp(\mathfrak{g})$-modules

$$\Gamma : L' \cong C^\sharp(\mathfrak{g}) \otimes L$$

(of course, this isomorphism must be the given one modulo the maximal ideal of $C^*(\mathfrak{g})$). The space of such isomorphisms is contractible. Thus, it suffices to verify that if we have a family of such isomorphisms, parametrized by the n-simplex, we get a family of L_∞ maps over the base ring $\Omega^*(\triangle^n)$. If

$$\widetilde{\Gamma} : L' \otimes C^\infty(\triangle^n) \cong C^\sharp(\mathfrak{g}) \otimes L \otimes C^\infty(\triangle^n)$$

is such a family, it extends by linearity to an isomorphism of graded $C^\sharp(\mathfrak{g}) \otimes \Omega^\sharp(\triangle^n)$-modules

$$L' \otimes \Omega^\sharp(\triangle^n) \cong C^\sharp(\mathfrak{g}) \otimes L \otimes \Omega^\sharp(\triangle^n).$$

The left hand side is equipped with the structure of a family of classical field theories over $C^*(\mathfrak{g}) \otimes \Omega^*(\triangle^n)$ (this family is constant in the $\triangle^n$-directions). As above, this structure can be interpreted as giving a Maurer-Cartan element α of $C^*(\mathfrak{g}) \otimes \Omega^*(\triangle^n) \otimes \mathscr{O}_{loc}(B\mathcal{L})[-1]$, which vanishes modulo the maximal ideal of $C^*(\mathfrak{g})$. Such a Maurer-Cartan element is precisely the same as a family of L_∞ maps $\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{L})[-1]$ over the base ring $\Omega^*(\triangle^n)$. □

4.17. Conserved quantities and Noether's theorem

There is a famous theorem of E. Noether that says, in essence, that to every infinitesimal local symmetry of a classical field theory corresponds a "conserved current." We will now describe that statement in our formalism. For simplicity, we will assume that our space-time manifold M is oriented. It is straightforward to modify our constructions to deal with the general case.

In the usual framework, a current J is something which associates to a field ϕ an $n-1$ form $J(\phi)$ on space-time, defined up to the addition of an exact $n-1$ form. Thus, we can integrate $J(\phi)$ over any oriented codimension 1 hypersurface in space-time. The association of $J(\phi)$ to ϕ must be local in nature: the value of $J(\phi)$ at a point $p \in M$ must only depend on the values of the derivatives of ϕ at p.

In our context, the space of fields is described by an elliptic L_∞ algebra $\mathcal{L}$ on space-time M. The space of $n-1$ forms is described by the Abelian L_∞ algebra $\Omega^{n-1}_M[-1]$, concentrated in degree 1. The space of $n-1$ forms modulo exact $n-1$ forms is best described by the truncated de Rham complex $\Omega^{\leq n-1}_M[n-2]$, shifted so that Ω^{n-1}_M is in degree 1. We will view $\Omega^{\leq n-1}_M[n-2]$ as an Abelian local L_∞ algebra.

4.17.0.2 Definition. *A current on the classical field theory described by $\mathcal{L}$ is a map of local L_∞ algebras*

$$J : \mathcal{L} \to \Omega_M^{\leq n-1}[n-2].$$

A map of local L_∞ algebras is a map of L_∞ algebras $\mathcal{L}(M) \to \Omega_M^{\leq n-1}[n-2]$, with the property that the constituent maps

$$\mathcal{L}^{\otimes n} \to \Omega_M^{\leq n-1}[n-2]$$

are polydifferential operators.

A map of local L_∞ algebras like this induces a map of sheaves of formal moduli problems. We will let $\mathrm{EL}(U)$ be the formal moduli problem associated to the L_∞ algebra $\mathcal{L}(U)$; so that if (R, m) is a dg Artinian algebra, $\mathrm{EL}(U)(R)$ is the simplicial set of Maurer-Cartan elements of $\mathcal{L}(U) \otimes m$.

The formal moduli problem associated to the Abelian L_∞ algebra $\Omega^{\leq n-1}(U)[n-2]$ sends a dg Artinian algebra R to the Dold-Kan simplicial set of the cochain complex $\Omega^{\leq n-1}(U)[n-1] \otimes m$. Recall that π_0 of this Dold-Kan simplicial set is $H^0(\Omega^{\leq n-1}(U)[n-1] \otimes m)$. Since R is concentrated in degrees ≤ 0, we see that π_0 of this simplicial set is

$$\left(\Omega^{n-1}(U)/\mathrm{d}\Omega^{n-2}(U)\right) \otimes H^0(m).$$

It follows that if J is a current, then for every dg Artinian ring (R, m), and for every open set $U \subset M$, we get a map

$$\pi_0 J : \pi_0(\mathrm{EL}(U)(m)) \to \left(\Omega^{n-1}(U)/\mathrm{d}\Omega^{n-2}(U)\right) \otimes H^0(m).$$

This map takes a homotopy class of solution to the Euler-Lagrange equations on U, and yields an $n-1$ form modulo an exact $n-1$ form.

4.17.1. Conserved currents. Suppose that $N \subset M$ is a compact oriented submanifold of codimension one. We will see how we can integrate J a current over N to yield a function on the formal moduli problem $\mathrm{EL}(M)$.

Let $\mathbb{A}^1$ denote the formal moduli problem which sends a dg Artinian ring (R, m) to the Dold-Kan simplicial set for the cochain complex m. The reason for this notation is that $\mathbb{A}^1$ is represented by the pro-Artinian algebra $\mathbb{R}[[t]]$. Thus, given any formal moduli problem F, a map of formal moduli problems $F \to \mathbb{A}^1$ should be regarded as a function on F.

Given a compact oriented codimension 1 submanifold N of M, a current J as above yields a map of formal moduli problems

$$\int_N J : \mathrm{EL}(M) \to \mathbb{A}^1.$$

This sends an n-simplex $\phi \in \mathrm{EL}(M)(R)[n]$ to $\int_N J(\phi)$, which is a closed degree 0 element of $m \otimes \Omega^*(\triangle^n)$.

Note that $\int_N J(\phi)$ only depends on the behavior of the field ϕ on a neighborhood of N in M. Thus, we will let $\mathrm{EL}(N)$ denote the formal moduli problem of germs of solutions to the Euler-Lagrange equation near N:

$$\mathrm{EL}(N)(R) = \varprojlim_{N \subset U \subset M} \mathrm{EL}(U)(R).$$

The integral $\int_N J$ is a function on the formal moduli problem $\mathrm{EL}(N)$.

In the usual treatment, a conserved current is a current with the property that, if we let a field evolve from one space-like hypersurface to another, the value of the current does not change.

In order to explain this concept more precisely, let us assume that our space-time manifold is of the form $M = N \times \mathbb{R}$, where N is compact. We will let N_t denote the hypersurface $N \times \{t\}$. Then, for each $t \in \mathbb{R}$ we have a map

$$\int_{N_t} J : \mathrm{EL}(N_t) \to \mathbb{A}^1.$$

The condition for $\int_{N_t} J$ to be a conserved quantity is that, if $\phi_t \in \mathrm{EL}(N_t)(R)$ is a family of fields which evolve according to the equations of motion, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{N_t} J(\phi_t) = 0.$$

To say that a family of fields ϕ_t evolves according to the equation of motion means precisely that this family arises from a global solution $\phi \in \mathrm{EL}(M)(R)$.

Thus, the statement that a current is conserved can be recast as saying that, for all solutions $\phi \in \mathrm{EL}(M)(R)$ of the equations of motion, $\int_{N_t} J(\phi)$ is independent of t. Of course, this will happen when the $n-1$ form $J(\phi)$ is closed.

A conserved current should thus be a map of local L_∞ algebras from $\mathcal{L}$ to the local L_∞ algebra describing *closed* $n-1$ forms, modulo total derivative. The latter is best described by the Abelian L_∞ algebra $\Omega^*_M[n-2]$ of all forms, shifted so that Ω^{n-1}_M is in degree 1.

4.17.1.1 Definition. *A conserved current is a map of local L_∞ algebras*

$$\mathcal{L} \to \Omega^*_M[n-2].$$

4.17.2. Noether's theorem. Now we are ready to state Noether's theorem, relating conserved currents and symmetries. We have seen that the dg Lie algebra controlling deformations and symmetries of the classical field theory $\mathcal{L}$ is $\mathscr{O}_{loc}(B\mathcal{L})$, the local functionals on $\mathcal{L}$.

In our framework, Noether's theorem is a statement about the simplicial set of conserved currents. An n-simplex in this simplicial set is simply a family of conserved currents over the base ring $\Omega^*(\triangle^n)$.

4.17.2.1 Theorem. *There is a natural weak homotopy equivalence between the simplicial set of conserved currents and the Dold-Kan simplicial set for the cochain complex $\mathscr{O}_{loc}(B\mathcal{L})[-1]$. In particular, homotopy classes of conserved currents are in bijection with the group $H^{-1}(\mathscr{O}_{loc}(B\mathcal{L}))$, which is the group of homotopy classes of infinitesimal symmetries of the classical field theory $\mathcal{L}$.*

PROOF. To give a map $\mathcal{L} \to \Omega^*_M[n-2]$ of local L_∞ algebras is the same as to give a map $J(L) \to J(\Omega^*_M)[n-2]$ of D_M L_∞ algebras.

In general, if $\mathfrak{g}$ and $\mathfrak{h}$ are L_∞ algebras, the L_∞ algebra controlling maps from $\mathfrak{g}$ to $\mathfrak{h}$ is $C^*_{red}(\mathfrak{g}) \otimes \mathfrak{h}$, the tensor product of the reduced cochain complex of $\mathfrak{g}$ with $\mathfrak{h}$.

It follows that to give a map of D_M L_∞ algebras $J(L) \to J(\Omega^*_M)[n-2]$ is the same as to give a flat section of

$$C^*_{red}(J(L)) \otimes_{C^\infty_M} J(\Omega^*_M)[n-1].$$

The sheaf of flat sections of this can be identified with

$$\Omega^*(M, C^*_{red}(J(L)))[n-1],$$

the de Rham complex of M with coefficients in the D_M-module $C^*_{red}(J(L))$, with a shift of $n-1$.

More generally, the simplicial set of maps of local L_∞ algebras $\mathcal{L} \to \Omega^*_M[n-2]$ is the Dold-Kan simplicial set associated to the cochain complex $\Omega^*(M, C^*_{red}(J(L)))[n-1]$.

Since M is oriented, and since $C^*_{red}(J(L))$ is a flat D_M-module, this cochain coincides with

$$\omega_M \otimes_{D_M} C^*_{red}(J(L))[-1] = \mathscr{O}_{loc}(B\mathcal{L})[-1].$$

Thus, the simplicial set of conserved currents is the Dold-Kan simplicial set for $\mathscr{O}_{loc}(B\mathcal{L})[-1]$, as desired. ☐

Note that is not difficult to modify these constructions to relate "higher conserved currents" (which associate to a field a closed $n-k$-form) and higher infinitesimal symmetries of a classical field theory, corresponding to the groups $H^{n-k}(\mathscr{O}_{loc}(B\mathcal{L}))$.

4.17.3. A simple example. Consider the free particle moving in the Euclidean vector space $V \cong \mathbb{R}^n$. We will show how to recover the usual conserved quantity of linear momentum using the formalism described above. We encode this physical system as a free theory on the real line $\mathbb{R}$. Let V also denote the trivial vector bundle over $\mathbb{R}^1$ with fiber V.

A section $\phi \in \Gamma(\mathbb{R}^1, V)$ satisfies the equation of motion if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi = 0$. Thus, the elliptic L_∞ algebra describing this classical field theory is

$$\mathcal{L} = \Gamma(\mathbb{R}^1, V)[-1] \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}t^2}} \Gamma(\mathbb{R}^1, V[-2]).$$

This L_∞ is Abelian. The pairing between $\mathcal{L}^1 = \Gamma(\mathbb{R}^1, V)$ and $\mathcal{L}^2 = \Gamma(\mathbb{R}^1, V)$ is

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^1} \langle \phi(t), \psi(t) \rangle_V \, \mathrm{d}t,$$

where $\langle -, - \rangle_V$ is the Euclidean inner product on V.

We now describe the obstruction-deformation complex $\mathscr{O}_{loc}(B\mathcal{L})$ for this free theory. Let J denote the infinite jet bundle of smooth functions on $\mathbb{R}^1$. Then the jet bundle $J(V)$ for sections of V is isomorphic to $J \otimes V$. Likewise, recall that $J^\vee := \mathrm{Hom}_{C^\infty}(J, C^\infty) = D_{\mathbb{R}}$, the ring of differential operators on $\mathbb{R}^1$, and so $J(V)^\vee \cong D \otimes V^\vee$. Hence the $D_{\mathbb{R}}$ L_∞ algebra associated to L is

$$J(L) = J \otimes (V[-1] \oplus V[-2]).$$

This system is invariant under the translation action of the vector space V. We will show how to construct the conserved current associated to translation by each $v \in V$. This conserved current plays the role of momentum.

The Lie algebra of symmetries and deformations of our theory is

$$\mathscr{O}_{loc}(B\mathcal{L})[-1] = \omega_{\mathbb{R}^1} \otimes_{D_{\mathbb{R}^1}} C^*_{red}(J(L))[-1],$$

which we can identify with

$$\Omega^*(\mathbb{R}^1, C^*_{red}(J(L))).$$

Since we are dealing with a free field theory, the differential on the Lie algebra cochain complex $C^*_{red}(J(L))$ preserves the grading into homogeneous components. Since momentum is a linear function on the vector space V, we are interested in symmetries coming from $J(L)^\vee[-2]$.

The de Rham complex of the D module $J(L)^\vee[-2]$ is concentrated in cohomological degrees -1, 0, and 1:

$$0 \to \Omega^0 \otimes D_{\mathbb{R}^1} \otimes V^\vee \to (\Omega^0 \otimes D_{\mathbb{R}^1} \otimes V_0^\vee) \oplus (\Omega^1 \otimes D_{\mathbb{R}^1} \otimes V_1^\vee) \to \Omega^1 \otimes D_{\mathbb{R}^1} \otimes V_0^\vee \to 0,$$

and the differential is $\nabla + Q$, where ∇ denotes the flat connection on the bundle $D_{\mathbb{R}^1}$, and Q is the differential on $J(L)^\vee[-1]$.

An element of degree 0 is a sum of terms

$$f \otimes \partial^k \otimes v,$$

with f a smooth function, $\partial = \partial/\partial x$, and $v \in V^\vee$, and of terms

$$f \, \mathrm{d}x \otimes \partial^k \otimes v.$$

We compute the differential as follows:

$$f \otimes \partial^k \otimes v + g \, \mathrm{d}x \otimes \partial^l \otimes w \mapsto (\partial f)\mathrm{d}x \otimes \partial^k \otimes v + f \, \mathrm{d}x \otimes \partial^{k+1} \otimes v + g \, \mathrm{d}x \otimes \partial^{l+2} \otimes w,$$

since we apply the connection on $D_{\mathbb{R}^1}$ to terms in de Rham degree 0 and the differential on $J(L)^\vee$ to terms in de Rham degree 1.

To each $v \in V$ we have a translation symmetry, given by the closed element of degree 0

$$X_v = 1 \otimes \partial \otimes v^\vee - \mathrm{d}x \otimes 1 \otimes v^\vee,$$

where $v^\vee \in V^\vee$ is the linear functional given by inner product with v.

The associated current $J(X_v)$ simply picks out the 0-form component of X_v. Thus, the current is

$$J(X_v) = 1 \otimes \partial \otimes v.$$

Applied to a field $\phi \in C^\infty(\mathbb{R}^1) \otimes V$, the current $J(X_v)$ yields

$$\langle v, \partial\phi \rangle_V \in C^\infty(\mathbb{R}^1).$$

Evaluated at a point $t \in \mathbb{R}^1$, we find $\langle v, \partial\phi \rangle_V (t)$, which is the momentum of ϕ in the v-direction at t.

CHAPTER 5

Quantum field theory

We have already explained in 1.7 the philosophy of this book: a classical field theory gives a P_0 factorization algebra, and a quantum field theory gives a quantization of this P_0 factorization algebra. In this section we will prove the following theorem.

5.0.3.1 Theorem. *Any quantum field theory on a manifold M, in the sense of* [**Cos11c**], *gives rise to a factorization algebra* Obs^q *on M of* quantum observables*. This is a factorization algebra over $\mathbb{C}[[\hbar]]$, valued in differentiable pro-cochain complexes, and it quantizes (in the sense of 1.7) the P_0 factorization algebra of classical observables of the corresponding classical field theory.*

The machinery of [**Cos11c**] allows one to construct many examples of quantum field theories. Given a classical field theory, the results of [**Cos11c**] allow one to describe possible quantizations in terms of certain cohomology groups of local functionals. By calculating these cohomology groups, one can construct many examples of quantum field theory. For example, in [**Cos11c**], the quantum Yang-Mills gauge theory is constructed.

Thus, this theorem, together with the results of [**Cos11c**], produces many interesting examples of factorization algebras.

Here is an overview of this chapter.

(1) In section 5.1 we recall the definition of a free theory in the BV formalism and construct the factorization algebra of quantum observables of a general free theory.
(2) In sections 5.2 to 5.6 we give an overview of the definition of QFT developed in [**Cos11c**].

(3) In section 5.7 we show how the definition of a QFT leads immediately to a construction of a BD algebra of "global observables" on the manifold M, which we denote $\mathrm{Obs}^q_{\mathscr{P}}(M)$.

(4) In section 5.8 we start the construction of the factorization algebra associated to a QFT. We construct a cochain complex $\mathrm{Obs}^q(M)$ of global observables, which is quasi-isomorphic to (but much smaller than) the BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$.

(5) In section 5.10 we construct, for every open subset $U \subset M$, the subspace $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U.

(6) Section 5.11 accomplishes the primary aim of the chapter. In it, we prove that the cochain complexes $\mathrm{Obs}^q(U)$ form a factorization algebra. The proof of this result is the most technical part of the chapter.

(7) In section 5.12 we show that translation-invariant theories have translation-invariant factorization algebras of observables, and we treat the holomorphic situation as well.

(8) In section 5.13 we explain how to interpret our definition of a QFT in the special case of a cotangent theory: roughly speaking, a quantization of the cotangent theory to an elliptic moduli problem yields a locally-defined volume form on the moduli problem we start with.

Remark: We forewarn the reader that our definitions and constructions involve a heavy use of functional analysis and (perhaps more surprisingly) simplicial sets. Making a quantum field theory typically requires many choices, and as mathematicians, we wish to pin down precisely how the quantum field theory depends on these choices. The machinery we use gives us very precise statements, but statements that can be forbidding at first pass. We encourage the reader, on a first pass through this material, to simply make all necessary choices (such as a parametrix) and focus on the output of our machine, namely the factorization algebra of quantum observables. Keeping track of the dependence on choices requires careful bookkeeping (aided by the machinery of simplicial sets, etc) but is straightforward once the primary construction is understood.

5.1. Free fields

Before we give our general construction of the factorization algebra associated to a quantum field theory, we will give the much easier construction of the factorization algebra for a free field theory.

Let us recall the definition of a free BV theory.

5.1.0.2 Definition. *A free BV theory on a manifold M consists of the following data:*

(1) a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that has finite rank;
(2) an antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of degree -1 that is fiberwise nondegenerate. It induces a symplectic pairing on compactly supported smooth sections $\mathscr{E}_c$ of E:

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(3) a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.

Remark: When we consider deforming free theories into interacting theories, we will need to assume the existence of a "gauge fixing operator": this is a degree -1 operator $Q^{GF} : \mathscr{E} \to E$ such that $[Q, Q^{GF}]$ is a generalized Laplacian in the sense of [BGV92].

On any open set $U \subset M$, the commutative dg algebra of classical observables supported in U is

$$\mathrm{Obs}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^\vee(U)), Q),$$

where

$$\mathscr{E}^\vee(U) = \overline{\mathscr{E}}^!_c(U)$$

denotes the distributions dual to $\mathscr{E}$ with compact support in U and Q is the derivation given by extending the natural action of Q on the distributions.

In section 4.14 we constructed a sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^!_c(U)), Q)$$

defined as the symmetric algebra on the compactly-supported smooth (rather than distributional) sections of the bundle $E^!$. We showed that the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \to$

$\mathrm{Obs}^{cl}(U)$ is a weak equivalence of factorization algebras. Further, $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a Poisson bracket of cohomological degree 1, defined on the generators by the natural pairing

$$\mathscr{E}_c^!(U) \otimes \mathscr{E}_c^!(U) \to \mathbb{R},$$

which arises from the dual pairing on $\mathscr{E}_c(U)$. In this section we will show how to construct a quantization of the P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$.

5.1.1. The Heisenberg algebra construction. Our quantum observables on an open set U will be built from a certain Heisenberg Lie algebra.

Recall the usual construction of a Heisenberg algebra. If V is a symplectic vector space, viewed as an abelian Lie algebra, then the Heisenberg algebra $\mathrm{Heis}(V)$ is the central extension

$$0 \to \mathbb{C} \cdot \hbar \to \mathrm{Heis}(V) \to V$$

whose bracket is $[x, y] = \hbar \langle x, y \rangle$.

Since the element $\hbar \in \mathrm{Heis}(V)$ is central, the algebra $\widehat{U}(\mathrm{Heis}(V))$ is an algebra over $\mathbb{C}[[\hbar]]$, the completed universal enveloping algebra of the Abelian Lie algebra $\mathbb{C} \cdot \hbar$.

In quantum mechanics, this Heisenberg construction typically appears in the study of systems with quadratic Hamiltonians. In this context, the space V can be viewed in two ways. Either it is the space of solutions to the equations of motion, which is a linear space because we are dealing with a free field theory; or it is the space of linear observables dual to the space of solutions to the equations of motion. The natural symplectic pairing on V gives an isomorphism between these descriptions. The algebra $\widehat{U}(\mathrm{Heis}(V))$ is then the algebra of non-linear observables.

Our construction of the quantum observables of a free field theory will be formally very similar. We will start with a space of linear observables, which (after a shift) is a cochain complex with a symplectic pairing of cohomological degree 1. Then, instead of applying the usual universal enveloping algebra construction, we will take Chevalley-Eilenberg chain complex, whose cohomology is the Lie algebra homology.[1] This fits with our operadic philosophy: Chevalley-Eilenberg chains are the E_0 analog of the universal enveloping algebra.

[1]As usual, we always use gradings such that the differential has degree $+1$.

5.1.2. The basic homological construction. Let us start with a 0-dimensional free field theory. Thus, let V be a cochain complex equipped with a symplectic pairing of cohomological degree -1. We will think of V as the space of fields of our theory. The space of linear observables of our theory is $V^\vee$; the Poisson bracket on $\mathscr{O}(V)$ induces a symmetric pairing of degree 1 on $V^\vee$. We will construct the space of all observables from a Heisenberg Lie algebra built on $V^\vee[-1]$, which has a symplectic pairing $\langle -, - \rangle$ of degree -1. Note that there is an isomorphism $V \cong V^\vee[-1]$ compatible with the pairings on both sides.

5.1.2.1 Definition. *The Heisenberg algebra* $\mathrm{Heis}(V)$ *is the Lie algebra central extension*

$$0 \to \mathbb{C} \cdot \hbar[-1] \to \mathrm{Heis}(V) \to V^\vee[-1] \to 0$$

whose bracket is

$$[v + \hbar a, w + \hbar b] = \hbar \langle v, w \rangle$$

The element $\hbar$ *labels the basis element of the center* $\mathbb{C}[-1]$*.*

Putting the center in degree 1 may look strange, but it is necessary to do this in order to get a Lie bracket of cohomological degree 0.

Let $\widehat{C}_*(\mathrm{Heis}(V))$ denote the completion[2] of the Lie algebra chain complex of $\mathrm{Heis}(V)$, defined by the product of the spaces $\mathrm{Sym}^n \mathrm{Heis}(V)$, instead of their sum.

In this zero-dimensional toy model, the classical observables are

$$\mathrm{Obs}^{cl} = \mathscr{O}(V) = \prod_n \mathrm{Sym}^n(V^\vee).$$

This is a commutative dg algebra equipped with the Poisson bracket of degree 1 arising from the pairing on V. Thus, $\mathscr{O}(V)$ is a P_0 algebra.

5.1.2.2 Lemma. *The completed Chevalley-Eilenberg chain complex* $\widehat{C}_*(\mathrm{Heis}(V))$ *is a BD algebra (section 2.4) which is a quantization of the* P_0 *algebra* $\mathscr{O}(V)$*.*

PROOF. The completed Chevalley-Eilenberg complex for $\mathrm{Heis}(V)$ has the completed symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$ as its underlying graded vector space. Note that

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1]) = \mathrm{Sym}(V^\vee \oplus \mathbb{C} \cdot \hbar) = \widehat{\mathrm{Sym}}(V^\vee)[[\hbar]],$$

[2]One doesn't need to take the completed Lie algebra chain complex. We do this to be consistent with our discussion of the observables of interacting field theories, where it is essential to complete.

so that $\widehat{C}_*(\mathrm{Heis}(V))$ is a flat $\mathbb{C}[[\hbar]]$ module which reduces to $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$. The Chevalley-Eilenberg chain complex $\widehat{C}_*(\mathrm{Heis}(V))$ inherits a product, corresponding to the natural product on the symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$. Further, it has a natural Poisson bracket of cohomological degree 1 arising from the Lie bracket on $\mathrm{Heis}(V)$, extended to be a derivation of $\widehat{C}_*(\mathrm{Heis}(V))$. Note that, since $\mathbb{C}\cdot\hbar[-1]$ is central in $\mathrm{Heis}(V)$, this Poisson bracket reduces to the given Poisson bracket on $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$.

In order to prove that we have a BD quantization, it remains to verify that, although the commutative product on $\widehat{C}_*(\mathrm{Heis}(V))$ is not compatible with the product, it satisfies the BD axiom:

$$\mathrm{d}(a\cdot b) = (\mathrm{d}a)\cdot b + (-1)^{|a|}a\cdot(\mathrm{d}b) + \hbar\{a,b\}.$$

This follows by definition. □

5.1.3. Cosheaves of Heisenberg algebras. Next, let us give the analog of this construction for a general free BV theory E on a manifold M. As above, our classical observables are defined by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = \widehat{\mathrm{Sym}}\,\mathscr{E}_c^!(U)$$

which has a Poisson bracket arising from the pairing on $\mathscr{E}_c^!(U)$. Recall that this is a factorization algebra.

To construct the quantum theory, we define, as above, a Heisenberg algebra $\mathrm{Heis}(U)$ as a central extension

$$0\to\mathbb{C}[-1]\cdot\hbar\to\mathrm{Heis}(U)\to\mathscr{E}_c^!(U)[-1]\to 0.$$

Note that $\mathrm{Heis}(U)$ is a pre-cosheaf of Lie algebras. The bracket in this Heisenberg algebra arises from the pairing on $\mathscr{E}_c^!(U)$.

We then define the quantum observables by

$$\mathrm{Obs}^q(U) = \widehat{C}_*(\mathrm{Heis}(U)).$$

The underlying cochain complex is, as before,

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1])$$

where the completed symmetric algebra is defined (as always) using the completed tensor product.

5.1.3.1 Proposition. *Sending U to $\mathrm{Obs}^q(U)$ defines a BD factorization algebra in the category of differentiable pro-cochain complexes over $\mathbb{R}[[\hbar]]$, which quantizes $\mathrm{Obs}^{cl}(U)$.*

PROOF. First, we need to define the filtration on $\mathrm{Obs}^q(U)$ making it into a differentiable pro-cochain complex. The filtration is defined, in the identification

$$\mathrm{Obs}^q(U) = \widehat{\mathrm{Sym}}\, \mathscr{E}_c^!(U)[[\hbar]]$$

by saying

$$F^n \mathrm{Obs}^q(U) = \prod_k \hbar^k \mathrm{Sym}^{\geq n-2k} \mathscr{E}_c^!(U).$$

This filtration is engineered so that the $F^n \mathrm{Obs}^q(U)$ is a subcomplex of $\mathrm{Obs}^q(U)$.

It is immediate that Obs^q is a BD pre-factorization algebra quantizing $\mathrm{Obs}^{cl}(U)$. The fact that it is a factorization algebra follows from the fact that $\mathrm{Obs}^{cl}(U)$ is a factorization algebra, and then a simple spectral sequence argument. (A more sophisticated version of this spectral sequence argument, for interacting theories, is given in section 5.11.) □

5.2. Overview of perturbative quantum field theory

In sections 5.2 to 5.6, we will give a summary of the definition of a QFT as developed in [**Cos11c**]. We will emphasize the aspects which we will use in our construction of the factorization algebra associated to a QFT. This means that important aspects of the story there — such as the concept of renormalizability — will not be mentioned.

The introductory chapter of [**Cos11c**] is a leisurely exposition of the main physical and mathematical ideas, and we encourage the reader to examine it before delving into this book. The approach is perturbative and hence has the flavor of formal geometry (that is, geometry with formal manifolds).

A perturbative field theory is defined to be a family of effective field theories parametrized by some notion of "scale". The notion of scale can be quite flexible: the simplest version is where the scale is a positive real number, the length. Then, the effective theory at a length scale L is obtained from the effective theory at scale ε by integrating out over fields with length scale between ε and L. In order to construct factorization algebras, we need a more refined notion of "scale", where there is a scale for every parametrix Φ of a certain elliptic operator. We denote such a family of effective field theories by $\{I[\Phi]\}$, where $I[\Phi]$ is the "interaction term" in the action functional $S[\Phi]$ at "scale" Φ. We always study families with respect to a fixed free theory.

A local action functional (see section 5.3) S is a real-valued function on the space of fields such tat $S(\phi)$ is given by integrating some function of the field and its derivatives over the base manifold (the "spacetime"). The main result of [**Cos11c**] states that the space of perturbative QFTs is the "same size" as the space of local action functionals. More precisely, the space of perturbative QFTs defined modulo $\hbar^{n+1}$ is a torsor over the space of QFTs defined modulo $\hbar^n$ for the abelian group of local action functionals.

The starting point for many physical constructions – such as the path integral – is a local action functional. However, a naive application of these constructions to such an action functional yields a nonsensical answer. Many of these constructions *do* work if, instead of applying them to a local action functional, they are applied to a family $\{I[\Phi]\}$ of effective action functionals. Thus, one can view the family of effective action functionals $\{I[\Phi]\}$ as a quantum version of the local action functional defining classical field theory. The results of [**Cos11c**] allow one to construct such families of action functionals. Many formal manipulations with path integrals in the physics literature apply rigorously to families $\{I[\Phi]\}$ of effective actions. Our strategy for constructing the factorization algebra of observables is to mimic path-integral definitions of observables one can find in the physics literature, but replacing local functionals by families of effective actions.

5.3. Local action functionals

In studying field theory, there is a special class of functions on the fields, known as local action functionals, that parametrize the possible classical physical systems. Let M be a smooth manifold. Let $\mathscr{E} = C^\infty(M, E)$ denote the smooth sections of a $\mathbb{Z}$-graded super vector bundle E on M, which has finite rank when all the graded components are included. We call $\mathscr{E}$ the *fields*.

Spaces of functions on the space of fields are defined in the appendix A.7.

5.3.0.2 Definition. *A functional F is an element of*

$$\mathscr{O}(\mathscr{E}) = \prod_{n=0}^{\infty} \operatorname{Hom}_{DVS}(\mathscr{E}^{\otimes n}, \mathbb{R})^{S_n},$$

namely the completed symmetric algebra of $\mathscr{E}^\vee$, the continuous dual space to the fields.

Let $\mathscr{O}_{red}(\mathscr{E}) = \mathscr{O}(\mathscr{E})/\mathbb{C}$ be the space of functionals on $\mathscr{E}$ modulo constants.

The local functionals depend only on the local behavior of a field, so that at each point of M, a local functional should only depend on the jet of the field at that point. In the Lagrangian formalism for field theory, their role is to describe the permitted actions, so we call them *local action functionals*. A local action functional is the essential datum of a *classical* field theory.

5.3.0.3 Definition. *A functional F is* local *if each homogeneous component F_n is a finite sum of terms of the form*

$$F_n(\phi) = \int_M (D_1\phi) \cdots (D_n\phi)\, d\mu,$$

where each D_i is a differential operator from $\mathscr{E}$ to $C^\infty(M)$ and $d\mu$ is a density on M.

We let

$$\mathscr{O}_{loc}(\mathscr{E}) \subset \mathscr{O}_{red}(\mathscr{E})$$

denote denote the space of local action functionals modulo constants.

dg algebra

5.4. The definition of a quantum field theory

In this section, we will give the formal definition of a quantum field theory. The definition is a little long and somewhat technical. The reader should consult the first chapter of [**Cos11c**] for physical motivations for this definition. We will provide some justification for the definition from the point of view of homological algebra shortly (section 5.7).

5.4.1. In section 5.1 we gave a definition of a free BV theory.

5.4.1.1 Definition. *A free BV theory on a manifold M consists of the following data:*

(1) a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that is of finite rank;

(2) *an antisymmetric map of vector bundles* $\langle -,-\rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ *of degree* -1 *that is fiberwise nondegenerate. It induces a symplectic pairing on compactly supported smooth sections* $\mathscr{E}_c$ *of* E*:*

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(3) *a square-zero differential operator* $Q : \mathscr{E} \to \mathscr{E}$ *of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

In our constructions, we require the existence of a *gauge-fixing operator* $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the following properties:

(1) it is a square-zero differential operator of cohomological degree -1 ;
(2) it is self adjoint for the symplectic pairing;
(3) $D = [Q, Q^{GF}]$ is a generalized Laplacian on M, in the sense of [**BGV92**]. This means that D is an order 2 differential operator whose symbol $\sigma(D)$, which is an endomorphism of the pullback bundle p^*E on the cotangent bundle $p : T^*M \to M$, is

$$\sigma(D) = g \, \mathrm{Id}_{p^*E}$$

where g is some Riemannian metric on M, viewed as a function on T^*M.

All our constructions vary homotopically with the choice of gauge fixing operator. In practice, there is a natural contractible space of gauge fixing operators, so that our constructions are independent (up to contractible choice) of the choice of gauge fixing operator.

5.4.2. Operators and kernels. Let us recall the relationship between kernels and operators on $\mathscr{E}$. Any continuous linear map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ can be represented by a kernel

$$K_F \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}^! = \mathrm{Hom}_{DVS}(\mathscr{E}_c, \overline{\mathscr{E}}).$$

The symplectic pairing on $\mathscr{E}$ gives an isomorphism between $\overline{\mathscr{E}}$ and $\overline{\mathscr{E}}^![-1]$. This allows us to view the kernel for any continuous linear map F as an element $K_F \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$. If F is of cohomological degree k, then the kernel K_F is of cohomological degree $k+1$.

If the map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ has image in $\overline{\mathscr{E}}_c$ and extends to a continuous linear map $\mathscr{E} \to \overline{\mathscr{E}}_c$, then the kernel K_F has compact support. If F has image in $\mathscr{E}$ and extends to a continuous linear map $\overline{\mathscr{E}}_c \to \mathscr{E}$, then the kernel K_F is smooth.

Our conventions are such that the following hold.

(1) $K_{[Q,F]} = QK_F$, where Q is the total differential on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(2) Suppose that $F : \mathscr{E}_c \to \mathscr{E}_c$ is skew-symmetric with respect to the degree -1 pairing on $\mathscr{E}_c$. Then K_F is symmetric. Similarly, if F is symmetric, then K_F is anti-symmetric.

5.4.3. The heat kernel. In this section we will discuss heat kernels associated to the generalized Laplacian $D = [Q, Q^{GF}]$. These generalized heat kernels will not be essential to our story; most of our constructions will work with a general parametrix for the operator D, and the heat kernel simply provides a convenient example.

Suppose that we have a free BV theory with a gauge fixing operator Q^{GF}. As above, let $D = [Q, Q^{GF}]$. If our manifold M is compact, then this leads to a heat operator e^{-tD} acting on sections $\mathscr{E}$. The heat kernel K_t is the corresponding kernel, which is an element of $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}} \otimes C^\infty(\mathbb{R}_{\geq 0})$. Further, if $t > 0$, the operator e^{-tD} is a smoothing operator, so that the kernel K_t is in $\mathscr{E} \otimes \mathscr{E}$. Since the operator e^{-tD} is skew symmetric for the symplectic pairing on $\mathscr{E}$, the kernel K_t is symmetric.

The kernel K_t is uniquely characterized by the following properties:

(1) The heat equation:
$$\frac{\mathrm{d}}{\mathrm{d}t} K_t + (D \otimes 1) K_t = 0.$$
(2) The initial condition that $K_0 \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is the kernel for the identity operator.

On a non-compact manifold M, there is more than one heat kernel satisfying these properties. We will only consider heat kernels on a compact manifold.

5.4.4. Parametrices. In [Cos11c], two equivalent definitions of a field theory are given: one based on the heat kernel, and one based on a general parametrix. We will use exclusively the parametrix version in this book.

Before we define the notion of parametrix, we need a technical definition.

5.4.4.1 Definition. *If M is a manifold, a subset $V \subset M^n$ is* proper *if all of the projection maps $\pi_1, \dots, \pi_n : V \to M$ are proper. We say that a function, distribution, etc. on M^n has proper support if its support is a proper subset of M^n.*

5.4.4.2 Definition. *A parametrix Φ is a distributional section*

$$\Phi \in \overline{\mathscr{E}}(M) \otimes \overline{\mathscr{E}}(M)$$

of the bundle $E \boxtimes E$ on $M \times M$ with the following properties.

(1) *Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}}(M) \otimes \overline{\mathscr{E}}(M)$.*
(2) *Φ is of cohomological degree 1.*
(3) *Φ is closed under the natural differential $Q \otimes 1 + 1 \otimes Q$ on $\overline{\mathscr{E}}(M) \otimes \overline{\mathscr{E}}(M)$.*
(4) *Φ has proper support.*
(5) *Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that*

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is a smooth section of $E \boxtimes E$ on $M \times M$. Thus,

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}} \in \mathscr{E}(M) \otimes \mathscr{E}(M).$$

(Here K_{Id} is the kernel corresponding to the identity operator).

Remark: For clarity's sake, note that our definition depends on a choice of Q^{GF}. Thus, we are defining here parametrices for the generalized Laplacian $[Q, Q^{GF}]$, *not* general parametrices for the elliptic complex $\mathscr{E}$.

Note that the parametrix Φ can be viewed (using the correspondence between kernels and operators described above) as a linear map $A_\Phi : \mathscr{E} \to \mathscr{E}$. This operator is of cohomological degree 0, and has the property that

$$A_\Phi [Q, Q^{GF}] = \mathrm{Id} + \text{a smoothing operator}$$
$$[Q, Q^{GF}] A_\Phi = \mathrm{Id} + \text{a smoothing operator.}$$

This property – being both a left and right inverse to the operator $[Q, Q^{GF}]$, up to a smoothing operator – is the standard definition of a parametrix.

An example of a parametrix is the following. For M compact, let $K_t \in \mathscr{E} \otimes \mathscr{E}$ be the heat kernel. Then, the kernel $\int_0^L K_t \mathrm{d}t$ is a parametrix, for any $L > 0$.

It is a standard result in the theory of pseudo-differential operators (see e.g. [Tar87]) that every elliptic operator admits a parametrix. Normally a parametrix is not assumed to have proper support; however, if Φ is a parametrix satisfying all conditions except that of proper support, and if $f \in C^\infty(M \times M)$ is a smooth function with proper support which is 1 in a neighborhood of the diagonal, then $f\Phi$ is a parametrix with proper support. This shows that parametrices with proper support always exist.

Here is a key property of parametrices that we will use repeatedly.

5.4.4.3 Lemma. *If Φ, Ψ are parametrices, then $\Phi - \Psi \in \mathscr{E} \otimes \mathscr{E}$ is smooth.*

PROOF. Indeed, $\Phi - \Psi$ is annihilated by the elliptic operator $([Q, Q^{GF}]) \otimes 1 + 1 \otimes ([Q, Q^{GF}])$ on $\mathscr{E}(M) \otimes \mathscr{E}(M)$. Hence it is a smooth section. □

If Φ, Ψ are parametrices, we say that $\Phi < \Psi$ if the support of Φ is contained in the support of Ψ. In this way, parametrices acquire a partial order.

5.4.5. The propagator for a parametrix. If Φ is a parametrix, we let

$$P(\Phi) = (Q^{GF} \otimes 1)\Phi \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}.$$

This is the propagator associated to Φ. We let

$$K_\Phi = K_{\mathrm{Id}} - ([Q, Q^{GF}] \otimes 1)\Phi.$$

To relate to section 5.4.3 and [**Cos11c**], we note that if M is a compact manifold and if

$$\Phi = \int_0^L K_t \mathrm{d}t$$

is the parametrix associated to the heat kernel, then

$$P(\Phi) = P(0, L) = \int_0^L (Q^{GF} \otimes 1) K_t \mathrm{d}t$$

and

$$K_\Phi = K_L.$$

5.4.6. Classes of functionals. In the appendix A.7 we define various classes of functions on the space $\mathscr{E}_c$ of compactly-supported fields. Here we give an overview of those classes. Many of the conditions seem somewhat technical at first, but they arise naturally as one attempts both to discuss the support of an observable and to extend the algebraic ideas of the BV formalism in this infinite-dimensional setting.

We are interested, firstly, in functions modulo constants, which we call $\mathscr{O}_{red}(\mathscr{E}_c)$. Every functional $F \in \mathscr{O}_{red}(\mathscr{E}_c)$ has a Taylor expansion in terms of symmetric continuous linear maps

$$F_k : \mathscr{E}_c^{\otimes k} \to \mathbb{C}$$

(for $k > 0$). We say that F has *proper support* if the support of each F_k (as defined above) is a proper subset of M^k. The space of functionals with proper support is denoted $\mathscr{O}_P(\mathscr{E}_c)$ (as always in this section, we work with functionals modulo constants). This condition equivalently means that, when we think of F_k as an operator

$$\mathscr{E}_c^{\otimes k-1} \to \overline{\mathscr{E}}^!,$$

it extends to a continuous linear map

$$F_k : \mathscr{E}^{\otimes k-1} \to \overline{\mathscr{E}}^!.$$

At various points in this book, we will need to consider *functionals with smooth first derivative*, which are functionals satisfying a certain technical regularity constraint. Functionals with smooth first derivative are needed in two places in the text: when we define the Poisson bracket on classical observables, and when we give the definition of a quantum field theory. In terms of the Taylor components F_k, viewed as multilinear operators $\mathscr{E}_c^{\otimes k-1} \to \overline{\mathscr{E}}^!$, this condition means that the F_k has image in $\mathscr{E}^!$. (For more detail, see Appendix A, section A.7.)

We are interested in the functionals with smooth first derivative and with proper support. We denote this space by $\mathscr{O}_{P,sm}(\mathscr{E})$. These are the functionals with the property that the Taylor components F_k, when viewed as operators, give continuous linear maps

$$\mathscr{E}^{\otimes k-1} \to \mathscr{E}^!.$$

5.4.7. The renormalization group flow. Suppose that Φ, Ψ are parametrices. Then $P(\Phi) - P(\Psi)$ is a smooth kernel with proper support.

Given any element $\alpha \in \mathscr{E} \otimes \mathscr{E}$ of cohomological degree 0, we define an operator

$$\partial_\alpha : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

This map is an order 2 differential operator, which, on components, is the map given by contraction with α:

$$\alpha \vee - : \operatorname{Sym}^n \mathscr{E}^\vee \to \operatorname{Sym}^{n-2} \mathscr{E}^\vee.$$

The operator ∂_α is the unique order 2 differential operator which is given by pairing with α on $\mathrm{Sym}^2 \mathscr{E}^\vee$ and which is zero on $\mathrm{Sym}^{\leq 1} \mathscr{E}^\vee$.

We define a map

$$\begin{aligned} W(\alpha, -) : \mathscr{O}^+(\mathscr{E})[[\hbar]] &\to \mathscr{O}^+(\mathscr{E})[[\hbar]] \\ F &\mapsto \hbar \log\left(e^{\hbar \partial_\alpha} e^{F/\hbar} \right), \end{aligned}$$

known as the RG flow with respect to α. (When $\alpha = P(\Phi) - P(\Psi)$, we call it the RG flow from Ψ to Φ.) This formula is a succinct way of summarizing a Feynman diagram expansion. In particular, $W(\alpha, F)$ can be written as a sum over Feynman diagrams with the Taylor components F_k of F labelling vertices of valence k, and with α as propagator. (All of this, and indeed everything else in this section, is explained in far greater detail in chapter 2 of [Cos11c].) For this map to be well-defined, the functional F must have only cubic and higher terms modulo $\hbar$. The notation $\mathscr{O}^+(\mathscr{E})[[\hbar]]$ denotes this restricted class of functionals.

If $\alpha \in \mathscr{E} \otimes \mathscr{E}$ has proper support, then the operator $W(\alpha, -)$ extends (uniquely, of course) to a continuous operator

$$W(\alpha, -) : \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]].$$

Our philosophy is that a parametrix Φ is like a choice of "scale" for our field theory. The renormalization group flow relating the scale given by Φ and that given by Ψ is $W(P(\Phi) - P(\Psi), -)$.

Because $P(\Phi)$ is not a smooth kernel, the operator $W(P(\Phi), -)$ is not well-defined. This is just because the definition of $W(P(\Phi), -)$ involves multiplying distributions. In physics terms, the singularities that appear when one tries to define $W(P(\Phi), -)$ are called ultraviolet divergences.

However, if $I \in \mathscr{O}^+_{P,sm}(\mathscr{E})$, the tree level part

$$W_0(P(\Phi), I) = W((P(\Phi), I) \mod \hbar$$

is a well-defined element of $\mathscr{O}^+_{P,sm}(\mathscr{E})$. The $\hbar \to 0$ limit of $W(P(\Phi), I)$ is called the tree-level part because, whereas the whole object $W(P(\Phi), I)$ is defined as a sum over graphs, the $\hbar \to 0$ limit $W_0(P(\Phi), I)$ is defined as a sum over trees. It is straightforward to see that $W_0(P(\Phi), I)$ only involves multiplication of distributions with transverse singular support, and so is well defined.

5.4.8. The BD algebra structure associated to a parametrix. A parametrix also leads to a BV operator

$$\triangle_\Phi = \partial_{K_\Phi} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Again, this operator preserves the subspace $\mathscr{O}_{P,sm}(\mathscr{E})$ of functions with proper support and smooth first derivative. The operator $\triangle_\Phi$ commutes with Q, and it satisfies $(\triangle_\Phi)^2 = 0$. In a standard way, we can use the BV operator $\triangle_\Phi$ to define a bracket on the space $\mathscr{O}(\mathscr{E})$, by

$$\{I, J\}_\Phi = \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J.$$

This bracket is a Poisson bracket of cohomological degree 1. If we give the graded-commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$ the standard product, the Poisson bracket $\{-,-\}_\Phi$, and the differential $Q + \hbar\triangle_\Phi$, then it becomes a BD algebra.

The bracket $\{-,-\}_\Phi$ extends uniquely to a continuous linear map

$$\mathscr{O}_P(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Further, the space $\mathscr{O}_{P,sm}(\mathscr{E})$ is closed under this bracket. (Note, however, that $\mathscr{O}_{P,sm}(\mathscr{E})$ is *not* a commutative algebra if M is not compact: the product of two functionals with proper support no longer has proper support.)

A functional $F \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is said to satisfy the Φ-quantum master equation if

$$QF + \hbar\triangle_\Phi F + \tfrac{1}{2}\{F, F\}_\Phi = 0.$$

It is shown in [Cos11c] that if F satisfies the Φ-QME, and if Ψ is another parametrix, then $W(P(\Psi) - P(\Phi), F)$ satisfies the Ψ-QME. This follows from the identity

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = \triangle_\Psi - \triangle_\Phi$$

of order 2 differential operators on $\mathscr{O}(\mathscr{E})$. This relationship between the renormalization group flow and the quantum master equation is a key part of the approach to QFT of [Cos11c].

5.4.9. The definition of a field theory. Our definition of a field theory is as follows.

5.4.9.1 Definition. *Let $(\mathscr{E}, Q, \langle -,- \rangle)$ be a free BV theory. Fix a gauge fixing condition Q^{GF}. Then a quantum field theory (with this space of fields) consists of the following data.*

(1) *For all parametrices Φ, a functional*

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]]$$

which we call the scale Φ effective interaction. As we explained above, the subscripts indicate that $I[\Phi]$ must have smooth first derivative and proper support. The superscript $+$ indicates that, modulo $\hbar$, $I[\Phi]$ must be at least cubic. Note that we work with functions modulo constants.

(2) *For two parametrices Φ, Ψ, $I[\Phi]$ must be related by the renormalization group flow:*

$$I[\Phi] = W\left(P(\Phi) - P(\Psi), I[\Psi]\right).$$

(3) *Each $I[\Phi]$ must satisfy the Φ-quantum master equation*

$$(Q + \hbar \triangle_\Phi) e^{I[\Phi]/\hbar} = 0.$$

Equivalently,

$$Q I[\Phi] + \hbar \triangle_\Phi I[\Phi] + \tfrac{1}{2}\{I[\Phi], I[\Phi]\}_\Phi.$$

(4) *Finally, we require that $I[\Phi]$ satisfies a locality axiom. Let*

$$I_{i,k}[\Phi] : \mathscr{E}_c^{\otimes k} \to \mathbb{C}$$

be the kth Taylor component of the coefficient of $\hbar^i$ in $I[\Phi]$.

Our locality axiom says that, as Φ tends to zero, the support of

$$I_{i,k}[\Phi]$$

becomes closer and closer to the small diagonal in M^k.

For the constructions in this book, it turns out to be useful to have precise bounds on the support of $I_{i,k}[\Phi]$. To give these bounds, we need some notation. Let $\operatorname{Supp}(\Phi) \subset M^2$ be the support of the parametrix Φ, and let $\operatorname{Supp}(\Phi)^n \subset M^2$ be the subset obtained by convolving $\operatorname{Supp}(\Phi)$ with itself n times. (Thus, $(x, y) \in \operatorname{Supp}(\Phi)^n$ if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \operatorname{Supp}(\Phi)$.)

Our support condition is that, if $e_j \in \mathscr{E}_c$, then

$$I_{i,k}(e_1, \ldots, e_k) = 0$$

unless, for all $1 \le r < s \le k$,

$$\operatorname{Supp}(e_r) \times \operatorname{Supp}(e_s) \subset \operatorname{Supp}(\Phi)^{3i+k}.$$

Remark: (1) The locality axiom condition as presented here is a little unappealing. An equivalent axiom is that for all open subsets $U \subset M^k$ containing the small diagonal $M \subset M^k$, there exists a parametrix Φ_U such that

$$\operatorname{Supp} I_{i,k}[\Phi] \subset U \text{ for all } \Phi < \Phi_U.$$

In other words, by choosing a small parametrix Φ, we can make the support of $I_{i,k}[\Phi]$ as close as we like to the small diagonal on M^k.

We present the definition with a precise bound on the size of the support of $I_{i,k}[\Phi]$ because this bound will be important later in the construction of the factorization algebra. Note, however, that the precise exponent $3i + k$ which appears in the definition (in $\operatorname{Supp}(\Phi)^{3i+k}$) is not important. What is important is that we have some bound of this form.

(2) It is important to emphasize that the notion of quantum field theory is only defined once we have chosen a gauge fixing operator. Later, we will explain in detail how to understand the dependence on this choice. More precisely, we will construct a simplicial set of QFTs and show how this simplicial set only depends on the homotopy class of gauge fixing operator (in most examples, the space of natural gauge fixing operators is contractible).

Suppose that $I_0 \in \mathscr{O}_{loc}(E)$ is a local functional (defined modulo constants) which satisfies the classical master equation

$$QI_0 + \tfrac{1}{2}\{I_0, I_0\} = 0.$$

Let us suppose that I_0 is at least cubic.

Then, as we have seen above, we can define a family of functionals

$$I_0[\Phi] = W_0\left(P(\Phi), I_0\right) \in \mathscr{O}_{P,sm}(\mathscr{E})$$

as the tree-level part of the renormalization group flow operator from scale 0 to the scale given by the parametrix Φ. The compatibility between this classical renormalization group flow and the classical master equation tells us that $I_0[\Phi]$ satisfies the Φ-classical master equation

$$QI_0[\Phi] + \tfrac{1}{2}\{I_0[\Phi], I_0[\Phi]\}_\Phi = 0.$$

5.4.9.2 Definition. *Let $I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$ be the collection of effective interactions defining a quantum field theory. Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional satisfying the classical* *master equation, and so defining a classical field theory. We say that the quantum field theory* $\{I[\Phi]\}$ *is a* quantization *of the classical field theory defined by* I_0 *if*

$$I[\Phi] = I_0[\Phi] \text{ mod } \hbar,$$

or, equivalently, if

$$\lim_{\Phi \to 0} I[\Phi] - I_0 \text{ mod } \hbar = 0.$$

5.4.10. Allowing families. Before discussing the interpretation of these axioms and also explaining the results of [Cos11c] that allow one to construct such quantum field theories, we will explain how to define families of quantum field theories over some base dg algebra. The fact that we can work in families in this way means that the moduli space of quantum field theories is something like a derived stack. For instance, by considering families over the base dg algebra of forms on the n-simplex, we see that the set of quantizations of a given classical field theory is a simplicial set.

One particularly important use of the families version of the theory is that it allows us to show that our constructions and results are independent, up to homotopy, of the choice of gauge fixing condition (provided one has a contractible — or at least connected — space of gauge fixing conditions, which happens in most examples).

In later sections, we will work implicitly over some base dg ring in the sense described here, although we will normally not mention this base ring explicitly.

5.4.10.1 Definition. *A* nilpotent dg manifold *is a manifold* X *(possibly with corners), equipped with a sheaf* $\mathscr{A}$ *of commutative differential graded algebras over the sheaf* Ω^*_X*, with the following properties.*

1. *(1)* $\mathscr{A}$ *is concentrated in finitely many degrees.*
2. *(2) Each* $\mathscr{A}^i$ *is a locally free sheaf of* Ω^0_X*-modules of finite rank. This means that* $\mathscr{A}^i$ *is the sheaf of sections of some finite rank vector bundle* A^i *on* X*.*
3. *(3) We are given a map of dg* Ω^*_X*-algebras* $\mathscr{A} \to C^\infty_X$*.*
   We will let $\mathscr{I} \subset \mathscr{A}$ *be the ideal which is the kernel of the map* $\mathscr{A} \to C^\infty_X$*: we require that* $\mathscr{I}$*, its powers* $\mathscr{I}^k$*, and each* $\mathscr{A} / \mathscr{I}^k$ *are locally free sheaves of* C^∞_X*-modules. Also, we require that* $\mathscr{I}^k = 0$ *for* k *sufficiently large.*

Note that the differential d *on* $\mathscr{A}$ *is necessary a differential operator.*

We will use the notation $A^\sharp$ to refer to the bundle of graded algebras on X whose smooth sections are $\mathscr{A}^\sharp$, the graded algebra underlying the dg algebra $\mathscr{A}$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are nilpotent dg manifolds, a map $(Y, \mathscr{B}) \to (X, \mathscr{A})$ is a smooth map $f : Y \to X$ together with a map of dg $\Omega^*(X)$-algebras $\mathscr{A} \to \mathscr{B}$.

Here are some basic examples.

(1) $\mathscr{A} = C^\infty(X)$ and $\mathscr{I} = 0$. This describes the smooth manifold X.
(2) $\mathscr{A} = \Omega^*(X)$ and $\mathscr{I} = \Omega^{>0}(X)$. This equips X with its de Rham complex as a structure sheaf. (Informally, we can say that "constant functions are the only functions on a small open" so that this dg manifold is sensitive to topological rather than smooth structure.)
(3) If R is a dg Artinian $\mathbb{C}$-algebra with maximal ideal m, then R can be viewed as giving the structure of nilpotent graded manifold on a point.
(4) If again R is a dg Artinian algebra, then for any manifold $(X, R \otimes \Omega^*(X))$ is a nilpotent dg manifold.
(5) If X is a complex manifold, then $\mathscr{A} = (\Omega^{0,*}(X), \overline{\partial})$ is a nilpotent dg manifold.

Remark: We study field theories in families over nilpotent dg manifolds for both practical and structural reasons. First, we certainly wish to discuss familes of field theories over smooth manifolds. Nilpotent dg manifolds provide a version of derived geometry that is a modest enlargement of smooth manifolds. In particular, by only allowing (derived) Artinian directions, we avoid many subtleties of "global" derived geometry and can rely on the ideas and techniques of deformation theory. Second, from a practical point of view, our arguments are tractable when working over nilpotent dg manifolds. Families over more sophisticated spaces might very well work, but nilpotent dg manifolds are well-suited to our methods. We find it convenient, for example, that the algebra of functions on a nilpotent dg manifold is a nice Fréchet space.

We can now give a precise notion of "family of field theories."

5.4.10.2 Definition. *Let M be a manifold. A family of free BV theories on M, parameterized by a nilpotent dg manifold $(X, \mathscr{A})$, is the following data.*

(1) A graded bundle E on $M \times X$ of locally free $A^\sharp$-modules. We will refer to global sections of E as $\mathscr{E}$. The space of those sections $s \in \Gamma(M \times X, E)$ with the property

that the map Supp $s \to X$ *is proper will be denoted* $\mathscr{E}_c$. *Similarly, we let* $\overline{\mathscr{E}}$ *denote the space of sections which are distributional on* M *and smooth on* X, *that is,*

$$\overline{\mathscr{E}} = \mathscr{E} \otimes_{C^\infty(M \times X)} (\mathcal{D}(M) \otimes C^\infty(X)).$$

(This is just the algebraic tensor product, which is reasonable as $\mathscr{E}$ *is a finitely generated projective* $C^\infty(M \times X)$*-module).*

As above, we let

$$E^! = \mathrm{Hom}_{A^\sharp}(E, A^\sharp) \otimes \mathrm{Dens}_M$$

denote the "dual" bundle. There is a natural $\mathscr{A}^\sharp$*-valued pairing between* $\mathscr{E}$ *and* $\mathscr{E}_c^!$.

(2) *A differential operator* $Q : \mathscr{E} \to \mathscr{E}$, *of cohomological degree* 1 *and square-zero, making* $\mathscr{E}$ *into a dg module over the dg algebra* $\mathscr{A}$.

(3) *A map*

$$E \otimes_{A^\sharp} E \to \mathrm{Dens}_M \otimes A^\sharp$$

which is of degree -1, *anti-symmetric, and leads to an isomorphism*

$$\mathrm{Hom}_{A^\sharp}(E, A^\sharp) \otimes \mathrm{Dens}_M \to E$$

of sheaves of $A^\sharp$*-modules on* $M \times X$.

This pairing leads to a degree -1 *anti-symmetric* $\mathscr{A}$*-linear pairing*

$$\langle -, - \rangle : \mathscr{E}_c \otimes \mathscr{E}_c \to \mathscr{A}.$$

We require it to be a cochain map. In other words, if $e, e' \in \mathscr{E}_c$,

$$\mathrm{d}_{\mathscr{A}} \langle e, e' \rangle = \langle Qe, e' \rangle + (-1)^{|e|} \langle e, Qe' \rangle.$$

5.4.10.3 Definition. *Let* $(E, Q, \langle -, - \rangle)$ *be a family of free BV theories on* M *parameterized by* $\mathscr{A}$. *A gauge fixing condition on* $\mathscr{E}$ *is an* $\mathscr{A}$*-linear differential operator*

$$Q^{GF} : \mathscr{E} \to \mathscr{E}$$

such that

$$\mathrm{D} = [Q, Q^{GF}] : \mathscr{E} \to \mathscr{E}$$

is a generalized Laplacian, in the following sense.

Note that D *is an* $\mathscr{A}$*-linear cochain map. Thus, we can form*

$$\mathrm{D}_0 : \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X) \to \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X)$$

by reducing modulo the maximal ideal $\mathscr{I}$ *of* $\mathscr{A}$.

Let $E_0 = E/I$ be the bundle on $M \times X$ obtained by reducing modulo the ideal I in the bundle of algebras A. Let

$$\sigma(\mathrm{D}_0) : \pi^* E_0 \to \pi^* E_0$$

be the symbol of the $C^\infty(X)$-linear operator D_0. Thus, $\sigma(\mathrm{D}_0)$ is an endomorphism of the bundle of $\pi^* E_0$ on $(T^*M) \times X$.

We require that $\sigma(\mathrm{D}_0)$ is the product of the identity on E_0 with a smooth family of metrics on M parameterized by X.

Throughout this section, we will fix a family of free theories on M, parameterized by $\mathscr{A}$. We will take $\mathscr{A}$ to be our base ring throughout; thus, tensor products, etc. will be taken over $\mathscr{A}$. All of our constructions will be functorial with respect to change of the base ring $\mathscr{A}$.

5.4.11. Now that we have defined free theories over a base ring $\mathscr{A}$, the definition of an interacting theory over $\mathscr{A}$ is very similar to the definition given when $\mathscr{A} = \mathbb{C}$. First, one defines a parametrix to be an element

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$$

with the same properties as before, but where now we take all tensor products (and so on) over $\mathscr{A}$. More precisely,

(1) Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(2) Φ is of cohomological degree 1.
(3) Φ is closed under the differential on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(4) Φ has proper support: this means that the map $\operatorname{Supp} \Phi \to M \times X$ is proper.
(5) Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is an element of $\mathscr{E} \otimes \mathscr{E}$ (where, as before, $K_{\mathrm{Id}} \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is the kernel for the identity map).

An interacting field theory is then defined to be a family of $\mathscr{A}$-linear functionals

$$I[\Phi] \in \mathscr{O}_{red}(\mathscr{E})[[\hbar]] = \prod_{n \geq 1} \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}^{\otimes_{\mathscr{A}} n}, \mathscr{A})_{S_n}[[\hbar]]$$

satisfying the renormalization group flow equation, quantum master equation, and locality condition, just as before. In order for the RG flow to make sense, we require that each $I[\Phi]$ has proper support and smooth first derivative. In this context, this means the following. Let $I_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathscr{A}$ be the kth Taylor component of the coefficient of $\hbar^i$ in $I_{i,k}[\Phi]$. Proper support means that any projection map

$$\operatorname{Supp} I_{i,k}[\Phi] \subset M^k \times X \to M \times X$$

is proper. Smooth first derivative means, as usual, that when we think of $I_{i,k}[\Phi]$ as an operator $\mathscr{E}^{\otimes k-1} \to \overline{\mathscr{E}}$, the image lies in $\mathscr{E}$.

If we have a family of theories over $(X, \mathscr{A})$, and a map

$$f : (Y, \mathscr{B}) \to (X, \mathscr{A})$$

of dg manifolds, then we can base change to get a family over $(Y, \mathscr{B})$. The space of fields for this family is

$$f^* \mathscr{E} = \mathscr{E} \otimes_{\mathscr{A}} \mathscr{B}.$$

The gauge fixing operator

$$Q^{GF} : \mathscr{E} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{E} \otimes_{\mathscr{A}} \mathscr{B}$$

is the $\mathscr{B}$-linear extension of the gauge fixing condition for the family of theories over $\mathscr{A}$.

If

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}} \subset f^* \overline{\mathscr{E}} \otimes_{\mathscr{B}} f^* \overline{\mathscr{E}}$$

is a parametrix for the family of free theories $\mathscr{E}$ over $\mathscr{A}$, then it defines a parametrix $f^*\Phi$ for the family of free theories $f^*\mathscr{E}$ over $\mathscr{B}$. For parametrices of this form, the effective action functionals

$$f^* I[f^*\Phi] \in \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]] = \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes_{\mathscr{A}} \mathscr{B}$$

is simply the image of the original effective action functional

$$I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \subset \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]].$$

For a general parametrix Ψ for $f^*\mathscr{E}$, the effective action functional is defined by the renormalization group equation

$$f^* I[\Psi] = W\left(P(\Psi) - P(f^*\Phi), f^* I[f^*\Phi]\right).$$

This is well-defined because

$$P(\Psi) - P(f^*\Phi) \in f^*\mathscr{E} \otimes_{\mathscr{B}} f^*\mathscr{E}$$

has no singularities.

The compatibility between the renormalization group equation and the quantum master equation guarantees that the effective action functionals $f^*I[\Psi]$ satisfy the QME for every parametrix Ψ. The locality axiom for the original family of effective action functionals $I[\Phi]$ guarantees that the pulled-back family $f^*I[\Psi]$ satisfy the locality axiom necessary to define a family of theories over $\mathscr{B}$.

5.5. The simplicial set of theories

One of the main reasons for introducing theories over a nilpotent dg manifold $(X, \mathscr{A})$ is that this allows us to talk about the simplicial set of theories. This is essential, because the main result we will use from [Cos11c] is homotopical in nature: it relates the simplicial set of theories to the simplicial set of local functionals.

Let us introduce some notation. Let us fix a family of classical field theories on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$. As above, the fields of such a theory are a dg $\mathscr{A}$-module $\mathscr{E}$ equipped with an $\mathscr{A}$-linear local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation $QI + \frac{1}{2}\{I, I\} = 0$.

By pulling back along the projection map

$$(X \times \triangle^n, \mathscr{A} \otimes C^\infty(\triangle^n)) \to (X, \mathscr{A}),$$

we get a new family of classical theories over the dg base ring $\mathscr{A} \otimes C^\infty(\triangle^n)$, whose fields are $\mathscr{E} \otimes C^\infty(\triangle^n)$. We can then ask for a gauge fixing operator

$$Q^{GF} : \mathscr{E} \otimes C^\infty(\triangle^n) \to \mathscr{E} \otimes C^\infty(\triangle^n).$$

for this family of theories. This is the same thing as a smooth family of gauge fixing operators for the original theory depending on a point in the n-simplex.

5.5.0.1 Definition. *Let $(\mathscr{E}, I)$ denote the classical theory we start with over $\mathscr{A}$. Let $\mathscr{GF}(\mathscr{E}, I)$ denote the simplicial set whose n-simplices are such families of gauge fixing operators over $\mathscr{A} \otimes C^\infty(\triangle^n)$. If there is no ambiguity as to what classical theory we are considering, we will denote this simplicial set by $\mathscr{GF}$.*

Any such gauge fixing operator extends, by $\Omega^*(\triangle^n)$-linearity, to a linear map $\mathscr{E} \otimes \Omega^*(\triangle^n) \to \mathscr{E} \otimes \Omega^*(\triangle^n)$, which thus defines a gauge fixing operator for the family of theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ pulled back via the projection

$$(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A}).$$

(Note that $\Omega^*(\triangle^n)$ is equipped with the de Rham differential.)

Example: Suppose that $\mathscr{A} = \mathbb{C}$, and the classical theory we are considering is Chern-Simons theory on a 3-manifold M, where we perturb around the trivial bundle. Then, the space of fields is $\mathscr{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$ and $Q = \mathrm{d}_{dR}$. For every Riemannian metric on M, we find a gauge fixing operator $Q^{GF} = \mathrm{d}^*$. More generally, if we have a smooth family

$$\{g_\sigma \mid \sigma \in \triangle^n\}$$

of Riemannian metrics on M, depending on the point σ in the n-simplex, we get an n-simplex of the simplicial set $\mathscr{GF}$ of gauge fixing operators.

Thus, if $\mathrm{Met}(M)$ denotes the simplicial set whose n-simplices are the set of Riemannian metrics on the fibers of the submersion $M \times \triangle^n \to \triangle^n$, then we have a map of simplicial sets

$$\mathrm{Met}(M) \to \mathscr{GF}.$$

Note that the simplicial set $\mathrm{Met}(M)$ is (weakly) contractible (which follows from the familiar fact that, as a topological space, the space of metrics on M is contractible).

A similar remark holds for almost all theories we consider. For example, suppose we have a theory where the space of fields

$$\mathscr{E} = \Omega^{0,*}(M, V)$$

is the Dolbeault complex on some complex manifold M with coefficients in some holomorphic vector bundle V. Suppose that the linear operator $Q : \mathscr{E} \to \mathscr{E}$ is the $\overline{\partial}$-operator. The natural gauge fixing operators are of the form $\overline{\partial}^*$. Thus, we get a gauge fixing operator for each choice of Hermitian metric on M together with a Hermitian metric on the fibers of V. This simplicial set is again contractible.

It is in this sense that we mean that, in most examples, there is a natural contractible space of gauge fixing operators.

5.5.1. We will use the shorthand notation $(\mathscr{E}, I)$ to denote the classical field theory over $\mathscr{A}$ that we start with; and we will use the notation $(\mathscr{E}_{\triangle^n}, I_{\triangle^n})$ to refer to the family of classical field theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ obtained by base-change along the projection $(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A})$.

5.5.1.1 Definition. *We let $\mathscr{T}^{(n)}$ denote the simplicial set whose k-simplices consist of the following data.*

1. *A k-simplex $Q^{GF}_{\triangle^k} \in \mathscr{GF}[k]$, defining a gauge-fixing operator for the family of theories $(\mathscr{E}_{\triangle^k}, I_{\triangle^k})$ over $\mathscr{A} \otimes \Omega^*(\triangle^k)$.*
2. *A quantization of the family of classical theories with gauge fixing operator $(\mathscr{E}_{\triangle^k}, I_{\triangle^k}, Q^{GF}_{\triangle^k})$, defined modulo $\hbar^{n+1}$.*

We let $\mathscr{T}^{(\infty)}$ denote the corresponding simplicial set where the quantizations are defined to all orders in $\hbar$.

Note that there are natural maps of simplicial sets $\mathscr{T}^{(n)} \to \mathscr{T}^{(m)}$, and that $\mathscr{T}^{(\infty)} = \varprojlim \mathscr{T}^{(n)}$. Further, there are natural maps $\mathscr{T}^{(n)} \to \mathscr{GF}$.

Note further that $\mathscr{T}^{(0)} = \mathscr{GF}$.

This definition describes the most sophisticated version of the set of theories we will consider. Let us briefly explain how to interpret this simplicial set of theories.

Suppose for simplicity that our base ring $\mathscr{A}$ is just $\mathbb{C}$. Then, a 0-simplex of $\mathscr{T}^{(0)}$ is simply a gauge-fixing operator for our theory. A 0-simplex of $\mathscr{T}^{(n)}$ is a gauge fixing operator, together with a quantization (defined with respect to that gauge-fixing operator) to order n in $\hbar$.

A 1-simplex of $\mathscr{T}^{(0)}$ is a homotopy between two gauge fixing operators. Suppose that we fix a 0-simplex of $\mathscr{T}^{(0)}$, and consider a 1-simplex of $\mathscr{T}^{(\infty)}$ in the fiber over this 0-simplex. Such a 1-simplex is given by a collection of effective action functionals

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E}) \otimes \Omega^*([0,1])[[\hbar]]$$

one for each parametrix Φ, which satisfy a version of the QME and the RG flow, as explained above.

We explain in some more detail how one should interpret such a 1-simplex in the space of theories. Let us fix a parametrix Φ on $\mathscr{E}$ and extend it to a parametrix for the family of theories over $\Omega^*([0,1])$. We can then expand our effective interaction $I[\Phi]$ as

$$I[\Phi] = J[\Phi](t) + J'[\Phi](t)\mathrm{d}t$$

where $J[\Phi](t), J'[\Phi](t)$ are elements

$$J[\Phi](t), J'[\Phi](t) \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}) \otimes C^\infty([0,1])[[\hbar]].$$

Here t is the coordinate on the interval $[0,1]$.

The quantum master equation implies that the following two equations hold, for each value of $t \in [0,1]$,

$$QJ[\Phi](t) + \tfrac{1}{2}\{J[\Phi](t), J[\Phi](t)\}_\Phi + \hbar\triangle_\Phi J[\Phi](t) = 0,$$
$$\frac{\partial}{\partial t}J[\Phi](t) + QJ'[\Phi](t) + \{J[\Phi](t), J'[\Phi](t)\}_\Phi + \hbar\triangle_\Phi J'[\Phi](t) = 0.$$

The first equation tells us that for each value of t, $J[\Phi](t)$ is a solution of the quantum master equation. The second equation tells us that the t-derivative of $J[\Phi](t)$ is homotopically trivial as a deformation of the solution to the QME $J[\Phi](t)$.

In general, if I is a solution to some quantum master equation, a transformation of the form

$$I \mapsto I + \varepsilon J = I + \varepsilon QI' + \{I, I'\} + \hbar\triangle I'$$

is often called a "BV canonical transformation" in the physics literature. In the physics literature, solutions of the QME related by a canonical transformation are regarded as equivalent: the canonical transformation can be viewed as a change of coordinates on the space of fields.

For us, this interpretation is not so important. If we have a family of theories over $\Omega^*([0,1])$, given by a 1-simplex in $\mathscr{T}^{(\infty)}$, then the factorization algebra we will construct from this family of theories will be defined over the dg base ring $\Omega^*([0,1])$. This implies that the factorization algebras obtained by restricting to 0 and 1 are quasi-isomorphic.

5.5.2. Generalizations. We will shortly state the theorem which allows us to construct such quantum field theories. Let us first, however, briefly introduce a slightly more general notion of "theory."

We work over a nilpotent dg manifold $(X, \mathscr{A})$. Recall that part of the data of such a manifold is a differential ideal $I \subset \mathscr{A}$ whose quotient is $C^\infty(X)$. In the above discussion, we assumed that our classical action functional S was at least quadratic; we then split S as

$$S = \langle e, Qe \rangle + I(e)$$

into kinetic and interacting terms.

We can generalize this to the situation where S contains linear terms, as long as they are accompanied by elements of the ideal $\mathscr{I} \subset \mathscr{A}$. In this situation, we also have some freedom in the splitting of S into kinetic and interacting terms; we require only that linear and quadratic terms in the interaction I are weighted by elements of the nilpotent ideal $\mathscr{I}$.

In this more general situation, the classical master equation $\{S, S\} = 0$ does not imply that $Q^2 = 0$, only that $Q^2 = 0$ modulo the ideal $\mathscr{I}$. However, this does not lead to any problems; the definition of quantum theory given above can be easily modified to deal with this more general situation.

In the L_∞-language used in Chapter 4, this more general situation describes a family of curved L_∞ algebras over the base dg ring $\mathscr{A}$ with the property that the curving vanishes modulo the nilpotent ideal $\mathscr{I}$.

Recall that ordinary (not curved) L_∞ algebras correspond to formal pointed moduli problems. These curved L_∞ algebras correspond to families of formal moduli problems over $\mathscr{A}$ which are pointed modulo $\mathscr{I}$.

5.6. The theorem on quantization

Let M be a manifold, and suppose we have a family of classical BV theories on M over a nilpotent dg manifold $(X, \mathscr{A})$. Suppose that the space of fields on M is the $\mathscr{A}$-module $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})$ be the dg $\mathscr{A}$-module of local functionals with differential $Q + \{I, -\}$.

Given a cochain complex C, we denote the Dold-Kan simplicial set associated to C by $\mathrm{DK}(C)$. Its n-simplices are the closed, degree 0 elements of $C \otimes \Omega^*(\triangle^n)$.

5.6.0.1 Theorem. *All of the simplicial sets $\mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan complexes and $\mathscr{T}^{(\infty)}(\mathscr{E}, I)$. The maps $p : \mathscr{T}^{(n+1)}(\mathscr{E}, I) \to \mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan fibrations.*

Further, there is a homotopy fiber diagram of simplicial sets

$$\begin{array}{ccc} \mathscr{T}^{(n+1)}(\mathscr{E}, I) & \longrightarrow & 0 \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathscr{T}^{(n)}(\mathscr{E}, I) & \xrightarrow{O} & \mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E})[1], Q + \{I, -\}) \end{array}$$

where O is the "obstruction map."

In more prosaic terms, the second part of the theorem says the following. If $\alpha \in \mathscr{T}^{(n)}(\mathscr{E}, I)[0]$ is a zero-simplex of $\mathscr{T}^{(n)}(\mathscr{E}, I)$, then there is an obstruction $O(\alpha) \in \mathscr{O}_{loc}(\mathscr{E})$. This obstruction is a closed degree 1 element. The simplicial set $p^{-1}(\alpha) \in \mathscr{T}^{(n+1)}(\mathscr{E}, I)$ of extensions of α to the next order in $\hbar$ is homotopy equivalent to the simplicial set of ways of making $O(\alpha)$ exact. In particular, if the cohomology class $[O(\alpha)] \in H^1(\mathscr{O}_{loc}(E), Q + \{I, -\})$ is non-zero, then α does not admit a lift to the next order in $\hbar$. If this cohomology class is zero, then the simplicial set of possible lifts is a torsor for the simplicial Abelian group $\mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E}))[1]$.

Note also that a first order deformation of the classical field theory $(\mathscr{E}, Q, I)$ is given by a closed degree 0 element of $\mathscr{O}_{loc}(\mathscr{E})$. Further, two such first order deformations are equivalent if they are cohomologous. Thus, this theorem tells us that the moduli space of QFTs is "the same size" as the moduli space of classical field theories: at each order in $\hbar$, the data needed to describe a QFT is a local action functional.

The first part of the theorem says can be interpreted as follows. A Kan simplicial set can be thought of as an "infinity-groupoid." Since we can consider families of theories over arbitrary nilpotent dg manifolds, we can consider $\mathscr{T}^{\infty}(\mathscr{E}, I)$ as a functor from the category of nilpotent dg manifolds to that of Kan complexes, or infinity-groupoids. Thus, the space of theories forms something like a "derived stack" [**Toë06**, **Lur11**].

This theorem also tells us in what sense the notion of "theory" is independent of the choice of gauge fixing operator. The simplicial set $\mathscr{T}^{(0)}(\mathscr{E}, I)$ is the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators. Since the map

$$\mathscr{T}^{(\infty)}(\mathscr{E}, I) \to \mathscr{T}^{(0)}(\mathscr{E}, I) = \mathscr{G}\mathscr{F}$$

is a fibration, a path between two gauge fixing conditions Q_0^{GF} and Q_1^{GF} leads to a homotopy between the corresponding fibers, and thus to an equivalence between the ∞-groupoids of theories defined using Q_0^{GF} and Q_1^{GF}.

As we mentioned several times, there is often a natural contractible simplicial set mapping to the simplicial set $\mathscr{GF}$ of gauge fixing operators. Thus, $\mathscr{GF}$ often has a canonical "homotopy point". From the homotopical point of view, having a homotopy point is just as good as having an actual point: if $S \to \mathscr{GF}$ is a map out of a contractible simplicial set, then the fibers in $\mathscr{T}^{(\infty)}$ above any point in S are canonically homotopy equivalent.

5.7. The BD algebra of global observables

In this section, we will try to motivate our definition of a quantum field theory from the point of view of homological algebra. All of the constructions we will explain will work over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, but to keep the notation simple we will not normally mention the base ring $\mathscr{A}$.

Thus, suppose that $(\mathscr{E}, I, Q, \langle -, - \rangle)$ is a classical field theory on a manifold M. We have seen (Chapter 4, section 4.13) how such a classical field theory gives immediately a commutative factorization algebra whose value on an open subset is

$$\mathrm{Obs}^{cl}(U) = (\mathscr{O}(\mathscr{E}(U)), Q + \{I, -\}).$$

Further, we saw that there is a P_0 sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\mathscr{O}_{sm}(\mathscr{E}(U)), Q + \{I, -\}).$$

In particular, we have a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ of global sections of this P_0 algebra. We can think of $\widetilde{\mathrm{Obs}}^{cl}(M)$ as the algebra of functions on the derived space of solutions to the Euler-Lagrange equations.

In this section we will explain how a quantization of this classical field theory will give a quantization (in a homotopical sense) of the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ into a BD algebra $\mathrm{Obs}^q(M)$ of global observables. This BD algebra has some locality properties, which we will exploit later to show that $\mathrm{Obs}^q(M)$ is indeed the global sections of a factorization algebra of quantum observables.

In the case when the classical theory is the cotangent theory to some formal elliptic moduli problem $B\mathcal{L}$ on M (encoded in an elliptic L_∞ algebra $\mathcal{L}$ on M), there is a particularly nice class of quantizations, which we call cotangent quantizations. Cotangent quantizations have a very clear geometric interpretation: they are locally-defined volume forms on the sheaf of formal moduli problems defined by $\mathcal{L}$.

5.7.1. The BD algebra associated to a parametrix. Suppose we have a quantization of our classical field theory (defined with respect to some gauge fixing condition, or family of gauge fixing conditions). Then, for every parametrix Φ, we have seen how to construct a cohomological degree 1 operator

$$\triangle_\Phi : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

and a Poisson bracket

$$\{-,-\}_\Phi : \mathscr{O}(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

such that $\mathscr{O}(\mathscr{E})[[\hbar]]$, with the usual product, with bracket $\{-,-\}_\Phi$ and with differential $Q + \hbar\triangle_\Phi$, forms a BD algebra.

Further, since the effective interaction $I[\Phi]$ satisfies the quantum master equation, we can form a new BD algebra by adding $\{I[\Phi],-\}_\Phi$ to the differential of $\mathscr{O}(\mathscr{E})[[\hbar]]$.

5.7.1.1 Definition. *Let $\mathrm{Obs}^q_\Phi(M)$ denote the BD algebra*

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi\right),$$

with bracket $\{-,-\}_\Phi$ and the usual product.

Remark: Note that $I[\Phi]$ is not in $\mathscr{O}(\mathscr{E})[[\hbar]]$, but rather in $\mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$. However, as we remarked earlier in 5.4.8, the bracket

$$\{I[\Phi],-\}_\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

is well-defined.

Remark: Note that we consider $\mathrm{Obs}^q_\Phi(M)$ as a BD algebra valued in the multicategory of differentiable pro-cochain complexes (see Appendix A). This structure includes a filtration on $\mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]]$. The filtration is defined by saying that

$$F^n\mathscr{O}(\mathscr{E})[[\hbar]] = \prod_i \hbar^i \,\mathrm{Sym}^{\geq(n-2i)}(\mathscr{E}^\vee);$$

it is easily seen that the differential $Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi$ preserves this filtration.

We will show that for varying Φ, the BD algebras $\mathrm{Obs}^q_\Phi(M)$ are canonically weakly equivalent. Moreover, we will show that there is a canonical weak equivalence of P_0 algebras

$$\mathrm{Obs}^q_\Phi(M) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \simeq \widetilde{\mathrm{Obs}}^{cl}(M).$$

To show this, we will construct a family of BD algebras over the dg base ring of forms on a certain contractible simplicial set of parametrices that restricts to $\mathrm{Obs}^q_\Phi(M)$ at each vertex.

Before we get into the details of the construction, however, let us say something about how this result allows us to interpret the definition of a quantum field theory.

A quantum field theory gives a BD algebra for each parametrix. These BD algebras are all canonically equivalent. Thus, at first glance, one might think that the data of a QFT is entirely encoded in the BD algebra for a single parametrix. However, this does not take account of a key part of our definition of a field theory, that of *locality*.

The BD algebra associated to a parametrix Φ has underlying commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$, equipped with a differential which we temporarily denote

$$\mathrm{d}_\Phi = Q + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi.$$

If $K \subset M$ is a closed subset, we have a restriction map

$$\mathscr{E} = \mathscr{E}(M) \to \mathscr{E}(K),$$

where $\mathscr{E}(K)$ denotes germs of smooth sections of the bundle E on K. There is a dual map on functionals $\mathscr{O}(\mathscr{E}(K)) \to \mathscr{O}(\mathscr{E})$. We say a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is *supported on K* if it is in the image of this map.

As $\Phi \to 0$, the effective interaction $I[\Phi]$ and the BV Laplacian $\triangle_\Phi$ become more and more local (i.e., their support gets closer to the small diagonal). This tells us that, for very small Φ, the operator d_Φ only increases the support of a functional in $\mathscr{O}(\mathscr{E})[[\hbar]]$ by a small amount. Further, by choosing Φ to be small enough, we can increase the support by an arbitrarily small amount.

Thus, a quantum field theory is

(1) A family of BD algebra structures on $\mathscr{O}(\mathscr{E})[[\hbar]]$, one for each parametrix, which are all homotopic (and which all have the same underlying graded commutative algebra).

(2) The differential d_Φ defining the BD structure for a parametrix Φ increases support by a small amount if Φ is small.

This property of d_Φ for small Φ is what will allow us to construct a factorization algebra of quantum observables. If d_Φ did not increase the support of a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ at all, the factorization algebra would be easy to define: we would just set $\mathrm{Obs}^q(U) = \mathscr{O}(\mathscr{E}(U))[[\hbar]]$, with differential d_Φ. However, because d_Φ does increase support by some amount (which we can take to be arbitrarily small), it takes a little work to push this idea through.

Remark: The precise meaning of the statement that d_Φ increases support by an arbitrarily small amount is a little delicate. Let us explain what we mean. A functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ has an infinite Taylor expansion of the form $f = \sum \hbar^i f_{i,k}$, where $f_{i,k} : \mathscr{E}^{\otimes k} \to \mathbb{C}$ is a symmetric linear map. We let $\mathrm{Supp}_{\leq (i,k)} f$ be the unions of the supports of $f_{r,s}$ where $(r,s) \leq (i,k)$ in the lexicographical ordering. If $K \subset M$ is a subset, let $\Phi^n(K)$ denote the subset obtained by convolving n times with $\mathrm{Supp}\,\Phi \subset M^2$. The differential d_Φ has the following property: there are constants $c_{i,k} \in \mathbb{Z}_{>0}$ of a purely combinatorial nature (independent of the theory we are considering) such that, for all $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$,

$$\mathrm{Supp}_{\leq (i,k)} d_\Phi f \subset \Phi^{c_{i,k}}(\mathrm{Supp}_{\leq (i,k)} f).$$

Thus, we could say that d_Φ increase support by an amount linear in $\mathrm{Supp}\,\Phi$. We will use this concept in the main theorem of this chapter.

5.7.2. Let us now turn to the construction of the equivalences between $\mathrm{Obs}^q_\Phi(M)$ for varying parametrices Φ. The first step is to construct the simplicial set $\mathscr{P}$ of parametrices; we will then construct a BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$ over the base dg ring $\Omega^*(\mathscr{P})$, which we define below.

Let

$$V \subset \mathscr{E} \otimes \mathscr{E}$$

denote the subspace of those elements which are cohomologically closed and of degree 1, symmetric, and have proper support.

Note that the set of parametrices has the structure of an affine space for V: if Φ, Ψ are parametrices, then

$$\Phi - \Psi \in V \subset \mathscr{E} \otimes \mathscr{E},$$

and, conversely, if Φ is a parametrix and $A \in V$, then $\Phi + A$ is a new parametrix.

Let $\mathscr{P}$ denote the simplicial set whose n-simplices are affine-linear maps from $\triangle^n$ to the affine space of parametrices. It is clear that $\mathscr{P}$ is contractible.

For any vector space V, let $V_\triangle$ denote the simplicial set whose k-simplices are affine linear maps $\triangle^k \to V$. For any convex subset $U \subset V$, there is a sub-simplicial set $U_\triangle \subset V_\triangle$ whose k-simplices are affine linear maps $\triangle^k \to U$. Note that $\mathscr{P}$ is a sub-simplicial set of $\overline{\mathscr{E}}^{\otimes 2}_\triangle$, corresponding to the convex subset of parametrices inside $\overline{\mathscr{E}}^{\otimes 2}$.

Let $\mathscr{C}\mathscr{P}[0] \subset \overline{\mathscr{E}}^{\otimes 2}$ denote the cone on the affine subspace of parametrices, with vertex the origin $\overline{0}$. An element of $\mathscr{C}\mathscr{P}[0]$ is an element of $\overline{\mathscr{E}}^{\otimes 2}$ of the form $t\Phi$, where Φ is a parametrix and $t \in [0,1]$. Let $\mathscr{C}\mathscr{P}$ denote the simplicial set whose k-simplices are affine linear maps to $\mathscr{C}\mathscr{P}[0]$.

Recall that the simplicial de Rham algebra $\Omega^*_\triangle(S)$ of a simplicial set S is defined as follows. Any element $\omega \in \Omega^i_\triangle(S)$ consists of an i-form

$$\omega(\phi) \in \Omega^i(\triangle^k)$$

for each k-simplex $\phi : \triangle^k \to S$. If $f : \triangle^k \to \triangle^l$ is a face or degeneracy map, then we require that

$$f^*\omega(\phi) = \omega(\phi \circ f).$$

The main results of this section are as follows.

5.7.2.1 Theorem. *There is a BD algebra* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *over* $\Omega^*(\mathscr{P})$ *which, at each* 0*-simplex* Φ*, is the BD algebra* $\mathrm{Obs}^q_\Phi(M)$ *discussed above.*

The underlying graded commutative algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *is* $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\mathscr{P})[[\hbar]]$.

For every open subset $U \subset M \times M$*, let* $\mathscr{P}_U$ *denote the parametrices whose support is in* U*. Let* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *denote the restriction of* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *to* U*. The differential on* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *increases support by an amount linear in* U *(in the sense explained precisely in the remark above).*

The bracket $\{-,-\}_{\mathscr{P}_U}$ *on* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *is also approximately local, in the following sense. If* $O_1, O_2 \in \mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *have the property that*

$$\operatorname{Supp} O_1 \times \operatorname{Supp} O_2 \cap U = \emptyset \in M \times M,$$

then $\{O_1, O_2\}_{\mathscr{P}_U} = 0$.

Further, there is a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ over $\Omega^*(\mathscr{C}\mathscr{P})$ equipped with a quasi-isomorphism of P_0 algebras over $\Omega^*(\mathscr{P})$,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\mathscr{P}} \simeq \mathrm{Obs}^q_{\mathscr{P}}(M) \text{ modulo } \hbar,$$

and with an isomorphism of P_0 algebras,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\bar{0}} \cong \widetilde{\mathrm{Obs}}^{cl}(M),$$

where $\widetilde{\mathrm{Obs}}^{cl}(M)$ is the P_0 algebra constructed in Chapter 4.

The underlying commutative algebra of $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is $\widetilde{\mathrm{Obs}}^{cl}(M) \otimes \Omega^*(\mathscr{C}\mathscr{P})$, the differential on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ increases support by an arbitrarily small amount, and the Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is approximately local in the same sense as above.

PROOF. We need to construct, for each k-simplex $\phi : \triangle^k \to \mathscr{P}$, a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$. We view the k-simplex as a subset of $\mathbb{R}^{k+1}$ by

$$\triangle^k := \left\{ (\lambda_0, \dots, \lambda_k) \subset [0,1]^{k+1} \ : \ \sum_i \lambda_i = 1 \right\}.$$

Since simplices in $\mathscr{P}$ are affine linear maps to the space of parametrices, the simplex ϕ is determined by $k+1$ parametrices $\Phi_0, \dots, \Phi_k$, with

$$\phi(\lambda_0, \dots, \lambda_k) = \sum_i \lambda_i \Phi_i$$

for $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$.

The graded vector space underlying our BD algebra is

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k).$$

The structure as a BD algebra will be encoded by an order two, $\Omega^*(\triangle^k)$-linear differential operator

$$\triangle_\phi : \mathrm{Obs}^q_\phi(M) \to \mathrm{Obs}^q_\phi(M).$$

We need to recall some notation in order to define this operator. Each parametrix Φ provides an order two differential operator $\triangle_\Phi$ on $\mathscr{O}(\mathscr{E})$, the BV Laplacian corresponding to Φ. Further, if Φ, Ψ are two parametrices, then the difference between the propagators $P(\Phi) - P(\Psi)$ is an element of $\mathscr{E} \otimes \mathscr{E}$, so that contracting with $P(\Phi) - P(\Psi)$ defines an order two differential operator $\partial_{P(\Phi)} - \partial_{P(\Psi)}$ on $\mathscr{O}(\mathscr{E})$. (This operator defines the infinitesimal version of the renormalization group flow from Ψ to Φ.) We have the equation

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = -\triangle_{\Phi} + \triangle_{\Psi}.$$

Note that although the operator $\partial_{P(\Phi)}$ is only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$, because $P(\Phi) \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$, the difference $\partial_{P(\Phi)}$ and $\partial_{P(\Psi)}$ is nonetheless well-defined on $\mathscr{O}(\mathscr{E})$ because $P(\Phi) - P(\Psi) \in \mathscr{E} \otimes \mathscr{E}$.

The BV Laplacian $\triangle_{\phi}$ associated to the k-simplex $\phi : \triangle^k \to \mathscr{P}$ is defined by the formula

$$\triangle_{\phi} = \sum_{i=0}^{k} \lambda_i \triangle_{\Phi_i} - \sum_{i=0}^{k} \mathrm{d}\lambda_i \partial_{P(\Phi_i)},$$

where the $\lambda_i \in [0,1]$ are the coordinates on the simplex $\triangle^k$ and, as above, the Φ_i are the parametrices associated to the vertices of the simplex ϕ.

It is not entirely obvious that this operator makes sense as a linear map $\mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$, because the operators $\partial_{P(\Phi)}$ are only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$. However, since $\sum \mathrm{d}\lambda_i = 0$, we have

$$\sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} = \sum \mathrm{d}\lambda_i (\partial_{P(\Phi_i)} - \partial_{P(\Phi_0)}),$$

and the right hand side is well defined.

It is immediate that $\triangle_{\phi}^2 = 0$. If we denote the differential on the classical observables $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^n)$ by $Q + \mathrm{d}_{dR}$, we have

$$[Q + \mathrm{d}_{dR}, \triangle_{\phi}] = 0.$$

To see this, note that

$$\begin{aligned}
[Q + \mathrm{d}_{dR}, \triangle_{\phi}] &= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} + \sum \mathrm{d}\lambda_i [Q, \partial_{\Phi_i} - \partial_{\Phi_0}] \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} - \sum \mathrm{d}\lambda_i (\triangle_{\Phi_i} - \triangle_{\Phi_0}) \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_0} \\
&= 0,
\end{aligned}$$

where we use various identities from earlier.

The operator $\triangle_{\phi}$ defines, in the usual way, an $\Omega^*(\triangle^k)$-linear Poisson bracket $\{-,-\}_{\phi}$ on $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$.

We have effective action functionals $I[\Psi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Ψ. Let

$$I[\phi] = I[\sum \lambda_i \Phi_i] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes C^\infty(\triangle^k).$$

The renormalization group equation tells us that $I[\sum \lambda_i \Phi_i]$ is smooth (actually polynomial) in the λ_i.

We define the structure of BD algebra on the graded vector space

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k)$$

as follows. The product is the usual one; the bracket is $\{-,-\}_\phi$, as above; and the differential is

$$Q + \mathrm{d}_{dR} + \hbar \triangle_\phi + \{I[\phi], -\}_\phi.$$

We need to check that this differential squares to zero. This is equivalent to the quantum master equation

$$(Q + \mathrm{d}_{dR} + \hbar \triangle_\phi) e^{I[\phi]/\hbar} = 0.$$

This holds as a consequence of the quantum master equation and renormalization group equation satisfied by $I[\phi]$. Indeed, the renormalization group equation tells us that

$$e^{I[\phi]/\hbar} = \exp\left(\hbar \sum \lambda_i \left(\partial_{P\Phi_i)} - \partial_{P(\Phi_0)}\right)\right) e^{I[\Phi_0]/\hbar}.$$

Thus,

$$\mathrm{d}_{dR} e^{I[\phi]/\hbar} = \hbar \sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} e^{I[\phi]/\hbar}$$

The QME for each $I[\sum \lambda_i \Phi_i]$ tells us that

$$(Q + \hbar \sum \lambda_i \triangle_{\Phi_i}) e^{I[\phi]/\hbar} = 0.$$

Putting these equations together with the definition of $\triangle_\phi$ shows that $I[\phi]$ satisfies the QME.

Thus, we have constructed a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$ for every simplex $\phi : \triangle^k \to \mathscr{P}$. It is evident that these BD algebras are compatible with face and degeneracy maps, and so glue together to define a BD algebra over the simplicial de Rham complex $\Omega^*_\triangle(\mathscr{P})$ of $\mathscr{P}$.

Let ϕ be a k-simplex of $\mathscr{P}$, and let

$$\mathrm{Supp}(\phi) = \cup_{\lambda \in \triangle^k} \mathrm{Supp}(\sum \lambda_i \Phi_i).$$

We need to check that the bracket $\{O_1, O_2\}_\phi$ vanishes for observables O_1, O_2 such that $(\operatorname{Supp} O_1 \times \operatorname{Supp})O_2 \cap \operatorname{Supp}\phi = \emptyset$. This is immediate, because the bracket is defined by contracting with tensors in $\mathscr{E} \otimes \mathscr{E}$ whose supports sit inside $\operatorname{Supp}\phi$.

Next, we need to verify that, on a k-simplex ϕ of $\mathscr{P}$, the differential $Q + \{I[\phi], -\}_\phi$ increases support by an amount linear in $\operatorname{Supp}(\phi)$. This follows from the support properties satisfied by $I[\Phi]$ (which are detailed in the definition of a quantum field theory, definition 5.4.9.1).

It remains to construct the P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$. The construction is almost identical, so we will not give all details. A zero-simplex of $\mathscr{C}\mathscr{P}$ is an element of $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ of the form $\Psi = t\Phi$, where Φ is a parametrix. We can use the same formulae we used for parametrices to construct a propagator $P(\Psi)$ and Poisson bracket $\{-,-\}_\Psi$ for each $\Psi \in \mathscr{C}\mathscr{P}$. The kernel defining the Poisson bracket $\{-,-\}_\Psi$ need not be smooth. This means that the bracket $\{-,-\}_\Psi$ is only defined on the subspace $\mathscr{O}_{sm}(\mathscr{E})$ of functionals with smooth first derivative. In particular, if $\Psi = 0$ is the vertex of the cone $\mathscr{C}\mathscr{P}$, then $\{-,-\}_0$ is the Poisson bracket defined in Chapter 4 on $\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E})$.

For each $\Psi \in \mathscr{C}\mathscr{P}$, we can form a tree-level effective interaction

$$I_0[\Psi] = W_0\left(P(\Psi), I\right) \in \mathscr{O}_{sm,P}(\mathscr{E}),$$

where $I \in \mathscr{O}_{loc}(\mathscr{E})$ is the classical action functional we start with. There are no difficulties defining this expression because we are working at tree-level and using functionals with smooth first derivative. If $\Psi = 0$, then $I_0[0] = I$.

The P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$ is defined in almost exactly the same way as we defined the BD algebra over $\Omega^*_{\mathscr{P}}$. The underlying commutative algebra is $\mathscr{O}_{sm}(\mathscr{E}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. On a k-simplex ψ with vertices $\Psi_0, \dots, \Psi_k$, the Poisson bracket is

$$\{-,-\}_\psi = \sum \lambda_i \{-,-\}_{\Psi_i} + \sum \mathrm{d}\lambda_i \{-,-\}_{P(\Psi_i)},$$

where $\{-,-\}_{P(\Psi_i)}$ is the Poisson bracket of cohomological degree 0 defined using the propagator $P(\Psi_i) \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ as a kernel. If we let $I_0[\psi] = I_0[\sum \lambda_i \Psi_i]$, then the differential is

$$\mathrm{d}_\psi = Q + \{I_0[\psi], -\}_\psi.$$

The renormalization group equation and classical master equation satisfied by the $I_0[\Psi]$ imply that $\mathrm{d}_\psi^2 = 0$. If $\Psi = 0$, this P_0 algebra is clearly the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ constructed in Chapter 4. When restricted to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$, this P_0 algebra is the sub P_0

algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$ obtained by restricting to functionals with smooth first derivative; the inclusion

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\mid_{\mathscr{P}} \hookrightarrow \mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$$

is thus a quasi-isomorphism, using proposition 4.15.2.4 of Chapter 4. □

5.8. Global observables

In the next few sections, we will prove the first version (section 1.7) of our quantization theorem. Our proof is by construction, associating a factorization algebra on M to a quantum field theory on M, in the sense of [**Cos11c**]. This is a quantization (in the weak sense) of the P_0 factorization algebra associated to the corresponding classical field theory.

More precisely, we will show the following.

5.8.0.2 Theorem. *For any quantum field theory on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$, there is a factorization algebra Obs^q on M, valued in the multicategory of differentiable pro-cochain complexes flat over $\mathscr{A}[[\hbar]]$.*

There is an isomorphism of factorization algebras

$$\mathrm{Obs}^q \otimes_{\mathscr{A}[[\hbar]]} \mathscr{A} \cong \mathrm{Obs}^{cl}$$

between Obs^q modulo $\hbar$ and the commutative factorization algebra Obs^{cl}.

Further, Obs^q is a weak quantization (in the sense of Chapter 1, section 1.7) of the P_0 factorization algebra Obs^{cl} of classical observables.

5.8.1. So far we have constructed a BD algebra $\mathrm{Obs}^q_\Phi(M)$ for each parametrix Φ; these BD algebras are all weakly equivalent to each other. In this section we will define a cochain complex $\mathrm{Obs}^q(M)$ of global observables which is independent of the choice of parametrix. For every open subset $U \subset M$, we will construct a subcomplex $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U. The complexes $\mathrm{Obs}^q(U)$ will form our factorization algebra.

Thus, suppose we have a quantum field theory on M, with space of fields $\mathscr{E}$ and effective action functionals $\{I[\Phi]\}$, one for each parametrix (as explained in section 5.2). An *observable* for a quantum field theory (that is, an element of the cochain complex

$\mathrm{Obs}^q(M)$) is simply a first-order deformation $\{I[\Phi] + \delta O[\Phi]\}$ of the family of effective action functionals $I[\Phi]$, which satisfies a renormalization group equation but does not necessarily satisfy the locality axiom in the definition of a quantum field theory. Definition 5.8.1.3 makes this idea precise.

Remark: This definition is motivated by a formal argument with the path integral. Let $S(\phi)$ be the action functional for a field ϕ, and let $O(\phi)$ be another function of the field, describing a measurement that one could make. Heuristically, the expectation value of the observable is

$$\langle O \rangle = \frac{1}{Z_S} \int O(\phi) e^{-S(\phi)/\hbar} \mathscr{D}\phi,$$

where Z_S denotes the partition function, simply the integral without O. A formal manipulation shows that

$$\langle O \rangle = \frac{d}{d\delta} \frac{1}{Z_S} \int e^{(-S(\phi)+\hbar\delta O(\phi))/\hbar} \mathscr{D}\phi.$$

In other words, we can view O as a first-order deformation of the action functional S and compute the expectation value as the change in the partition function. Because the book [Cos11c] gives an approach to the path integral that incorporates the BV formalism, we can define and compute expectation values of observables by exploiting the second description of $\langle O \rangle$ given above.

Earlier we defined cochain complexes $\mathrm{Obs}^q_\Phi(M)$ for each parametrix. The underlying graded vector space of $\mathrm{Obs}^q_\Phi(M)$ is $\mathscr{O}(\mathscr{E})[[\hbar]]$; the differential on $\mathrm{Obs}^q_\Phi(M)$ is

$$\widehat{Q}_\Phi = Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi.$$

5.8.1.1 Definition. *Define a linear map*

$$W^\Phi_\Psi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

by requiring that, for an element $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ of cohomological degree i,

$$I[\Phi] + \delta W^\Phi_\Psi(f) = W\left(P(\Phi) - P(\Psi), I[\Psi] + \delta f\right)$$

where δ is a square-zero parameter of cohomological degree $-i$.

5.8.1.2 Lemma. *The linear map*

$$W^\Phi_\Psi : \mathrm{Obs}^q_\Psi(M) \to \mathrm{Obs}^q_\Phi(M)$$

is an isomorphism of differentiable pro-cochain complexes.

PROOF. The fact that W_Ψ^Φ intertwines the differentials $\widehat{Q}_\Phi$ and $\widehat{Q}_\Psi$ follows from the compatibility between the quantum master equation and the renormalization group equation described in [Cos11c], Chapter 5 and summarized in section 5.2. It is not hard to verify that W_Ψ^Φ is a map of differentiable pro-cochain complexes. The inverse to W_Ψ^Φ is W_Φ^Ψ. □

5.8.1.3 Definition. *A global observable O of cohomological degree i is an assignment to every parametrix Φ of an element*

$$O[\Phi] \in \mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathcal{E})[[\hbar]]$$

of cohomological degree i such that

$$W_\Psi^\Phi O[\Psi] = O[\Phi].$$

If O is an observable of cohomological degree i, we let $\widehat{Q}O$ be defined by

$$\widehat{Q}(O)[\Phi] = \widehat{Q}_\Phi(O[\Phi]) = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi].$$

This makes the space of observables into a differentiable pro-cochain complex, which we call $\mathrm{Obs}^q(M)$.

Thus, if $O \in \mathrm{Obs}^q(M)$ is an observable of cohomological degree i, and if δ is a square-zero parameter of cohomological degree $-i$, then the collection of effective interactions $\{I[\Phi] + \delta O[\Phi]\}$ satisfy most of the axioms needed to define a family of quantum field theories over the base ring $\mathbb{C}[\delta]/\delta^2$. The only axiom which is not satisfied is the locality axiom: we have not imposed any constraints on the behavior of the $O[\Phi]$ as $\Phi \to 0$.

5.9. Local observables

So far, we have defined a cochain complex $\mathrm{Obs}^q(M)$ of global observables on the whole manifold M. If $U \subset M$ is an open subset of M, we would like to isolate those observables which are "supported on U".

The idea is to say that an observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for sufficiently small parametrices, $O[\Phi]$ is supported on U. The precise definition is as follows.

5.9.0.4 Definition. *An observable* $O \in \mathrm{Obs}^q(M)$ *is supported on* U *if, for each* $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$*, there exists a compact subset* $K \subset U^k$ *and a parametrix* Φ*, such that for all parametrices* $\Psi \leq \Phi$

$$\operatorname{Supp} O_{i,k}[\Psi] \subset K.$$

Remark: Recall that $O_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathbb{C}$ is the kth term in the Taylor expansion of the coefficient of $\hbar^i$ of the functional $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$.

Remark: As always, the definition works over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, even though we suppress this from the notation. In this generality, instead of a compact subset $K \subset U^k$ we require $K \subset U^k \times X$ to be a set such that the map $K \to X$ is proper.

We let $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ be the sub-graded vector space of observables supported on U.

5.9.0.5 Lemma. $\mathrm{Obs}^q(U)$ *is a sub-cochain complex of* $\mathrm{Obs}^q(M)$*. In other words, if* $O \in \mathrm{Obs}^q(U)$*, then so is* $\widehat{Q}O$*.*

PROOF. The only thing that needs to be checked is the support condition. We need to check that, for each (i,k), there exists a compact subset K of U^k such that, for all sufficiently small Φ, $\widehat{Q}O_{i,k}[\Phi]$ is supported on K.

Note that we can write

$$\widehat{Q}O_{i,k}[\Phi] = QO_{i,k}[\Phi] + \sum_{\substack{a+b=i \\ r+s=k+2}} \{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi + \Delta_\Phi O_{i-1,k+2}[\Phi].$$

We now find a compact subset K for $\widehat{Q}O_{i,k}[\Phi]$. We know that, for each (i,k) and for all sufficiently small Φ, $O_{i,k}[\Phi]$ is supported on $\tilde{K}$, where $\tilde{K}$ is some compact subset of U^k. It follows that $QO_{i,k}[\Phi]$ is supported on $\tilde{K}$.

By making $\tilde{K}$ bigger, we can assume that for sufficiently small Φ, $O_{i-1,k+2}[\Phi]$ is supported on L, where L is a compact subset of U^{k+2} whose image in U^k, under every projection map, is in $\tilde{K}$. This implies that $\Delta_\Phi O_{i-1,k+2}[\Phi]$ is supported on $\tilde{K}$.

The locality condition for the effective actions $I[\Phi]$ implies that, by choosing Φ to be sufficiently small, we can make $I_{i,k}[\Phi]$ supported as close as we like to the small diagonal in M^k. It follows that, by choosing Φ to be sufficiently small, the support of

$\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ can be taken to be a compact subset of U^k. Since there are only a finite number of terms like $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ in the expression for $(\widehat{Q}O)_{i,k}[\Phi]$, we see that for Φ sufficiently small, $(\widehat{Q}O)_{i,k}[\Phi]$ is supported on a compact subset K of U^k, as desired. □

5.9.0.6 Lemma. $\mathrm{Obs}^q(U)$ *has a natural structure of differentiable pro-cochain complex space.*

PROOF. Our general strategy for showing that something is a differentiable vector space is to ensure that everything works in families over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. Thus, suppose that the theory we are working with is defined over $(X, \mathscr{A})$. If Y is a smooth manifold, we say a smooth map $Y \to \mathrm{Obs}^q(U)$ is an observable for the family of theories over $(X \times Y, \mathscr{A} \otimes C^\infty(Y))$ obtained by base-change along the map $X \times Y \to X$ (so this family of theories is constant over Y).

The filtration on $\mathrm{Obs}^q(U)$ (giving it the structure of pro-differentiable vector space) is inherited from that on $\mathrm{Obs}^q(M)$. Precisely, an observable $O \in \mathrm{Obs}^q(U)$ is in $F^k \mathrm{Obs}^q(U)$ if, for all parametrices Φ,

$$O[\Phi] \in \prod \hbar^i \mathrm{Sym}^{\geq(2k-i)} \mathscr{E}^\vee.$$

The renormalization group flow W_Φ^Ψ preserves this filtration.

So far we have verified that $\mathrm{Obs}^q(U)$ is a pro-object in the category of pre-differentiable cochain complexes. The remaining structure we need is a flat connection

$$\nabla : C^\infty(Y, \mathrm{Obs}^q(U)) \to \Omega^1(Y, \mathrm{Obs}^q(U))$$

for each manifold Y, where $C^\infty(Y, \mathrm{Obs}^q(U))$ is the space of smooth maps $Y \to \mathrm{Obs}^q(U)$.

This flat connection is equivalent to giving a differential on

$$\Omega^*(Y, \mathrm{Obs}^q(U)) = C^\infty(Y, \mathrm{Obs}^q(U)) \otimes_{C^\infty(Y)} \Omega^*(Y)$$

making it into a dg module for the dg algebra $\Omega^*(Y)$. Such a differential is provided by considering observables for the family of theories over the nilpotent dg manifold $(X \times Y, \mathscr{A} \otimes \Omega^*(Y))$, pulled back via the projection map $X \times Y \to Y$. □

5.10. Local observables form a prefactorization algebra

At this point, we have constructed the cochain complex $\mathrm{Obs}^q(M)$ of global observables of our factorization algebra. We have also constructed, for every open subset $U \subset M$, a sub-cochain complex $\mathrm{Obs}^q(U)$ of observables supported on U.

In this section we will see that the local quantum observables $\mathrm{Obs}^q(U)$ of a quantum field on a manifold M form a prefactorization algebra.

The definition of local observables makes it clear that they form a pre-cosheaf: there are natural injective maps of cochain complexes

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$$

if $U \subset U'$ is an open subset.

Let U, V be disjoint open subsets of M. The structure of prefactorization algebra on the local observables is specified by the pre-cosheaf structure mentioned above, and a bilinear cochain map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

These product maps need to be a map of cochain complexes which is compatible with the pre-cosheaf structure and with reordering of the disjoint opens.

5.10.1. Defining the product map. Suppose that $O \in \mathrm{Obs}^q(U)$ and $O' \in \mathrm{Obs}^q(V)$ are observables on U and V respectively. Note that $O[\Phi]$ and $O'[\Phi]$ are elements of the cochain complex

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], \widehat{Q}_\Phi\right)$$

which is a BD algebra and so a commutative algebra (ignoring the differential, of course). (The commutative product is simply the usual product of functions on $\mathscr{E}$.) In the definition of the prefactorization product, we will use the product of $O[\Phi]$ and $O'[\Phi]$ taken in the commutative algebra $\mathcal{O}(\mathcal{E})$. This product will be denoted $O[\Phi] * O'[\Phi] \in \mathcal{O}(\mathcal{E})$.

Recall (see definition 5.8.1.1) that we defined a linear renormalization group flow operator W^Ψ_Φ, which is an isomorphism between the cochain complexes $\mathrm{Obs}^q_\Phi(M)$ and $\mathrm{Obs}^q_\Psi(M)$.

The main result of this section is the following.

5.10.1.1 Theorem. *For all observables $O \in \mathrm{Obs}^q(U)$, $O' \in \mathrm{Obs}^q(V)$, where U and V are disjoint, the limit*

$$\lim_{\Psi \to 0} W_\Psi^\Phi \left(O[\Psi] * O'[\Psi] \right) \in \mathscr{O}(\mathscr{E})[[\hbar]]$$

exists. Further, this limit satisfies the renormalization group equation, so that we can define an observable $m(O, O')$ by

$$m(O,O')[\Phi] = \lim_{\Psi \to 0} W_\Psi^\Phi \left(O[\Psi] * O'[\Psi] \right).$$

The map

$$\begin{aligned} \mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) &\mapsto \mathrm{Obs}^q(U \amalg V) \\ O \times O' &\mapsto m(O, O') \end{aligned}$$

is a smooth bilinear cochain map, and it makes Obs^q into a prefactorization algebra in the multicategory of differentiable pro-cochain complexes.

PROOF. We will show that, for each i, k, the Taylor term

$$W_\Phi^\Psi (O[\Phi] * O'[\Phi])_{i,k} : \mathscr{E}^{\otimes k} \to \mathbb{C}$$

is independent of Ψ for Ψ sufficiently small.

Note that

$$W_\Gamma^\Psi \left(W_\Phi^\Gamma \left(O[\Phi] * O'[\Phi] \right) \right) = W_\Phi^\Psi \left(O[\Phi] * O'[\Phi] \right).$$

Thus, to show that the limit $\lim_{\Phi \to 0} W_\Phi^\Psi \left(O[\Phi] * O'[\Phi] \right)$ is eventually constant, it suffices to show that, for all sufficiently small Φ, Γ satisfying $\Phi < \Gamma$,

$$W_\Phi^\Gamma (O[\Phi] * O'[\Phi])_{i,k} = (O[\Gamma] * O'[\Gamma])_{i,k}.$$

This turns out to be an exercise in the manipulation of Feynman diagrams. In order to prove this, we need to recall a little about the Feynman diagram expansion of $W_\Phi^\Gamma(O[\Phi])$. (Feynman diagram expansions of the renormalization group flow are discussed extensively in [Cos11c].)

We have a sum of the form

$$W_\Phi^\Gamma (O[\Phi])_{i,k} = \sum_G \frac{1}{|\mathrm{Aut}(G)|} w_G \left(O[\Phi] * O'[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi) \right).$$

The sum is over all connected graphs G with the following decorations and properties.

(1) The vertices v of G are labelled by an integer $g(v) \in \mathbb{Z}_{\geq 0}$, which we call the genus of the vertex.

(2) The first Betti number of G, plus the sum of over all vertices of the genus $g(v)$, must be i (the "total genus").
(3) G has one special vertex.
(4) G has k tails (or external edges).

The weight $w_G\left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi)\right)$ is computed by the contraction of a collection of symmetric tensors. One places $O[\Phi]_{r,s}$ at the special vertex, when that vertex has genus r and valency s; places $I[\Phi]_{g,v}$ at every other vertex of genus g and valency v; and puts the propagator $P(\Gamma) - P(\Phi)$ on each edge.

Let us now consider $W_\Phi^\Gamma(O[\Phi] * O'[\Phi])$. Here, we a sum over graphs with one special vertex, labelled by $O[\Phi] * O'[\Phi]$. This is the same as having two special vertices, one of which is labelled by $O[\Phi]$ and the other by $O'[\Phi]$. Diagrammatically, it looks like we have split the special vertex into two pieces. When we make this maneuver, we introduce possibly disconnected graphs; however, each connected component must contain at least one of the two special vertices.

Let us now compare this to the graphical expansion of

$$O[\Gamma] * O'[\Gamma] = W_\Phi^\Gamma(O[\Phi]) * W_\Phi^\Gamma(O'[\Phi]).$$

The Feynman diagram expansion of the right hand side of this expression consists of graphs with two special vertices, labelled by $O[\Phi]$ and $O'[\Phi]$ respectively (and, of course, any number of other vertices, labelled by $I[\Phi]$, and the propagator $P(\Gamma) - P(\Phi)$ labelling each edge). Further, the relevant graphs have precisely two connected components, each of which contains one of the special vertices.

Thus, we see that

$$W_\Phi^\Gamma(O[\Phi] * O'[\Phi]) - W_\Phi^\Gamma(O[\Phi]) * W_\Phi^\Gamma(O'[\Phi]).$$

is a sum over *connected* graphs, with two special vertices, one labelled by $O[\Phi]$ and the other by $O'[\Phi]$. We need to show that the weight of such graphs vanish for Φ, Γ sufficiently small, with $\Phi < \Gamma$.

Graphs with one connected component must have a chain of edges connecting the two special vertices. (A chain is a path in the graph with no repeated vertices or edges.) For a graph G with "total genus" i and k tails, the length of any such chain is bounded by an expression involving only i and k. (It is important to note here that we require a vertex of genus zero to have valence at least three and a vertex of genus one to have valence at least one. See [Cos11c] for more discussion.) Each step along such a chain involves a tensor with some support that depends on the choice of parametrix Φ. As we move from the special vertex O toward the other O', we extend the support, and our aim is to show that we can choose Φ small enough that the support of the chain, excluding $O'[\Phi]$, is disjoint from the support of $O'[\Phi]$. The contraction of a distribution and function with disjoint supports is zero, so that the weight will vanish. We now make this idea precise.

Let us choose arbitrarily a metric on M. By taking Φ and Γ to be sufficiently small, we can assume that the support of the propagator on each edge is within ε of the diagonal in this metric. By choosing Γ to be sufficiently small, we can take ε as small as we like. Similarly, the support of the $I_{r,s}[\Gamma]$ labelling a vertex of genus r and valency s can be taken to be within $c_{r,s}\varepsilon$ of the diagonal, where $c_{r,s}$ is a combinatorial constant. In addition, by choosing Φ to be small enough we can ensure that the supports of $O[\Phi]$ and $O'[\Phi]$ are disjoint.

Now let G' denote the graph G with the special vertex for O' removed. This graph corresponds to a symmetric tensor whose support is within some distance $C_G\varepsilon$ of the small diagonal, where C_G is a combinatorial constant depending on the graph G'. As the supports K and K' (of O and O', respectively) have a finite distance d between them, we can choose ε small enough that $C_G\varepsilon < d$. It follows that, by choosing Φ and Γ to be sufficiently small, the weight of any connected graph is obtained by contracting a distribution and a function which have disjoint support. The graph hence has weight zero.

As there are finitely many such graphs with total genus i and k tails, we see that we can choose Γ small enough that for any $\Phi < \Gamma$, the weight of all such graphs vanishes.

Thus we have proved the first part of the theorem and have produced a bilinear map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

It is a straightforward to show that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization algebra. The fact that this is a smooth map of differentiable pro-vector spaces follows from the fact that this construction works for families of theories over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. □

5.11. Local observables form a factorization algebra

We have seen how to define a prefactorization algebra Obs^q of observables for our quantum field theory. In this section we will show that this prefactorization algebra is in fact a factorization algebra. In the course of the proof, we show that modulo $\hbar$, this factorization algebra is isomorphic to Obs^{cl}.

5.11.0.2 Theorem. *(1) The prefactorization algebra* Obs^q *of quantum observables is, in fact, a factorization algebra.*
(2) Further, there is an isomorphism

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$, and the factorization algebra of classical observables.

5.11.1. Proof of the theorem.

This theorem will be a corollary of a more technical proposition.

5.11.1.1 Proposition. *For any open subset $U \subset M$, filter $\mathrm{Obs}^q(U)$ by saying that the k-th filtered piece $G^k \mathrm{Obs}^q(U)$ is the sub $\mathbb{C}[[\hbar]]$-module consisting of those observables which are zero modulo $\hbar^k$. Note that this is a filtration by sub prefactorization algebras over the ring $\mathbb{C}[[\hbar]]$.*

Then, there is an isomorphism of prefactorization algebras (in differentiable pro-cochain complexes)

$$\mathrm{Gr}\,\mathrm{Obs}^q \simeq \mathrm{Obs}^{cl} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]].$$

This isomorphism makes $\mathrm{Gr}\,\mathrm{Obs}^q$ into a factorization algebra.

Remark: We can give $G^k \mathrm{Obs}^q(U)$ the structure of a pro-differentiable cochain complex, as follows. The filtration on $G^k \mathrm{Obs}^q(U)$ that defines the pro-structure is obtained by intersecting $G^k \mathrm{Obs}^q(U)$ with the filtration on $\mathrm{Obs}^q(U)$ defining the pro-structure. Then the inclusion $G^k \mathrm{Obs}^q(U) \hookrightarrow \mathrm{Obs}^q(U)$ is a cofibration of differentiable pro-vector spaces (see definition A.5.0.7).

PROOF OF THE THEOREM, ASSUMING THE PROPOSITION. We need to show that for every open U and for every factorizing cover $\mathfrak{U}$, the natural map

$$(\dagger) \qquad \check{C}(\mathfrak{U}, \mathrm{Obs}^q) \to \mathrm{Obs}^q(U)$$

is a quasi-isomorphism of differentiable pro-cochain complexes.

The basic idea is that the filtration induces a spectral sequence for both $\check{C}(\mathfrak{U}, \mathrm{Obs}^q)$ and $\mathrm{Obs}^q(U)$, and we will show that the induced map of spectral sequences is an isomorphism on the first page. Because we are working with differentiable pro-cochain complexes, this is a little subtle. The relevant statements about spectral sequences in this context are developed in this context in Appendix A.

Note that $\check{C}(\mathcal{U}, \mathrm{Obs}^q)$ is filtered by $\check{C}(\mathfrak{U}, G^k \mathrm{Obs}^q)$. The map (†) preserves the filtrations. Thus, we have a maps of inverse systems

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q / G^k \mathrm{Obs}^q) \to \mathrm{Obs}^q(U)/G^k \mathrm{Obs}^q(U).$$

These inverse systems satisfy the properties of Appedix A, lemma A.5.0.11. Further, it is clear that

$$\mathrm{Obs}^q(U) = \varprojlim \mathrm{Obs}^q(U)/G^k \mathrm{Obs}^q(U).$$

We also have

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q) = \varprojlim \check{C}(\mathfrak{U}, \mathrm{Obs}^q / G^k \mathrm{Obs}^q).$$

This equality is less obvious, and uses the fact that the Čech complex is defined using the completed direct sum as described in Appendix A, section A.5.

Using lemma A.5.0.11, we need to verify that the map

$$\check{C}(\mathfrak{U}, \mathrm{Gr}\,\mathrm{Obs}^q) \to \mathrm{Gr}\,\mathrm{Obs}^q(U)$$

is an equivalence. This follows from the proposition because $\mathrm{Gr}\,\mathrm{Obs}^q$ is a factorization algebra. □

PROOF OF THE PROPOSITION. The first step in the proof of the proposition is the following lemma.

5.11.1.2 Lemma. *Let $\mathrm{Obs}^q_{(0)}$ denote the prefactorization algebra of observables which are only defined modulo $\hbar$. There is an isomorphism of prefactorization algebras*

$$\mathrm{Obs}^q_{(0)} \simeq \mathrm{Obs}^{cl}$$

of differential graded prefactorization algebras.

PROOF OF LEMMA. Let $O \in \mathrm{Obs}^{cl}(U)$ be a classical observable. Thus, O is an element of the cochain complex $\mathscr{O}(\mathscr{E}(U))$ of functionals on the space of fields on U.

We need to produce an element of $\mathrm{Obs}^q_{(0)}$ from O. An element of $\mathrm{Obs}^q_{(0)}$ is a collection of functionals $O[\Phi] \in \mathscr{O}(\mathscr{E})$, one for every parametrix Φ, satisfying a classical version of the renormalization group equation and an axiom saying that $O[\Phi]$ is supported on U for sufficiently small Φ.

Given an element

$$O \in \mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

we define an element

$$\{O[\Phi]\} \in \mathrm{Obs}^q_{(0)}$$

by the formula

$$O[\Phi] = \lim_{\Gamma \to 0} W^{\Phi}_{\Gamma}(O) \text{ modulo } \hbar.$$

The Feynman diagram expansion of the right hand side only involves trees, since we are working modulo $\hbar$. As we are only using trees, the limit exists. The limit is defined by a sum over trees with one special vertex, where each edge is labelled by the propagator $P(\Phi)$, the special vertex is labelled by O, and every other vertex is labelled by the classical interaction $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ of our theory.

The map

$$\mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^q_{(0)}(U)$$

we have constructed is easily seen to be a map of cochain complexes, compatible with the structure of prefactorization algebra present on both sides. (The proof is a variation on the argument in section 11, chapter 5 of [Cos11c], about the scale 0 limit of a deformation of I modulo $\hbar$.)

A simple inductive argument on the degree shows this map is an isomorphism.

Because the construction works over an arbitrary nilpotent dg manifold, it is clear that these maps are maps of differentiable cochain complexes. □

The next (and most difficult) step in the proof of the proposition is the following lemma. We use it to work inductively with the filtration of quantum observables.

Let $\mathrm{Obs}^q_{(k)}$ denote the prefactorization algebra of observables defined modulo $\hbar^{k+1}$.

5.11.1.3 Lemma. *For all open subsets $U \subset M$, the natural quotient map of differentiable pro-cochain complexes*

$$\mathrm{Obs}^q_{(k+1)}(U) \to \mathrm{Obs}^q_{(k)}(U)$$

is a fibration of differentiable pro-cochain complexes (see Appendix A, Definition A.5.0.7 for the definition of a fibration). The fiber is isomorphic to $\mathrm{Obs}^{cl}(U)$*.*

PROOF OF LEMMA. We give the set $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lexicographical ordering, so that $(i,k) > (r,s)$ if $i > r$ or if $i = r$ and $k > s$.

We will let $\mathrm{Obs}^q_{\leq (i,k)}(U)$ be the quotient of $\mathrm{Obs}^q_{(i)}$ consisting of functionals

$$O[\Phi] = \sum_{(r,s) \leq (i,k)} \hbar^r O_{(r,s)}[\Phi]$$

satisfying the renormalization group equation and locality axiom as before, but where $O_{(r,s)}[\Phi]$ is only defined for $(r,s) \leq (i,k)$. Similarly, we will let $\mathrm{Obs}^q_{<(i,k)}(U)$ be the quotient where the $O_{(r,s)}[\Phi]$ are only defined for $(r,s) < (i,k)$.

We will show that the quotient map

$$\mathsf{q} : \mathrm{Obs}^q_{\leq (i,k)}(U) \to \mathrm{Obs}^q_{<(i,k)}(U)$$

is a fibration. The result will follow.

Recall what it means for a map $f : V \to W$ of differentiable cochain complexes to be a fibration. For X a manifold, let $C^\infty_X(V)$ denote the sheaf of cochain complexes on X of smooth maps to V. We say f is a fibration if for every manifold X, the induced map of sheaves $C^\infty_X(V) \to C^\infty_X(W)$ is surjective in each degree. Equivalently, we require that for all smooth manifolds X, every smooth map $X \to W$ lifts locally on X to a map to V.

Now, by definition, a smooth map from X to $\mathrm{Obs}^q(U)$ is an observable for the constant family of theories over the nilpotent dg manifold $(X, C^\infty(X))$. Thus, in order to show q is a fibration, it suffices to show the following. For any family of theories over the nilpotent dg manifold $(X, \mathscr{A})$, any open subset $U \subset M$, and any observable α in the $\mathscr{A}$-module $\mathrm{Obs}^q_{<(i,k)}(U)$, we can lift α to an element of $\mathrm{Obs}^q_{\leq(i,k)}(U)$ locally on X.

To prove this, we will first define, for every parametrix Φ, a map

$$L_\Phi : \mathrm{Obs}^q_{<(i,k)}(U) \to \mathrm{Obs}^q_{\leq(i,k)}(M)$$

with the property that the composed map

$$\mathrm{Obs}^q_{<(i,k)}(U) \xrightarrow{L_\Phi} \mathrm{Obs}^q_{\leq(i,k)}(M) \to \mathrm{Obs}^q_{<(i,k)}(M)$$

is the natural inclusion map. Then, for every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we will show that $L_\Phi(O)$ is supported on U, for sufficiently small parametrices Φ, so that $L_\Phi(O)$ provides the desired lift.

For
$$O \in \mathrm{Obs}^q_{<(i,k)}(U),$$
we define
$$L_\Phi(O) \in \mathrm{Obs}^q_{\leq(i,k)}(M)$$
by
$$L_\Phi(O)_{r,s}[\Phi] = \begin{cases} O_{r,s}[\Phi] & \text{if } (r,s) < (i,k) \\ 0 & \text{if } (r,s) = (i,k) \end{cases}.$$
For $\Psi \neq \Phi$, we obtain $L_\Phi(O)_{r,s}[\Psi]$ by the renormalization group flow from $L_\Phi(O)_{r,s}[\Phi]$. If $(r,s) < (i,k)$, then
$$L_\Phi(O)_{r,s}[\Psi] = O_{r,s}[\Psi].$$
But
$$I_{i,k}[\Psi] + \delta\left(L_\Phi(O)_{i,k}[\Psi]\right) = W_{i,k}\left(P(\Psi) - P(\Phi), I[\Phi] + \delta O[\Phi]\right)$$
for δ a square-zero parameter of cohomological degree opposite to that of O. Hence $L_\Phi(O)_{r,s}$ satisfies the relevant RGE.

To complete the proof of this lemma, we prove the required local lifting property in the sublemma below. □

5.11.1.4 Sub-lemma. *For each $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we can find a parametrix Φ — locally over the parametrizing manifold X — so that $L_\Phi O$ lies in $\mathrm{Obs}^q_{\leq(i,k)}(U) \subset \mathrm{Obs}^q_{\leq(i,k)}(M)$.*

PROOF. Although the observables Obs^q form a factorization algebra on the manifold M, they also form a sheaf on the parametrizing base manifold X. That is, for every open subset $V \subset X$, let $\mathrm{Obs}^q(U) \mid_V$ denote the observables for our family of theories restricted to V. In other words, $\mathrm{Obs}^q(U) \mid_V$ denotes the sections of this sheaf $\mathrm{Obs}^q(U)$ on V.

The map L_Φ constructed above is then a map of sheaves on X.

For every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we need to find an open cover
$$X = \bigcup_\alpha Y_\alpha$$

of X, and on each Y_α a parametrix Φ_α (for the restriction of the family of theories to Y_α) such that

$$L_{\Phi_\alpha}(O\mid_{Y_\alpha}) \in \mathrm{Obs}^q_{\leq(i,k)}(U)\mid_{Y_\alpha}.$$

More informally, we need to show that locally in X, we can find a parametrix Φ such that for all sufficiently small Ψ, the support of $L_\Phi(O)_{(i,k)}[\Psi]$ is in a subset of $U^k \times X$ which maps properly to X.

This argument resembles previous support arguments (e.g., the product lemma from section 5.10). The proof involves an analysis of the Feynman diagrams appearing in the expression

$$(\star) \qquad L_\Phi(O)_{i,k}[\Psi] = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} w_\gamma\left(O[\Phi]; I[\Phi]; P(\Psi) - P(\Phi)\right).$$

The sum is over all connected Feynman diagrams of genus i with k tails. The edges are labelled by $P(\Psi) - P(\Phi)$. Each graph has one special vertex, where $O[\Phi]$ appears. More explicitly, if this vertex is of genus r and valency s, it is labelled by $O_{r,s}[\Phi]$. Each non-special vertex is labelled by $I_{a,b}[\Phi]$, where a is the genus and b the valency of the vertex. Note that only a finite number of graphs appear in this sum.

By assumption, O is supported on U. This means that there exists some parametrix Φ_0 and a subset $K \subset U \times X$ mapping properly to X such that for all $\Phi < \Phi_0$, $O_{r,s}[\Phi]$ is supported on K^r. (Here by $K^s \subset U^s \times X$ we mean the fibre product over X.)

Further, each $I_{a,b}[\Phi]$ is supported as close as we like to the small diagonal $M \times X$ in $M^k \times X$. We can find precise bounds on the support of $I_{a,b}[\Phi]$, as explained in section 5.4. To describe these bounds, let us choose metrics for X and M. For a parametrix Φ supported within ε of the diagonal $M \times X$ in $M \times M \times X$, the effective interaction $I_{a,b}[\Phi]$ is supported within $(2a+b)\varepsilon$ of the diagonal.

(In general, if $A \subset M^n \times X$, the ball of radius ε around A is defiend to be the union of the balls of radius ε around each fibre A_x of $A \to X$. It is in this sense that we mean that $I_{a,b}[\Phi]$ is supported within $(2a+b)\varepsilon$ of the diagonal.)

Similarly, for every parametrix Ψ with $\Psi < \Phi$, the propagator $P(\Psi) - P(\Phi)$ is supported within ε of the diagonal.

In sum, there exists a set $K \subset U \times X$, mapping properly to X, such that for all $\varepsilon > 0$, there exists a parametrix Φ_ε, such that

(1) $O[\Phi_\varepsilon]_{r,s}$ is supported on K^s for all $(r,s) < (i,k)$.
(2) $I_{a,b}[\Phi_\varepsilon]$ is supported within $(2a+b)\varepsilon$ of the small diagonal.
(3) For all $\Psi < \Phi_\varepsilon$, $P(\Psi) - P(\Phi_\varepsilon)$ is supported within ε of the small diagonal.

The weight w_γ in the graphical expansion $(\star)$ above (using the parametrices Φ_ε and any $\Psi < \Phi_\varepsilon$) is thus supported in the ball of radius $c\varepsilon$ around K^k (where c is some combinatorial constant, depending on the number of edges and vertices in γ). There are a finite number of such graphs in the sum, so we can choose the combinatorial constant c uniformly over the graphs.

Since $K \subset U \times X$ maps properly to X, locally on X, we can find an ε so that the closed ball of radius $c\varepsilon$ is still inside $U^k \times X$. This completes the proof. □

□

5.12. Translation-invariant factorization algebras from translation-invariant quantum field theories

In this section, we will show that a translation-invariant quantum field theory on $\mathbb{R}^n$ gives rise to a smoothly translation-invariant factorization algebra on $\mathbb{R}^n$ (see section 2.7). We will also show that a holomorphically translation-invariant field theory on $\mathbb{C}^n$ gives rise to a holomorphically translation-invariant factorization algebra.

5.12.1. First, we need to define what it means for a field theory to be translation-invariant. Let us consider a classical field theory on $\mathbb{R}^n$. Recall that this is given by

(1) A graded vector bundle E whose sections are $\mathscr{E}$;
(2) An antisymmetric pairing $E \otimes E \to \mathrm{Dens}_{\mathbb{R}^n}$;
(3) A differential operator $Q : \mathscr{E} \to \mathscr{E}$ making $\mathscr{E}$ into an elliptic complex, which is skew-self adjoint;
(4) A local action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

A classical field theory is translation-invariant if

(1) The graded bundle E is translation-invariant, so that we are given an isomorphism between E and the trivial bundle with fibre E_0.

(2) The pairing, differential Q, and local functional I are all translation-invariant.

It takes a little more work to say what it means for a quantum field theory to be translation-invariant. Suppose we have a translation-invariant classical field theory, equipped with a translation-invariant gauge fixing operator Q^{GF}. As before, a quantization of such a field theory is given by a family of interactions $I[\Phi] \in \mathscr{O}_{sm,P}(\mathscr{E})$, one for each parametrix Φ.

5.12.1.1 Definition. *A translation-invariant quantization of a translation-invariant classical field theory is a quantization with the property that, for all translation-invariant parametrices Φ, $I[\Phi]$ is translation-invariant.*

Remark: In general, in order to give a quantum field theory on a manifold M, we do not need to give an effective interaction $I[\Phi]$ for all parametrices. We only need to specify $I[\Phi]$ for a collection of parametrices such that the intersection of the supports of Φ is the small diagonal $M \subset M^2$. The functional $I[\Psi]$ for all other parametrices Ψ is defined by the renormalization group flow. It is easy to construct a collection of translation-invariant parametrices satisfying this condition.

5.12.1.2 Proposition. *The factorization algebra associated to a translation-invariant quantum field theory is smoothly translation-invariant (section 2.7).*

Proof. Let Obs^q denote the factorization algebra of quantum observables for our translation-invariant theory. An observable supported on $U \subset \mathbb{R}^n$ is defined by a family $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$, one for each translation-invariant parametrix, which satisfiese the RG flow and (in the sense we explained in section 5.9) is supported on U for sufficiently small parametrices. The renormalization group flow

$$W_\Psi^\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

for translation-invariant parametrices Ψ, Φ commutes with the action of $\mathbb{R}^n$ on $\mathscr{O}(\mathscr{E})$ by translation, and therefore acts on $\mathrm{Obs}^q(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, let $T_x U$ denote the x-translate of U. It is immediate that the action of $x \in \mathbb{R}^n$ on $\mathrm{Obs}^q(\mathbb{R}^n)$ takes $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(\mathbb{R}^n)$ to $\mathrm{Obs}^q(T_x U)$. It is not difficult to verify that the resulting map

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(T_x U)$$

is an isomorphism of differentiable pro-cochain complexes and that it is compatible with the structure of a factorization algebra.

We need to verify the smoothness hypothesis of a smoothly translation-invariant factorization algebra. This is the following. Suppose that $U_1, \dots, U_k$ are disjoint open subsets of $\mathbb{R}^n$, all contained in an open subset V. Let $A' \subset \mathbb{R}^{nk}$ be the subset consisting of those $x_1, \dots, x_k$ such that the closures of $T_{x_i} U_i$ remain disjoint and in V. Let A be the connected component of 0 in A'. We need only examine the case where A is non-empty.

We need to show that the composed map

$$m_{x_1,\dots,x_k} : \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to \mathrm{Obs}^q(T_{x_1} U_1) \times \cdots \times \mathrm{Obs}^q(T_{x_k} U_k) \to \mathrm{Obs}^q(V)$$

varies smoothly with $(x_1, \dots, x_k) \in A$. In this diagram, the first map is the product of the translation isomorphisms $\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i)$, and the second map is the product map of the factorization algebra.

The smoothness property we need to check says that the map $m_{x_1,\dots,x_k}$ lifts to a multilinear map of differentiable pro-cochain complexes

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to C^\infty(A, \mathrm{Obs}^q(V)),$$

where on the right hand side the notation $C^\infty(A, \mathrm{Obs}^q(V))$ refers to the smooth maps from A to $\mathrm{Obs}^q(V)$.

This property is local on A, so we can replace A by a smaller open subset if necessary.

Let us assume (replacing A by a smaller subset if necessary) that there exist open subsets U_i' containing U_i, which are disjoint and contained in V and which have the property that for each $(x_1, \dots, x_k) \in A$, $T_{x_i} U_i \subset U_i'$.

Then, we can factor the map $m_{x_1,\dots,x_k}$ as a composition

$$(\dagger) \quad \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \xrightarrow{i_{x_1} \times \cdots \times i_{x_k}} \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V).$$

Here, the map $i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$ is the composition

$$\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$$

of the translation isomorphism with the natural inclusion map $\mathrm{Obs}^q(T_{x_i}U_i) \to \mathrm{Obs}^q(U_i')$. The second map in equation (†) is the product map associated to the disjoint subsets $U_1', \ldots, U_k' \subset V$.

By possibly replacing A by a smaller open subset, let us assume that $A = A_1 \times \cdots \times A_k$, where the A_i are open subsets of $\mathbb{R}^n$ containing the origin. It remains to show that the map

$$i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$$

is smooth in x_i, that is, extends to a smooth map

$$\mathrm{Obs}^q(U_i) \to C^\infty(A_i, \mathrm{Obs}^q(U_i')).$$

Indeed, the fact that the product map

$$m : \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V)$$

is a smooth multilinear map implies that, for every collection of smooth maps $\alpha_i : Y_i \to \mathrm{Obs}^q(U_i')$ from smooth manifolds Y_i, the resulting map

$$\begin{aligned} Y_1 \times \cdots \times Y_k &\to \mathrm{Obs}^q(V) \\ (y_1, \ldots y_k) &\mapsto m(\alpha_1(y), \ldots, \alpha_k(y)) \end{aligned}$$

is smooth.

Thus, we have reduced the result to the following statement: for all open subsets $A \subset \mathbb{R}^n$ and for all $U \subset U'$ such that $T_x U \subset U'$ for all $x \in A$, the map $i_x : \mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$ is smooth in $x \in A$.

But this statement is tractable. Let

$$O \in \mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(U') \subset \mathrm{Obs}^q(\mathbb{R}^n)$$

be an observable. It is obvious that the family of observables $T_x O$, when viewed as elements of $\mathrm{Obs}^q(\mathbb{R}^n)$, depends smoothly on x. We need to verify that it depends smoothly on x when viewed as an element of $\mathrm{Obs}^q(U')$.

This amounts to showing that the support conditions which ensure an observable is in $\mathrm{Obs}^q(U')$ hold uniformly on x.

The fact that O is in $\mathrm{Obs}^q(U)$ means the following. For each (i, k), there exists a compact subset $K \subset U$ and $\varepsilon > 0$ such that for all translation-invariant parametrices

Φ supported within ε of the diagonal and for all $(r,s) \leq (i,k)$ in the lexicographical ordering, the Taylor coefficient $O_{r,s}[\Phi]$ is supported on K^s.

We need to enlarge K to a subset $L \subset U' \times A$, mapping properly to A, such that $T_x O$ is supported on L in this sense (again, for $(r,s) \leq (i,k)$). Taking $L = K \times A$, embedded in $U' \times A$ by

$$(k, x) \mapsto (T_x k, x)$$

suffices. □

Remark: Essentially the same proof will give us the somewhat stronger result that for any manifold M with a smooth action of a Lie group G, the factorization algebra corresponding to a G-equivariant field theory on M is smoothly G-equivariant.

5.12.2. Similarly, we can talk about holomorphically translation-invariant classical and quantum field theories on $\mathbb{C}^n$. In this context, we will take our space of fields to be $\Omega^{0,*}(\mathbb{C}^n, V)$, where V is some translation-invariant holomorphic vector bundle on $\mathbb{C}^n$. The pairing must arise from a translation-invariant map of holomorphic vector bundles

$$V \otimes V \to K_{\mathbb{C}^n}$$

of cohomological degree $n-1$, where $K_{\mathbb{C}^n}$ denotes the canonical bundle. This means that the composed map

$$\Omega_c^{0,*}(\mathbb{C}^n, V)^{\otimes 2} \to \Omega_c^{0,*}(\mathbb{C}^n, K_{\mathbb{C}^n}) \xrightarrow{\int} \mathbb{C}$$

is of cohomological degree -1.

Let

$$\eta_i = \frac{\partial}{\partial \overline{z}_i} \vee - : \Omega^{0,k}(\mathbb{C}^n, V) \to \Omega^{0,k-1}(\mathbb{C}^n, V)$$

be the contraction operator. The cohomological differential operator Q on $\Omega^{0,*}(V)$ must be of the form

$$Q = \overline{\partial} + Q_0$$

where Q_0 is translation-invariant and

$$[Q_0, \eta_i] = 0,$$

for $i = 1, \ldots, n$. Of course, Q must be skew self-adjoint with respect to the pairing on $\Omega_c^{0,*}(\mathbb{C}^n, V)$.

The other piece of data of a classical field theory is the local action functional $I \in \mathscr{O}_{loc}(\Omega^{0,*}(\mathbb{C}^n, V))$. We assume that I is translation-invariant, of course, but also that

$$\eta_i I = 0,$$

where the linear map η_i on $\Omega^{0,*}(\mathbb{C}^n, V)$ is extended in the natural way to a derivation of the algebra $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$ preserving the subspace of local functionals.

We then take our gauge fixing operator to be

$$\overline{\partial}^* = \sum \eta_i \frac{\partial}{\partial \overline{z}_i}.$$

Since $[\eta_i, Q_0] = 0$, we see that $[Q, \overline{\partial}^*] = [\overline{\partial}, \overline{\partial}^*]$ is the Laplacian. (More generally, we can consider a family of gauge fixing operators coming from the $\overline{\partial}^*$ operator for a family of flat Hermitian metrics on $\mathbb{C}^n$. Since the space of such metrics is $GL(n, \mathbb{C})/U(n)$ and thus contractible, we see that everything is independent up to homotopy of the choice of gauge fixing operator.)

We say a translation-invariant parametrix

$$\Phi \in \overline{\Omega}^{0,*}(\mathbb{C}^n, V)^{\otimes 2}$$

is *holomorphically translation-invariant* if

$$(\eta_i \otimes 1 + 1 \otimes \eta_i)\Phi = 0$$

for $i = 1, \dots, n$. For example, if $f \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ is a smooth function with proper support which is 1 near the diagonal, then

$$\Phi = f \int_0^L (\overline{\partial}^* \otimes 1) K_t \mathrm{d}t$$

defines such a parametrix. Clearly, we can find holomorphically translation-invariant parametrices which are supported arbitrarily close to the diagonal. This means that we can define a field theory by only considering $I[\Phi]$ for holomorphically translation-invariant parametrices Φ.

5.12.2.1 Definition. *A holomorphically translation-invariant quantization of a holomorphically translation-invariant classical field theory as above is a translation-invariant quantization such that for each holomorphically translation-invariant parametrix Φ, the effective interaction $I[\Phi]$ satisfies*

$$\eta_i I[\Phi] = 0$$

for $i = 1, \dots, n$. Here η_i abusively denotes the natural extension of the contraction η_i to a derivation on $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$.

5.12.2.2 Proposition. *A holomorphically translation-invariant quantum field theory on $\mathbb{C}^n$ leads to a holomorphically translation-invariant factorization algebra.*

PROOF. This follows immediately from proposition 5.12.1.2. Indeed, quantum observables form a smoothly translation-invariant factorization algebra. Such an observable O is specified by a family $O[\Phi] \in \mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ of functionals defined for each holomorphically translation-invariant parametrix Φ. The operators $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial \bar{z}_i}$, η_i act in a natural way on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ by derivations, and each commutes with the renormalization group flow W_Ψ^Φ for holomorphically translation-invariant parametrices Ψ, Φ. Thus, $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial \bar{z}_i}$ and η_i define derivations of the factorization algebra Obs^q. Explicitly, if $O \in \mathrm{Obs}^q(U)$ is an observable, then for each holomorphically translation-invariant parametrix Φ,

$$\left(\frac{\partial}{\partial z_i} O\right)[\Phi] = \frac{\partial}{\partial z_i}(O[\Phi]),$$

and similarly for $\frac{\partial}{\partial \bar{z}_i}$ and η_i.

By definition (Definition 2.7.6.1), a holomorphically translation-invariant factorization algebra is a translation-invariant factorization algebra where the derivation operator $\frac{\partial}{\partial \bar{z}_i}$ on observables is homotopically trivialized.

Note that, for a holomorphically translation-invariant parametrix Φ, $[\eta_i, \triangle_\Phi] = 0$ and η_i is a derivation for the Poisson bracket $\{-,-\}_\Phi$. It follows that

$$[Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi, \eta_i] = [Q, \eta_i]$$

as operators on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$. Since we wrote $Q = \bar{\partial} + Q_0$ and required that $[Q_0, \eta_i] = 0$, we have

$$[Q, \eta_i] = [\bar{\partial}, \eta_i] = \frac{\partial}{\partial \bar{z}_i}.$$

Since the differential on $\mathrm{Obs}^q(U)$ is defined by

$$(\widehat{Q}O)[\Phi] = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar \triangle_\Phi O[\Phi],$$

we see that $[\widehat{Q}, \eta_i] = \frac{\partial}{\partial \bar{z}_i}$, as desired. □

5.13. Cotangent theories and volume forms

In this section we will examine the case of a cotangent theory, in which our definition of a quantization of a classical field theory acquires a particularly nice interpretation. Suppose that $\mathcal{L}$ is an elliptic L_∞ algebra on a manifold M describing an elliptic moduli problem, which we denote by $B\mathcal{L}$. As we explained in Chapter 4, section 4.12, we can construct a classical field theory from $\mathcal{L}$, whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$. The main observation of this section is that a quantization of this classical field theory can be interpreted as a kind of "volume form" on the elliptic moduli problem $B\mathcal{L}$. This point of view was developed in [Cos11b], and used in [Cos11a] to provide a geometric interpretation of the Witten genus.

5.13.1. A finite dimensional model.

We first need to explain an algebraic interpretation of a volume form in finite dimensions. Let X be a manifold (or complex manifold or smooth algebraic variety; nothing we will say will depend on which geometric category we work in). Let $\mathscr{O}(X)$ denote the smooth functions on X, and let $\mathrm{Vect}(X)$ denote the vector fields on X.

If ω is a volume form on X, then it gives a divergence map

$$\mathrm{Div}_\omega : \mathrm{Vect}(X) \to \mathscr{O}(X)$$

defined via the Lie derivative:

$$\mathrm{Div}_\omega(V)\omega = \mathcal{L}_V \omega$$

for $V \in \mathrm{Vect}(X)$. Note that the divergence operator Div_ω satisfies the equations

$$(\dagger) \qquad \begin{aligned} \mathrm{Div}_\omega(fV) &= f\,\mathrm{Div}_\omega V + V(f). \\ \mathrm{Div}_\omega([V,W]) &= V\,\mathrm{Div}_\omega W - W\,\mathrm{Div}_\omega V. \end{aligned}$$

The volume form ω is determined up to a constant by the divergence operator Div_ω.

Conversely, to give an operator $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ satisfying equations ($\dagger$) is the same as to give a flat connection on the canonical bundle K_X of X, or, equivalently, to give a right D-module structure on the structure sheaf $\mathscr{O}(X)$.

5.13.1.1 Definition. *A* projective volume form *on a space X is an operator* $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ *satisfying equations ($\dagger$).*

The advantage of this definition is that it makes sense in many contexts where more standard definitions of a volume form are hard to define. For example, if A is a quasi-free differential graded commutative algebra, then we can define a projective volume form on the dg scheme $\operatorname{Spec} A$ to be a cochain map $\operatorname{Der}(A) \to A$ satisfying equations (†). Similarly, if $\mathfrak{g}$ is a dg Lie or L_∞ algebra, then a projective volume form on the formal moduli problem $B\mathfrak{g}$ is a cochain map $C^*(\mathfrak{g}, \mathfrak{g}[1]) \to C^*(\mathfrak{g})$ satisfying equations (†).

5.13.2. There is a generalization of this notion that we will use where, instead of vector fields, we take any Lie algebroid.

5.13.2.1 Definition. *Let A be a commutative differential graded algebra over a base ring k. A Lie algebroid L over A is a dg A-module with the following extra data.*

(1) A Lie bracket on L making it into a dg Lie algebra over k. This Lie bracket will be typically not A-linear.
(2) A homomorphism of dg Lie algebras $\alpha : L \to \operatorname{Der}^*(A)$, called the anchor map.
(3) These structures are related by the Leibniz rule

$$[l_1, f l_2] = (\alpha(l_1)(f))\, l_2 + (-1)^{|l_1||f|} f [l_1, l_2]$$

for $f \in A$, $l_i \in L$.

In general, we should think of L as providing the derived version of a foliation: an ordinary foliation consists of an ordinary commutative algebra A with a projective A-module L and an injective anchor map.

5.13.2.2 Definition. *If A is a commutative dg algebra and L is a Lie algebroid over A, then an L-projective volume form on A is a cochain map*

$$\operatorname{Div} : L \to A$$

satisfying

$$\operatorname{Div}(al) = a \operatorname{Div} l + (-1)^{|l||a|} \alpha(l) a.$$
$$\operatorname{Div}([l_1, l_2]) = l_1 \operatorname{Div} l_2 - (-1)^{|l_1||l_2| l_2} \operatorname{Div} l_1.$$

Of course, if the anchor map is an isomorphism, then this structure is the same as a projective volume form on A. In the more general case, we should think of an L-projective volume form as giving a projective volume form on the leaves of the derived foliation.

5.13.3. Let us explain how this definition relates to the notion of quantization of P_0 algebras.

5.13.3.1 Definition. *Give the operad P_0 a $\mathbb{C}^\times$ action where the product has weight 0 and the Poisson bracket has weight 1. A graded P_0 algebra is a $\mathbb{C}^\times$-equivariant differential graded algebra over this dg operad.*

Note that, if X is a manifold, $\mathscr{O}(T^*[-1]X)$ has the structure of graded P_0 algebra, where the $\mathbb{C}^\times$ action on $\mathscr{O}(T^*[-1]X)$ is given by rescaling the cotangent fibers.

Similarly, if L is a Lie algebroid over a commutative dg algebra A, then $\operatorname{Sym}_A^* L[1]$ is a $\mathbb{C}^\times$-equivariant P_0 algebra. The P_0 bracket is defined by the bracket on L and the L-action on A; the $\mathbb{C}^\times$ action gives $\operatorname{Sym}^k L[1]$ weight $-k$.

5.13.3.2 Definition. *Give the operad BD over $\mathbb{C}[[\hbar]]$ a $\mathbb{C}^\times$ action, covering the $\mathbb{C}^\times$ action on $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, the product has weight 0, and the Poisson bracket has weight 1.*

Note that this $\mathbb{C}^\times$ action respects the differential on the operad BD, which is defined on generators by

$$\mathrm{d}(- * -) = \hbar\{-,-\}.$$

Note also that by describing the operad BD as a $\mathbb{C}^\times$-equivariant family of operads over $\mathbb{A}^1$, we have presented BD as a filtered operad whose associated graded operad is P_0.

5.13.3.3 Definition. *A filtered BD algebra is a BD algebra A with a $\mathbb{C}^\times$ action compatible with the $\mathbb{C}^\times$ action on the ground ring $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, and compatible with the $\mathbb{C}^\times$ action on BD.*

5.13.3.4 Lemma. *If L is Lie algebroid over a dg commutative algebra A, then every L-projective volume form yields a filtered BD algebra structure on $\operatorname{Sym}_A^*(L[1])[[\hbar]]$, quantizing the graded P_0 algebra $\operatorname{Sym}_A^*(L[1])$.*

PROOF. If $\mathrm{Div} : L \to A$ is an L-projective volume form, then it extends uniquely to an order two differential operator $\triangle$ on $\mathrm{Sym}^*_A(L[1])$ which maps

$$\mathrm{Sym}^i_A(L[1]) \to \mathrm{Sym}^{i-1}_A(L[1]).$$

Then $\mathrm{Sym}^*_A L[1][[\hbar]]$, with differential $\mathrm{d} + \hbar\triangle$, gives the desired filtered BD algebra.

□

5.13.4. Let $B\mathcal{L}$ be an elliptic moduli problem on a compact manifold M. The main result of this section is that there exists a special kind of quantization of the cotangent field theory for $B\mathcal{L}$ that gives a projective volume on this formal moduli problem $B\mathcal{L}$. Projective volume forms arising in this way have a special "locality" property, reflecting the locality appearing in our definition of a field theory.

Thus, let $\mathcal{L}$ be an elliptic L_∞ algebra on M. This gives rise to a classical field theory whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$, as described in Chapter 4, section 4.12. Let us give the space $\mathscr{E}$ a $\mathbb{C}^\times$-action where $\mathcal{L}[1]$ has weight 0 and $\mathcal{L}^![-1]$ has weight 1. This induces a $\mathbb{C}^\times$ action on all associated spaces, such as $\mathscr{O}(\mathscr{E})$ and $\mathscr{O}_{loc}(\mathscr{E})$.

This $\mathbb{C}^\times$ action preserves the differential $Q + \{I, -\}$ on $\mathscr{O}(\mathscr{E})$, as well as the commutative product. Recall (Chapter 4, section 4.13) that the subspace

$$\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E}) \subset \mathscr{O}(\mathscr{E})$$

of functionals with smooth first derivative has a Poisson bracket of cohomological degree 1, making it into a P_0 algebra. This Poisson bracket is of weight 1 with respect to the $\mathbb{C}^\times$ action on $\widetilde{\mathrm{Obs}}^{cl}(M)$, so $\widetilde{\mathrm{Obs}}^{cl}(M)$ is a graded P_0 algebra.

We are interested in quantizations of our field theory where the BD algebra $\mathrm{Obs}^q_\Phi(M)$ of (global) quantum observables (defined using a parametrix Φ) is a filtered BD algebra.

5.13.4.1 Definition. *A cotangent quantization of a cotangent theory is a quantization, given by effective interaction functionals $I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Φ, such that $I[\Phi]$ is of weight -1 under the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ of functionals.*

This $\mathbb{C}^\times$ action gives $\hbar$ weight -1. Thus, this condition means that if we expand

$$I[\Phi] = \sum \hbar^i I_i[\Phi],$$

then the functionals $I_i[\Phi]$ are of weight $i-1$.

Since the fields $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ decompose into spaces of weights 0 and 1 under the $\mathbb{C}^\times$ action, we see that $I_0[\Phi]$ is linear as a function of $\mathcal{L}^![-2]$, that $I_1[\Phi]$ is a function only of $\mathcal{L}[1]$, and that $I_i[\Phi] = 0$ for $i > 1$.

Remark:

(1) The quantization $\{I[\Phi]\}$ is a cotangent quantization if and only if the differential $Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi$ preserves the $\mathbb{C}^\times$ action on the space $\mathscr{O}(\mathscr{E})[[\hbar]]$ of functionals. Thus, $\{I[\Phi]\}$ is a cotangent quantization if and only if the BD algebra $\mathrm{Obs}^q_\Phi(M)$ is a filtered BD algebra for each parametrix Φ.
(2) The condition that $I_0[\Phi]$ is of weight -1 is automatic.
(3) It is easy to see that the renormalization group flow

$$W\left(P(\Phi) - P(\Psi), -\right)$$

commutes with the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$.

5.13.5. Let us now explain the volume-form interpretation of cotangent quantization. Let $\mathcal{L}$ be an elliptic L_∞ algebra on M, and let $\mathscr{O}(B\mathcal{L}) = C^*(\mathcal{L})$ be the Chevalley-Eilenberg cochain complex of M. The cochain complexes $\mathscr{O}(B\mathcal{L}(U))$ for open subsets $U \subset M$ define a commutative factorization algebra on M.

As we have seen in Chapter 4, section 4.2, we should interpret modules for an L_∞ algebra $\mathfrak{g}$ as sheaves on the formal moduli problem $B\mathfrak{g}$. The $\mathfrak{g}$-module $\mathfrak{g}[1]$ corresponds to the tangent bundle of $B\mathfrak{g}$, and so vector fields on $\mathfrak{g}$ correspond to the $\mathscr{O}(B\mathfrak{g})$-module $C^*(\mathfrak{g}, \mathfrak{g}[1])$.

Thus, we use the notation

$$\mathrm{Vect}(B\mathcal{L}) = C^*(\mathcal{L}, \mathcal{L}[1]);$$

this is a dg Lie algebra and acts on $C^*(\mathcal{L})$ by derivations (see Appendix A, section A.8, for details).

For any open subset $U \subset M$, the $\mathcal{L}(U)$-module $\mathcal{L}(U)[1]$ has a sub-module $\mathcal{L}_c(U)[1]$ given by compactly supported elements of $\mathcal{L}(U)[1]$. Thus, we have a sub-$\mathscr{O}(B\mathcal{L}(U))$-module

$$\mathrm{Vect}_c(B\mathcal{L}(U)) = C^*(\mathcal{L}(U), \mathcal{L}_c(U)[1]) \subset \mathrm{Vect}(B\mathcal{L}(U)).$$

This is in fact a sub-dg Lie algebra, and hence a Lie algebroid over the dg commutative algebra $\mathscr{O}(B\mathcal{L}(U))$. Thus, we should view the subspace $\mathcal{L}_c(U)[1] \subset \mathcal{L}_c(U)[1]$ as providing a foliation of the formal moduli problem $B\mathcal{L}(U)$, where two points of $B\mathcal{L}(U)$ are in the same leaf if they coincide outside a compact subset of U.

If $U \subset V$ are open subsets of M, there is a restriction map of L_∞ algebras $\mathcal{L}(V) \to \mathcal{L}(U)$. The natural extension map $\mathcal{L}_c(U)[1] \to \mathcal{L}_c(V)[1]$ is a map of $\mathcal{L}(V)$-modules. Thus, by taking cochains, we find a map

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \to \mathrm{Vect}_c(B\mathcal{L}(V)).$$

Geometrically, we should think of this map as follows. If we have an R-point α of $B\mathcal{L}(V)$ for some dg Artinian ring R, then any compactly-supported deformation of the restriction $\alpha \mid_U$ of α to U extends to a compactly supported deformation of α.

We want to say that a cotangent quantization of $\mathcal{L}$ leads to a "local" projective volume form on the formal moduli problem $B\mathcal{L}(M)$ if M is compact. If M is compact, then $\mathrm{Vect}_c(B\mathcal{L}(M))$ coincides with $\mathrm{Vect}(B\mathcal{L}(M))$. A local projective volume form on $B\mathcal{L}(M)$ should be something like a divergence operator

$$\mathrm{Div} : \mathrm{Vect}(B\mathcal{L}(M)) \to \mathscr{O}(B\mathcal{L}(M))$$

satisfying the equations (†), with the locality property that Div maps the subspace

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \subset \mathrm{Vect}(B\mathcal{L}(M))$$

to the subspace $\mathscr{O}(B\mathcal{L}(U)) \subset \mathscr{O}(B\mathcal{L}(M))$.

Note that a projective volume form for the Lie algebroid $\mathrm{Vect}_c(B\mathcal{L}(U))$ over $\mathscr{O}(B\mathcal{L}(U))$ is a projective volume form on the leaves of the foliation of $B\mathcal{L}(U)$ given by $\mathrm{Vect}_c(B\mathcal{L}(U))$. The leaf space for this foliation is described by the L_∞ algebra

$$\mathcal{L}_\infty(U) = \mathcal{L}(U)/\mathcal{L}_c(U) = \operatorname*{colim}_{K\subset U} \mathcal{L}(U \setminus K).$$

(Here the colimit is taken over all compact subsets of U.) Consider the one-point compactification U_∞ of U. Then the formal moduli problem $\mathcal{L}_\infty(U)$ describes the germs at ∞ on U_∞ of sections of the sheaf on U of formal moduli problems given by $\mathcal{L}$.

Thus, the structure we're looking for is a projective volume form on the fibers of the maps $B\mathcal{L}(U) \to B\mathcal{L}_\infty(U)$ for every open subset $U \subset M$, where the divergence operators describing these projective volume forms are all compatible in the sense described above.

What we actually find is something a little weaker. To state the result, recall (section 5.7) that we use the notation $\mathscr{P}$ for the contractible simplicial set of parametrices, and $\mathscr{C}\mathscr{P}$ for the cone on $\mathscr{P}$. The vertex of the cone $\mathscr{C}\mathscr{P}$ will denoted $\bar{0}$.

5.13.5.1 Theorem. *A cotangent quantization of the cotangent theory associated to the elliptic L_∞ algebra $\mathcal{L}$ leads to the following data.*

(1) *A commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$. The underlying graded algebra of this commutative dg algebra is $\mathscr{O}(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. The restriction of this commutative dg algebra to the vertex $\bar{0}$ of $\mathscr{C}\mathscr{P}$ is the commutative dg algebra $\mathscr{O}(B\mathcal{L})$.*
(2) *A dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$, whose underlying graded $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$-module is $\mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. At the vertex $\bar{0}$ of $\mathscr{C}\mathscr{P}$, the dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ coincides with the dg Lie algebroid $\mathrm{Vect}_c(B\mathcal{L})$.*
(3) *We let $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ be the restrictions of $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$. Then we have a divergence operator*

$$\mathrm{Div}_{\mathscr{P}} : \mathscr{O}_{\mathscr{P}}(B\mathcal{L}) \to \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$$

defining the structure of a $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ projective volume form on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.

Further, when restricted to the sub-simplicial set $\mathscr{P}_U \subset \mathscr{P}$ of parametrices with support in a small neighborhood of the diagonal $U \subset M \times M$, all structures increase support by an arbitrarily small amount (more precisely, by an amount linear in U, in the sense explained in section 5.7).

PROOF. This follows almost immediately from theorem 5.7.2.1. Indeed, because we have a cotangent theory, we have a filtered BD algebra

$$\mathrm{Obs}^q_{\mathscr{P}}(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P}), \widehat{Q}_{\mathscr{P}}, \{-,-\}_{\mathscr{P}} \right).$$

Let us consider the sub-BD algebra $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$, which, as a graded vector space, is $\mathscr{O}_{sm}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P})$ (as usual, $\mathscr{O}_{sm}(\mathscr{E})$ indicates the space of functionals with smooth first derivative).

Because we have a filtered BD algebra, there is a $\mathbb{C}^\times$-action on this complex $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$. We let

$$\mathscr{O}_{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^0$$

be the weight 0 subspace. This is a commutative differential graded algebra over $\Omega^*(\mathscr{P})$, whose underlying graded algebra is $\mathscr{O}(B\mathcal{L})$; further, it extends (using again the results of 5.7.2.1) to a commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$, which when restricted to the vertex is $\mathscr{O}(B\mathcal{L})$.

Next, consider the weight -1 subspace. As a graded vector space, this is

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{P}) \oplus \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

We thus let

$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} / \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The Poisson bracket on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$ is of weight 1, and it makes the space $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ into a sub Lie algebra.

We have a natural decomposition of graded vector spaces

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \oplus \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The dg Lie algebra structure on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ gives us

(1) The structure of a dg Lie algebra on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ (as the quotient of $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ by the differential Lie algebra ideal $\hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L})$).
(2) An action of $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ by derivations; this defines the anchor map for the Lie algebroid structure on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.
(3) A cochain map
$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \hbar \mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
This defines the divergence operator.

It is easy to verify from the construction of theorem 5.7.2.1 that all the desired properties hold. □

5.13.6. The general results about quantization of [**Cos11c**] thus apply to this situation, to show that the following.

5.13.6.1 Theorem. *Consider the cotangent theory $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ to an elliptic moduli problem described by an elliptic L_∞ algebra $\mathcal{L}$ on a manifold M.*

The obstruction to constructing a cotangent quantization is an element in

$$H^1(\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^\times}) = H^1(\mathscr{O}_{loc}(B\mathcal{L})).$$

If this obstruction vanishes, then the simplicial set of cotangent quantizations is a torsor for the simplicial Abelian group arising from the cochain complex $\mathscr{O}_{loc}(B\mathcal{L})$ by the Dold-Kan correspondence.

As in Chapter 4, section 4.7, we are using the notation $\mathscr{O}_{loc}(B\mathcal{L})$ to refer to a "local" Chevalley-Eilenberg cochain for the elliptic L_∞ algebra $\mathcal{L}$. If L is the vector bundle whose sections are $\mathcal{L}$, then as we explained in [**Cos11c**], the jet bundle $J(L)$ is a D_M L_∞ algebra and

$$\mathscr{O}_{loc}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

There is a de Rham differential (see section 4.9) mapping $\mathscr{O}_{loc}(B\mathcal{L})$ to the complex of local 1-forms,

$$\Omega^1_{loc}(B\mathcal{L}) = C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

The de Rham differential maps $\mathscr{O}_{loc}(B\mathcal{L})$ isomorphically to the subcomplex of $\Omega^1_{loc}B\mathcal{L})$ of closed local one-forms.

Thus, we should view the obstruction to quantizing the cotangent theory, which is a class in $H^1(\Omega^1_{loc}(B\mathcal{L})_{cl})$, as being the local version of the first Chern class of the canonical bundle of $\mathcal{L}$.

APPENDIX A

Homological algebra with differentiable vector spaces

The factorization algebras we consider take values in vector spaces of an analytical nature, like the space of smooth functions on a manifold. We would thus like to perform homological algebra in this setting. The standard approach to working with objects of this nature is to treat them as topological vector spaces. However, it is not completely obvious how one should set up homological algebra when using topological vector spaces. It is also not straightforward to construct the topology on vector spaces which appear in our most important examples of factorization algebras: the observables of a quantum field theory.

Thus we will work with a weaker and more flexible concept, that of *differentaible vector space*. This Appendix develops homological algebra in the category of differentiable vector spaces. Related approaches to functional analysis are developed in [**Pau10**] and in [**KM97**].

A.1. Diffeological vector spaces

Let us remind the reader of the concept of diffeological space [**?**].

A.1.0.2 Definition. *The site of smooth manifolds is the site whose objects are smooth manifolds, morphisms are smooth maps, and where a map $M \to N$ is an open covering if it is a surjective local diffeomorphism.*

A diffeological space *X is a sheaf of sets on the site of smooth manifolds with the property that, for all smooth manifolds M, the map*

$$X(M) \to \mathrm{Hom}_{Sets}(M, X(*))$$

is injective. (On the right hand side, $X(*)$ is the value of X on a point.)

A map of diffeological spaces is a map of sheaves of sets on the smooth site.

We will sometimes refer to maps of diffeological spaces as smooth maps, to distinguish them from maps of the underlying sets.

We can rewrite the axioms of a diffeological space in more explicit terms, as follows. A diffeological space is determined by a set $X = X(*)$, and for each smooth manifold M, a subset $X(M) \subset \mathrm{Maps}(M, X)$ of *smooth maps* from M to X. These subsets must satisfy the following conditions. If $f : M \to X$ is a smooth map and if $g : N \to M$ is a smooth map of ordinary manifolds, then $f \circ g : N \to X$ is a smooth map. Further, a map $f : M \to X$ is smooth if and only if it is smooth locally on M. Finally, all constant maps to X are smooth. We call this collection of smooth maps the diffeology of X.

Note that if X, Y are diffeological spaces, then so is $X \times Y$: a map $M \to X \times Y$ is smooth if the composition with both projection maps is smooth.

A.1.0.3 Definition. *A diffeological vector space is a vector space V together with a diffeology compatible with the vector space structure. Thus, the sum map $V \times V \to V$ and the scalar multiplication map $\mathbb{R} \times V \to V$ are maps of diffeological spaces (where $\mathbb{R}$ is given the standard diffeology).*

A diffeological vector space V has enough structure to talk about smooth maps from a manifold M to V. We also want to be able to differentiate such maps. This requires extra structure.

We use the notation $C^\infty(M, V)$ to denote the $C^\infty(M)$-module of smooth maps from M to V. Thus, $C^\infty(M, V)$ is, as a vector space, just $V(M)$, equipped with the natural structure of module over the (discrete) algebra $C^\infty(M)$.

Similarly, we let

$$\Omega^k(M, V) = \Omega^k(M) \otimes_{C^\infty(M)} C^\infty(M, V)$$

denote the space of k-forms with values in V. This is just the *algebraic* tensor product. This is a reasonable thing to do because $\Omega^k(M)$ is a finitely-generated projective $C^\infty(M)$ module: it is a direct summand of a free finite rank $C^\infty(M)$-module. This implies that $\Omega^k(M, V)$ is a direct summand of $C^\infty(M, V)^{\oplus l}$ for some l.

A.1.0.4 Definition. *A differentiable vector space is a diffeological vector space together with, for each smooth manifold M, a flat connection*

$$\nabla_{M,V} : C^\infty(M,V) \to \Omega^1(M,V)$$

such that, for all smooth maps $f : N \to M$,

$$f^*\nabla_{M,V} = \nabla_{N,V}.$$

To say that $\nabla_{M,V}$ is a flat connection means, of course, that it satisfies the Leibniz rule,

$$\nabla_{M,V}(f \cdot s) = (\mathrm{d}f)s + f\nabla_{M,V}s,$$

and that the curvature

$$F(\nabla_{M,V}) = (\nabla_{M,V})^2 : C^\infty(M,V) \to \Omega^2(M,V)$$

vanishes.

The flat connection $\nabla_{M,V}$ allows us to differentiate smooth maps $M \to V$. If $f : M \to V$ is a smooth map and if $X \in \mathrm{Vect}(M)$ is a vector field on M, we define

$$X(f) = \langle X, \nabla_{M,V} f\rangle \in C^\infty(M,V),$$

where $\langle -,-\rangle$ indicates the $C^\infty(M)$-linear pairing

$$\mathrm{Vect}(M) \times \Omega^1(M,V) \to C^\infty(M,V).$$

Differentiable vector spaces form a category that we denote DVS. An object is a differentiable vector space V. A morphism $\phi : V \to W$ is a linear map such that for every smooth map $f : M \to V$, the map $\phi \circ f$ is smooth, and which is compatible with connections in the sense that, for all smooth manifolds M,

$$\phi \circ \nabla_{M,V} = \nabla_{M,V} \circ \phi.$$

We will often refer to morphisms of differentiable vector spaces as smooth linear maps.

Differentiable vector spaces appear naturally in geometry. In section A.2 below, we show that for M a manifold and E is a vector bundle on M, the space $C^\infty(N,E)$ of smooth sections of E has a natural structure of differentiable vector space. The same holds for the space of compactly supported or distributional sections of E. Most of our examples of differentiable vector spaces arise in this way.

Below, we will examine various properties and examples of differentiable vector spaces. Many of these constructions work equally well for diffeological vector spaces.

Remark: One of the main reasons we consider differentiable vector spaces instead of topological vector spaces is that homological algebra for sheaves of vector spaces on a site is relatively standard, whereas homological algebra for topological vector spaces is trickier.

Another reason is that, when we consider our construction of factorization algebras in families, we will only use families where the base is a smooth manifold. Differentiable vector spaces have just enough structure to talk about such families.

A.1.1. Limits and colimits. Let V be a differentiable vector space, and let $i : W \subset V$ be a sub-vector space. The *subspace diffeology* on W is defined by saying that a map $f : M \to W$ is smooth if the composed map $i \circ f : M \to V$ is smooth. We say that the diffeological subspace W is a *differentiable subspace* if, for all smooth manifolds M, the connection $\nabla_{M,V}$ maps $C^\infty(M, W) \subset C^\infty(M, V)$ to $\Omega^1(M, W) \subset \Omega^1(M, V)$.

Differentiable subspaces have the usual universal property: if A is another differentiable vector space, a linear map $A \to W$ is smooth if and only if the composed map $A \to V$ is.

If $W \subset V$ is a differentiable subspace, then we can form the quotient V/W. A map from M to V/W is smooth if, locally on M, it lifts to a smooth map to V. The connection on V/W is uniquely determined by the requirement that the map $V \to V/W$ is compatible with connections (and so a map of differentiable spaces). We call V/W a differentiable quotient of V.

Again, this has the usual universal property: if A is another differentiable space, a linear map $V/W \to A$ is smooth if and only if the composed map $V \to A$ is smooth.

A.1.1.1 Lemma. *The category of differentiable spaces admits all products and coproducts.*

These can be described explicitly as follows. Let $\{V_i \mid i \in I\}$ be some family of differentiable spaces indexed by a set I. The product $\prod_i V_i$ differentiable space has, as underlying vector space, the product vector space $\prod V_i$. A map $M \to \prod_i V_i$ is smooth if and only if the composed maps $M \to V_i$ are smooth for all i. The connection map

$$\nabla_{M,\prod V_i} : C^\infty(M, \prod V_i) = \prod C^\infty(M, V_i) \to \Omega^1(M, \prod V_i) = \prod \Omega^1(M, V_i)$$

is the product of the connections ∇_{M,V_i}.

Similarly, the differentiable coproduct of the V_i *has, as underlying vector space, the ordinary direct sum* $\oplus V_i$. *A map* $f : M \to \oplus V_i$ *is smooth if, locally on* M, f *can be written as a finite sum of smooth maps to some* $V_{i_1}, \ldots, V_{i_k}$. *The connection*

$$\nabla_{M,\oplus V_i} : C^\infty(M, \oplus V_i) \to \Omega^1(M, \oplus V_i)$$

is the unique connection which restricts to ∇_{M,V_i} *on the subspace* $C^\infty(M, V_i)$.

PROOF. We need to verify that the product and coproduct as described above have the desired universal properties. For the product, this is immediate. Let's verify it for the coproduct. Let A be another differentiable vector space. Let $f : \oplus V_i \to A$ be a linear map. Suppose that the maps $f_i : V_i \to A$ are all smooth. We need to show that f is smooth. Let $\phi : M \to \oplus V_i$ be a smooth map. To show that $f \circ \phi$ is smooth, it suffices to do so locally on M. Thus, we can assume that ϕ can be written as a finite sum of smooth maps $\phi_i : M \to V_i$. Then, $f \circ \phi$ is a finite sum of $f \circ \phi_i$, and by assumption, $f \circ \phi_i : M \to A$ are smooth. It is straightforward to verify that the fact that the maps f_i are compatible with the connections on V_i and A imply that f is compatible with connections. □

A.1.1.2 Corollary. *The category of differentiable vector spaces admits all limits (and so is complete).*

PROOF. Arbitrary limits are obtained from products and kernels.Thus, we need to verify that the category of differentiable vector spaces admits kernels.

Let $f : V \to W$ be a map of differentiable vector spaces. Let us consider the kernel $\operatorname{Ker} f \subset V$, just as an ordinary vector space. We say that a map $M \to \operatorname{Ker} f$ is smooth if and only if the composed map to V is smooth: this gives $\operatorname{Ker} f$ the subspace diffeology. Then, the sequence

$$0 \to C^\infty(M, \operatorname{Ker} f) \to C^\infty(M, V) \to C^\infty(M, W)$$

is exact.

We need to give $\operatorname{Ker} f$ a connection. Since the map $C^\infty(M, V) \to C^\infty(M, W)$ is compatible with connections, the connection $\nabla_{M,V}$ on $C^\infty(M, V)$ must map $C^\infty(M, \operatorname{Ker} f)$ to $\Omega^1(M, \operatorname{Ker} f)$.

It is easy to verify that $\operatorname{Ker} f$ satisfies the universal property of a kernel. □

Note that the forgetful functor DVS $\to$ Vect preserves all limits.

A.1.2. Cokernels and exact sequences. The category of differentiable spaces *does not* admit all cokernels. Here is the prime example. Let W be a differentiable space, and let $V \subset W$ be an arbitrary linear subspace. We equip V with the initial diffeology, by saying that the space of smooth maps $M \to V$ is the algebraic tensor product $C^\infty(M) \otimes_{alg} V$, rather than the subspace diffeology. Suppose these two diffeologies differ. The quotient W/V has a natural diffeology, by saying that a map $M \to W/V$ is smooth if locally it lifts to a smooth map to W. The fact that the sequence

$$C^\infty(M) \otimes_{alg} V = C^\infty(M, V) \to C^\infty(M, W) \to C^\infty(M, W/V) \to 0$$

is not exact means that the connection on $C^\infty(M, W)$ need not descend to one on $C^\infty(M, V/W)$.

Happily, this example is the only way that things can go wrong.

A.1.2.1 Definition. *A map $f : V \to W$ of differentiable spaces is* admissible *if, for all manifolds M and all maps $\phi : M \to \operatorname{Im} f$, the composed map $M \to W$ is smooth if and only if ϕ lifts locally to a smooth map to V.*

In other words, f is admissible if the two natural pre-diffeologies on $\operatorname{Im} f$ *(where we view it as a quotient of V or a subspace of W) coincide.*

A.1.2.2 Lemma. *Cokernels of admissible maps exist.*

PROOF. If $f : V \to W$ is an admissible map, we give $\operatorname{Coker} f$ the quotient diffeology: a map $M \to \operatorname{Coker} f$ is smooth if locally it lifts to a smooth map to W. Let $C^\infty_M(V)$ denote the sheaf on M which sends $U \subset M$ to $C^\infty(U, V)$. Then, the sequence of sheaves

$$C^\infty_M(V) \to C^\infty_M(W) \to C^\infty_M(\operatorname{Coker} f) \to 0$$

is exact. This implies that the connection on $C^\infty(M, W)$ descends uniquely to one on $C^\infty(M, \operatorname{Coker} f)$. □

Another simple class of colimits that exist is the following.

A.1.2.3 Lemma. *The category of differentiable vector spaces is closed under taking sequential colimits of injective maps.*

By injective, we just mean that the map on the underlying vector space is injective.

PROOF. Let V_i for $i \in \mathbb{Z}_{\geq 0}$ be a sequence of differentiable vector spaces, and let $f_{ij} : V_i \to V_j$ be injective maps with $f_{jk} f_{ij} = f_{ik}$. Let V denote the ordinary vector space

$$V = \operatorname{colim} V_i = \cup V_i.$$

We say a map from a smooth manifold M to V is smooth if, locally on M, it comes from a smooth map to one of the V_i. Let $C^\infty_M(V)$ denote the sheaf on M of smooth maps to V; then

$$C^\infty_M(V) = \operatorname{colim} C^\infty_M(V_i).$$

This identification uses the fact that the maps in our directed system are injective. Recall also that the colimit in the category of sheaves is defined to be the sheafification of the colimit in the category of presheaves.

Now, we define the flat connection $\nabla_{M,V}$ to be the map of sheaves

$$\nabla_{M,V} : C^\infty_M(V) \to \Omega^1_M(V)$$

which arises as the colimit of the maps of sheaves

$$C^\infty_M(V_i) \to \Omega^1_M(V_i).$$

□

A.1.2.4 Definition. *A sequence* $0 \to A \to B \to C \to 0$ *of differentiable spaces is exact if it is exact as a sequence of ordinary vector spaces,* $A \subset B$ *is a differentiable subspace, and* C *is a differentiable quotient.*

Equivalently, the sequence is exact if A *is the kernel of the map* $B \to C$ *and* C *is the cokernel of the map* $A \to B$.

Let V be a differentiable vector space. By evaluating V on open subsets of $\mathbb{R}^n$, V becomes a sheaf on $\mathbb{R}^n$. We can thus define the stalk

$$\operatorname{Stalk}_n(V) = \operatorname{colim}_{0 \in U \subset \mathbb{R}^n} V(U)$$

of V at the origin in $\mathbb{R}^n$. The colimit above is taken over open subsets of $\mathbb{R}^n$ containing the origin.

Note that the stalk of V at a point in any manifold can be defined in the same way, but the stalk at a point in a n-dimensional manifold is the same as the stalk at the origin in $\mathbb{R}^n$.

A.1.2.5 Lemma. *A sequence of differentiable vector spaces* $0 \to A \to B \to C \to 0$ *is exact if and only if, for all* n*, the sequence*

$$0 \to \mathrm{Stalk}_n(A) \to \mathrm{Stalk}_n(B) \to \mathrm{Stalk}_n(C) \to 0$$

of vector spaces is exact.

PROOF. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of differentiable vector spaces. Then, for any manifold M, the sequence $0 \to C^\infty(M, A) \to C^\infty(M, B) \to C^\infty(M, C)$ is exact. Further, a map $M \to C$ is smooth if locally on M it lifts to a smooth map to B. Thus, if $C^\infty_M(B)$ denotes the sheaf on M of smooth maps to B, the sequence

$$0 \to C^\infty_M(A) \to C^\infty_M(B) \to C^\infty_M(C) \to 0 \tag{†}$$

of sheaves on M is exact. This implies that the corresponding sequence on stalks is exact.

Conversely, if the sequence of stalks is exact for all n, then the sequence (†) of sheaves is exact for all manifolds M. This implies that the sequence

$$0 \to \Omega^1_M(A) \to \Omega^1_M(B) \to \Omega^1_M(C) \to 0$$

is also exact, where we define

$$\Omega^1_M(A) = \Omega^1_M \otimes_{C^\infty_M} C^\infty_M(A).$$

It follows that the connection $\nabla_{M,C}$ on C is the unique connection which descends from that on B, and that the connection on A is the restriction of the connection on B to A. These are the conditions we imposed for A to be a differentiable subspace of B and for C to be a differentiable quotient. □

A.1.3. The multicategory structure. We will consider the category of differentiable vector spaces as a multicategory, instead of a symmetric monoidal category.

A.1.3.1 Definition. *If $V_1, \dots, V_k, W$ are differentiable vector spaces, then a smooth multilinear map*

$$\phi : V_1 \times \cdots \times V_k \to W$$

is a multilinear map with the following properties.

(1) *It is smooth: if $f_i : M \to V_i$ are smooth maps from a manifold M, then*

$$\phi(f_1, \dots, f_k) : M \to W$$

is a smooth map.

(2) *It is compatible with flat connections: if $f_i : M \to V$ are smooth and if X is a vector field on M, then*

$$\nabla_X \phi(f_1, \dots, f_k) = \sum_i \phi(f_1, \dots, \nabla_X f_i, \dots, f_k).$$

Differentiable vector spaces form a multicategory where the multi-morphisms $\mathrm{Hom}(V_1, \dots V_k; W)$ *are smooth multilinear maps.*

A.1.4. In general, we can not tensor differentiable vector spaces together. Nonetheless, certain tensor products arise naturally and we will use them repeatedly.

First, we can tensor a differentiable vector space V with the algebra of smooth functions on a manifold: we use the notation

$$C^\infty(M) \otimes V = C^\infty(M, V)$$

when V is a differentiable vector space.

Similarly, if E is a vector bundle on M, then $C^\infty(M, E)$ is a projective module over $C^\infty(M)$. Thus, it's reasonable to form the algebraic tensor product

$$C^\infty(M, E) \otimes_{C^\infty(M)} C^\infty(M, V).$$

We interpret the output as a kind of completed tensor product of V with $C^\infty(M, E)$. We will use the notation $C^\infty(M, E \otimes V)$ to denote this tensor product. This notation is natural: we can tensor a differentiable vector space with a finite-dimensional vector space, so that $E \otimes V$ can be thought of as a bundle of differentiable vector spaces on M.

A.1.4.1 Lemma. *Let E, F be vector bundles on M. Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator. Let V be a differentiable vector space. Then, the map*

$$D \otimes 1 : C^\infty(M, E) \otimes_{alg} V \to C^\infty(M, F) \otimes_{alg} V$$

extends canonically to a map

$$C^\infty(M, E \otimes V) \to C^\infty(M, F \otimes V).$$

PROOF. Let's start with the case when E and F are trivial of rank 1. Then we are asserting that differential operators on M act naturally on $C^\infty(M, V)$. This action arises in the standard way from the connection $\nabla_{M,V} : C^\infty(M, V) \to \Omega^1(M, V)$ which we are given as part of the structure of a differentiable vector space.

In the case when E and F are non-trivial, one constructs the desired map in local trivializations and then verifies that it is independent of the choice of local trivializations. This is standard. □

Here is a formal way to describe these structures. Let $\mathcal{C}$ denote the following symmetric monoidal category: the objects are smooth manifolds M equipped with a vector bundle E; a morphism $(M, E) \to (N, F)$ is a smooth map $f : M \to N$ together with a differential operator $C^\infty(M, f^*F) \to C^\infty(M, E)$; and the tensor product is defined by

$$(M, E) \otimes (N, F) = (M \times N, E \boxtimes F).$$

Then, the category of differentiable vector spaces DVS is tensored over $\mathcal{C}^{op}$.

A.2. Differentiable vector spaces from sections of a vector bundle

In this section, we will describe various classes of sections of a vector bundle on a manifold M and show that each is equipped with a natural diffeology and flat connection. There are various ways to express this construction; we begin in a more geometric language and then rephrase in the language of topological vector spaces.

The most basic example is as follows. Let M be a manifold, and let E be a vector bundle on M. Then the space

$$\mathscr{E} = \Gamma(M, E)$$

of smooth sections of E is a differentiable vector space. We let

$$\mathscr{E}_c = \Gamma_c(M, E)$$

be the space of compactly supported smooth sections of E on M.

A.2.0.2 Definition. *Equip $\mathscr{E}$ with the diffeology where, for N a smooth manifold, a smooth map $f : N \to \mathscr{E}$ is a section of the pullback bundle $\pi_M^* E$ on $N \times M$ arising from the projection map $\pi_M : N \times M \to M$.*

Similarly, give $\mathscr{E}_c$ a diffeology by saying that a smooth map $N \to \mathscr{E}_c$ is a section s of $\pi_M^* E$ on $N \times M$ with the property that the map $\mathrm{Supp}(s) \to N$ is proper (where $\mathrm{Supp}(s)$ is the closure of the locus on which s is non-vanishing).

Note that the spaces $\mathscr{E}$ and $\mathscr{E}_c$ are complete locally-convex topological vector spaces, using the standard topologies for these spaces. As is explained in [**KM97**], for example, one has a notion of smooth map $N \to V$ for any manifold N and for any such topological vector space V. Thus, V defines a diffeological space. The diffeologies described above on $\mathscr{E}$ and $\mathscr{E}_c$ arise from the standard topologies on these spaces.

Notice as well that $C^\infty(N, \mathscr{E})$ is the vector space given by the completed projective tensor product $C^\infty(N) \otimes \mathscr{E}$, which is their natural tensor product as nuclear spaces.

Next, we explain the flat connections on the spaces $\mathscr{E}$ and $\mathscr{E}_c$.

A.2.0.3 Definition. *Let N be a smooth manifold. Equip the pullback bundle $\pi_M^* E$ on $N \times M$ with the natural flat connection along the fibers of the projection map $\pi_M : N \times M \to M$. We thus obtain a map*

$$\nabla_{N,\mathscr{E}} : \Gamma(N \times M, \pi_M^* E) \to \Gamma(N \times M, T^* N \boxtimes E)$$

or, equivalently, a map

$$\nabla_{N,\mathscr{E}} : C^\infty(N, \mathscr{E}) \to \Omega^1(N, \mathscr{E}).$$

This defines the flat connection on $C^\infty(N, \mathscr{E})$ and so gives $\mathscr{E}$ the structure of a differentiable vector space.

This flat connection preserves the subspace $C^\infty(N, \mathscr{E}_c)$, giving $\mathscr{E}_c$ the structure of a differentiable vector space.

A.2.1. We are also interested in distributional sections of a vector bundle E. Let $\mathcal{D}(M)$ denote the space of distributions on M, that is, the continuous dual of the space $C_c^\infty(M)$. Let $\mathcal{D}_c(M)$ denote the space of compactly supported distributions on M, which is the continuous dual of $C^\infty(M)$.

We let

$$\overline{\mathscr{E}}(M) = \mathscr{E}(M) \otimes_{C^\infty(M)} \mathcal{D}(M)$$

be the space of distributional sections of E. (These are sections whose coefficients are distributions rather than functions.) We let

$$\overline{\mathscr{E}}_c(M) = \mathscr{E}_c(M) \otimes_{C_c^\infty(M)} \mathcal{D}_c(M)$$

be the space of compactly supported distributional sections of E.

A.2.1.1 Definition. *Equip the space $\mathcal{D}(M)$ of distributions on M with the diffeology where a smooth map $N \to \mathcal{D}(M)$ is a continuous linear map $C_c^\infty(M) \to C^\infty(N)$. Similarly, give $\mathcal{D}_c(M)$ a diffeology by saying that a smooth map $N \to \mathcal{D}_c(M)$ is a continuous linear map $C^\infty(M) \to C^\infty(N)$.*

Equip $\overline{\mathscr{E}}(M)$ with the diffeology in which the vector space of smooth maps $N \to \overline{\mathscr{E}}(M)$ is

$$C^\infty(N, \overline{\mathscr{E}}(M)) = C^\infty(N, \mathscr{E}(M)) \otimes_{C^\infty(M\times N)} C^\infty(N, \mathcal{D}(M))$$

(where we use the notation $C^\infty(N, V)$ to indicate the space of smooth maps from N to a diffeological vector space V).

Similarly, give $\overline{\mathscr{E}}_c(M)$ a diffeology by saying that

$$C^\infty(N, \overline{\mathscr{E}}_c(M)) = C^\infty(N, \mathscr{E}_c(M)) \otimes_{C^\infty(N, C_c^\infty(M))} C^\infty(N, \mathcal{D}_c(M)).$$

Remark: The diffeologies we have defined on these spaces of distributions again arise from the standard topology on these spaces. In particular, note that

$$C^\infty(N, \mathcal{D}(M)) = C^\infty(N) \otimes \mathcal{D}(M)$$

as vector spaces, where $\otimes$ denotes the completed projective tensor product. Compare this to the analogous definition of $C^\infty(N, C^\infty(M))$ from earlier.

We need to equip these diffeological vector spaces with flat connections to make them into differentiable vector spaces. There is a natural choice.

A.2.1.2 Definition. *We extend the diffeological vector space $\mathcal{D}(M)$ to a differentiable vector space as follows. There is a natural inclusion*

$$C^\infty(N, \mathcal{D}(M)) \hookrightarrow \mathcal{D}(N \times M).$$

The Lie algebra $\mathrm{Vect}(N)$ *of vector fields on N acts naturally on $\mathcal{D}(N \times M)$; this action preserves the subspace $C^\infty(N, \mathcal{D}(M))$, giving this subspace the desired flat connection.*

The action of $\mathrm{Vect}(N)$ *also preserves the smaller subspace*

$$C^\infty(N, \mathcal{D}_c(M)) \subset \mathcal{D}(N \times M)$$

and so gives $C^\infty(N, \mathcal{D}_c(M))$ *the structure of differentiable vector space.*

Similarly, the space $\mathcal{D}(N \times M, \pi_M^* E)$ *of distributional sections on* $N \times M$ *of the vector bundle* $\pi_M^* E$ *has a natural action of vector fields on* M*. This preserves the subspaces* $C^\infty(N, \overline{E}(M))$ *and* $C^\infty(N, \overline{E}_c(M))$*, and gives those the structure of differentiable vector spaces.*

Let $E^!$ denote the vector bundle $E \otimes \mathrm{Dens}_M$, where Dens_M denotes the bundle of densities on M. Then, as above, we can define vector spaces

$$\begin{aligned} \mathscr{E}^! &= \Gamma(M, E^!) \\ \mathscr{E}_c^! &= \Gamma_c(M, E^!). \end{aligned}$$

The vector spaces have natural diffeologies and topologies. There are natural identifications

$$\begin{aligned} \overline{\mathscr{E}}(N) &= \mathrm{Hom}_{cont}(\mathscr{E}_c^!, C^\infty(N)) \\ \overline{\mathscr{E}}_c(N) &= \mathrm{Hom}_{cont}(\mathscr{E}^!, C^\infty(N)) \end{aligned}$$

where Hom_{cont} denotes the vector space of continuous linear maps.

A.3. Differentiable cochain complexes

A.3.0.3 Definition. *A differentiable cochain complex is a cochain complex* V *where each* V^i *is a differentiable vector space and each differential* $\mathrm{d} : V^i \to V^{i+1}$ *is a smooth map.*

A map of differentiable cochain complexes is simply a cochain map $V \to W$ *whose constituent maps* $V^i \to W^i$ *are all smooth.*

A cochain homotopy of such maps is a cochain homotopy whose constituent maps $V^i \to W^{i-1}$ *are all smooth.*

We want to perform standard constructions from homological algebra with differentiable cochain complexes. Of course, we need to make sure the definitions take into account the diffeological structure.

A.3.0.4 Definition. *(1) A sequence $0 \to A \to B \to C \to 0$ of differentiable cochain complexes is exact if the component sequences $0 \to A^i \to B^i \to C^i \to 0$ are exact.*

(2) A map $f : A \to B$ of differentiable cochain complexes is a cofibration if it fits into an exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$.

(3) Similarly, a map $f : A \to B$ is a fibration if it fits into an exact sequence $0 \to C \to A \xrightarrow{f} B \to 0$.

Note that $A \to B$ is a cofibration (respectively, fibration) if and only if, for all n, the map $\mathrm{Stalk}_n(A) \to \mathrm{Stalk}_n(B)$ is an injective (respectively, surjective) map of cochain complexes. Equivalently, the map $A \to B$ is a cofibration if in each cohomological degree, A^i is a differentiable subspace of B^i. This means that $A^i \to B^i$ is injective and that a map $M \to A^i$ is smooth if and only if the composed map to B^i is smooth. Similarly, $A \to B$ is a fibration if and only if every $A^i \to B^i$ is surjective and a map $M \to B^i$ is smooth if and only if it locally lifts to a map to A^i.

We have seen above that one can take kernels and cokernels in the category of differentiable vector spaces. Thus, we can define the cohomology groups $H^i(A)$ of any differentiable cochain complex. These are differentiable vector spaces.

A.3.0.5 Definition. *A map $A \to B$ is a weak equivalence if and only if, for all n, the map of cochain complexes $\mathrm{Stalk}_n A \to \mathrm{Stalk}_n B$ is a quasi-isomorphism.*

Standard constructions and lemmas from ordinary homological algebra hold in this setting. For example, if $f : V \to W$ is a map of differentiable cochain complexes, we can form the cone $\mathrm{Cone}(f)$, whose underlying graded differentiable space is $V[1] \oplus W$, but equipped with differential

$$\begin{pmatrix} \mathrm{d}_{V[1]} & f \\ 0 & \mathrm{d}_W \end{pmatrix}.$$

If f is a fibration, then the map

$$\mathrm{Ker}\, f[1] \to \mathrm{Cone}(f)$$

is an equivalence. If f is a cofibration, then the map

$$\mathrm{Cone}(f) \to \mathrm{Coker}(f)$$

is an equivalence.

A.3.1. We will often use versions of spectral sequence arguments in the category of differentiable complexes.

Suppose that V_i is a directed system of differentiable cochain complexes indexed by $i \in \mathbb{Z}_{\geq 0}$. Thus, we have maps $f_i : V_i \to V_{i+1}$.

Let us suppose that the maps f_i are cofibrations. Then, since the category of differentiable vector spaces is closed under colimits of cofibrant maps, we can form the differentiable cochain complex $\operatorname{colim}_i V$, which in cohomological degree k is $\operatorname{colim}_i V_i^k$.

A.3.1.1 Lemma. *Let V_*, W_* be sequential directed systems where the maps $V_i \to V_{i+1}$, $W_i \to W_{i+1}$ are cofibrations. Let $V_* \to W_*$ be a map of directed systems.*

Suppose that the maps $V_i / V_{i-1} \to W_i / W_{i-1}$ are all weak equivalences.

Then the map

$$\operatorname{colim} V_i \to \operatorname{colim} W_j$$

is a weak equivalence.

PROOF. We need to verify that the maps are equivalences at the level of stalks. The forgetful functors

$$\mathrm{Stalk}_n : \mathrm{DVS} \to \mathrm{Vect}$$

commute with all colimits of cofibrations. It follows that, in the situation above,

$$\operatorname{colim} \mathrm{Stalk}_n V_i = \mathrm{Stalk}_n \operatorname{colim} V_i.$$
$$\mathrm{Stalk}_n (V_i / V_{i-1}) = \mathrm{Stalk}_n V_i / \mathrm{Stalk}_n V_{i-1}.$$

Now, $\mathrm{Stalk}_n V_i$ is a directed system of cochain complexes where the maps are injective, and likewise for $\mathrm{Stalk}_n W_i$. Therefore, by the usual spectral sequence argument, if the map

$$\mathrm{Stalk}_n V_i / \mathrm{Stalk}_n V_{i-1} \to \mathrm{Stalk}_n W_i \, \mathrm{Stalk}_n W_{i-1}$$

is a quasi-isomorphism for all n, the map

$$\operatorname{colim} \mathrm{Stalk}_n V_i \to \operatorname{colim} \mathrm{Stalk}_n W_i$$

is also a weak equivalence, giving the desired result. □

Similarly, we have the following.

A.3.1.2 Lemma. *Let V_*, W_* be sequential directed systems of differentiable cochain complexes, where the maps $V_i \to V_{i+1}$, $W_i \to W_{i+1}$ are cofibrations. Let $V_* \to W_*$ be a map of systems, such that the constituent maps $V_i \to W_i$ are quasi-isomorphisms. Then the map* $\operatorname{colim} V_i \to \operatorname{colim} W_i$ *is a quasi-isomorphism.*

PROOF. The proof is almost identical to that of the previous lemma. □

We have a similar statement for inverse systems, but only under some stronger hypothesis.

A.3.1.3 Lemma. *Let V_*, W_* be sequential inverse systems of differentiable cochain complexes. Thus, there are maps $f_i : V_i \to V_{i-1}$ and $g_i : W_i \to W_{i-1}$. Suppose that these maps are fibrations and that the systems V_i, W_i are eventually constant. In other words, the maps f_i, g_i are isomorphisms for i sufficiently large.*

Let $V_* \to W_*$ be a map of inverse systems, which induces a quasi-isomorphism

$$\operatorname{Ker} f_i \to \operatorname{Ker} g_i$$

for each i.

Then the map $\varprojlim V \to \varprojlim W$ is a quasi-isomorphism.

PROOF. The functor of stalks commutes with finite limits of fibrations. The fact that the maps $V_i \to V_{i-1}$ are isomorphisms for sufficiently large $i \geq N$ thus implies that

$$\operatorname{Stalk}_n \varprojlim V_i = \varprojlim \operatorname{Stalk}_n V_i.$$

Because the maps $V_i \to V_{i-1}$ are fibrations, the maps $\operatorname{Stalk}_n V_i \to \operatorname{Stalk}_n V_j$ are surjective maps of cochain complexes. The spectral sequence argument implies that the map

$$\varprojlim \operatorname{Stalk}_n V_i \to \varprojlim \operatorname{Stalk}_n W_i$$

is a quasi-isomorphism as desired. □

A.3.2. With these definitions, we can define a factorization algebra valued in the multicategory of differentiable cochain complexes. Indeed, we have already given a general definition of factorization algebra valued in a multicategory. The definition presented here is just an exegesis of the general definition.

A.3.2.1 Definition. *A prefactorization algebra on a manifold M valued in the multicategory of differentiable cochain complexes is the assignment of a differentiable cochain complex $\mathcal{F}(U)$ to every open subset $U \subset M$, together with smooth multilinear cochain maps*

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

if $U_1, \ldots, U_n$ are disjoint open subsets of V, and satisfying the coherence axioms explained earlier.

Given any such prefactorization algebra $\mathcal{F}$ and any factorizing cover

$$\mathfrak{U} = \{U_i \mid i \in I\}$$

of an open set $V \subset M$, we can form the Čech complex

$$\check{C}(\mathfrak{U}, \mathcal{F}).$$

As usual, this is a direct sum

$$\check{C}(\mathfrak{U}, \mathcal{F}) = \oplus_{\alpha_1,\ldots,\alpha_k \subset PI} \mathcal{F}(U_{\alpha_1} \cap \cdots \cap U_{\alpha_n})[k-1].$$

Here, PI indicates the set of finite subsets $\alpha \subset I$ with the property that the open sets U_i for $i \in \alpha$ are disjoint. The set U_α indicates the disjoint union of the U_i for $i \in \alpha$.

Since differentiable cochain complexes admit all coproducts, this Čech complex is again a differentiable cochain complex.

A.3.2.2 Definition. *A differentiable factorization algebra on M is a differentiable prefactorization algebra $\mathcal{F}$ on M with the property that, for every factorizing cover $\mathfrak{U}$ of an open subset $V \subset M$, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(V)$$

is a weak equivalence of differentiable cochain complex (as defined above).

A.4. Pro-cochain complexes

This section involves ordinary vector spaces, not differentiable vector spaces, but it prepares us for an important notion we use throughout the book.

Most of the examples of factorization algebras we construct will take values not in the category of ordinary cochain complexes but in the category of pro-cochain complexes or, equivalently, of complete filtered cochain complexes.

A.4.0.3 Definition. *A complete filtered cochain complex is a cochain complex V equipped with a decreasing filtration $F^iV \subset V$ by sub-cochain complexes indexed by $i \in \mathbb{Z}_{\geq 0}$, such that $F^0V = V$ and*

$$V = \varprojlim V/F^iV.$$

A map of complete filtered cochain complexes is a map $V \to W$ which preserves the filtration.

Such a map is a weak equivalence if the map $\operatorname{Gr}^i V \to \operatorname{Gr}^i W$ is a quasi-isomorphism for all i. (Note that this implies that the map $V \to W$ is a quasi-isomorphism.)

Some care is needed when defining colimits of complete filtered cochain complexes.

A.4.0.4 Definition. *Let $\{V_\alpha \mid \alpha \in A\}$ be a collection of complete filtered cochain complexes indexed by some set A. Then the direct sum $\bigoplus_{\alpha \in A} V_\alpha$ is defined by*

$$\bigoplus_{\alpha \in A} V_\alpha = \varprojlim_{i \in \mathbb{Z}_{\geq 0}} \left(\bigoplus_{\alpha} V_\alpha / F^i V_\alpha \right).$$

(On the right hand side of this equation, $\bigoplus V_\alpha / F^i V_\alpha$ indicates the ordinary direct sum of cochain complexes.)

The reason for making this definition is that the filtration on the naive direct sum of the V_α is not complete. It is easy to verify that the direct sum defined above is a coproduct in the category of complete filtered cochain complexes.

Similarly, the tensor product of complete filtered cochain complexes needs to be completed.

A.4.0.5 Definition. *Let V, W be complete filtered cochain complexes. The tensor product $V \otimes W$ is defined as the limit*

$$V \otimes W = \varprojlim_{i,j} (V/F^iV) \otimes (V/F^jW).$$

The filtration on $V \otimes W$ is defined by

$$F^k(V \otimes W) = \operatorname*{colim}_{i+j \geq k} F^iV \otimes F^jW,$$

where the tensor product $F^iV \otimes F^jW$ is defined as the limit of $F^iV/F^rV \otimes F^jW/F^sV$.

Again, the reason for this definition is that the filtration on the naive tensor product of $V \otimes W$ is not complete.

With this definition of completed direct sum and tensor product, it is straightforward to modify our definition of factorization algebra to take values in the category of complete filtered cochain complexes.

A.4.0.6 Definition. *A complete filtered factorization algebra $\mathcal{F}$ on X is a prefactorization algebra $\mathcal{F}$ taking values in the symmetric monoidal category of complete filtered cochain complexes, using the tensor product described above, such that, for every factorizing open cover $\mathfrak{U}$ of an open subset U of X, the map*

$$\check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is an equivalence. The direct sums and tensor products appearing in the definition of the Čech complex are completed, as above.

A.5. Differentiable pro-cochain complexes

As a final elaboration on the concept of cochain complex, we will put together the two ideas described above.

A.5.0.7 Definition. *A differentiable pro-cochain complex is a differentiable cochain complex V equipped with a decreasing filtration by differentiable subcomplexes $F^i V$ with the following properties.*

(1) $F^0 V = V$.
(2) The maps $F^i V \to F^j V$ if $i > j$ are cofibrations. This means that they are injective and that in each cohomological degree, the map $F^i V^k \to F^j V^k$ has the property that a map $M \to F^j V^k$ is smooth if it lifts to a smooth map to $F^i V^k$.
This implies that we can form the quotient differentiable vector space $V / F^i V$, and that the maps
$$V / F^i V \to V / F^j V$$
are cofibrations.
(3) We require that
$$V = \varprojlim V / F^i V.$$

A map of differentiable pro-cochain complexes is a filtration-preserving map $V \to W$. Such a map is a weak equivalence if the maps $\mathrm{Gr}^i V \to \mathrm{Gr}^i W$ are weak equivalences of differentiable cochain complexes. Note that this implies that the maps $V/F^iV \to W/F^iW$ are weak equivalences of differentiable cochain complexes.

A map $V \to W$ of differentiable pro-cochain complexes is a fibration (respectively, a cofibration) if the map $V/F^iV \to W/F^iW$ are fibrations (cofibrations) for all i.

As before, we need to define the completed direct sum and multilinear maps of differentiable cochain complexes.

A.5.0.8 Definition. *If $\{V_\alpha \mid \alpha \in A\}$ is a collection of complete filtered differentiable cochain complexes, indexed by some set A, then the completed direct sum $\bigoplus_{\alpha\in A} V_\alpha$ is defined to be the inverse limit*

$$\bigoplus_{\alpha\in A} V_\alpha = \varprojlim_{i\in\mathbb{Z}_{\geq 0}} \bigoplus_{\alpha\in A}\left(V_\alpha/F^iV_\alpha\right),$$

where on the right hand side we use the ordinary direct sum of differentiable spaces.

We can define the stalks of a differentiable pro-cochain complex $\mathrm{Stalk}_n(V)$ as the colimit

$$\mathrm{Stalk}_n(V) = \operatorname*{colim}_{0\in U\subset\mathbb{R}^n} V(U),$$

where the colimit of the pro-cochain complexes $V(U)$ is completed as above. Thus, $\mathrm{Stalk}_n(V)$ is a pro-cochain complex, and

$$\mathrm{Stalk}_n(V)/F^i\,\mathrm{Stalk}_n(V) = \mathrm{Stalk}_n(V/F^iV).$$

A.5.0.9 Lemma. *A map $V \to W$ of differentiable pro-cochain complexes is an equivalence if and only if the maps $\mathrm{Stalk}_n(V) \to \mathrm{Stalk}_n(W)$ are weak equivalences of pro-cochain complexes.*

PROOF. Immediate. □

A.5.0.10 Lemma. *Let V_*, W_* be sequential directed systems of differentiable pro-cochain complexes, and let $V_* \to W_*$ be a map of directed systems.*

Suppose that the maps $V_i \to V_j$ and $W_i \to W_j$ are all cofibrations and suppose that the maps $V_i \to W_i$ are all equivalences.

Then the map

$$\operatorname{colim} V_i \to \operatorname{colim} W_j$$

is an equivalence.

PROOF. The proof is almost identical to the proof of lemma A.3.1.1. □

Similarly, we can have spectral sequences for inverse systems, but only under some more restrictive hypotheses.

A.5.0.11 Lemma. *Let V_*, W_* be sequential inverse systems of differentiable pro-cochain complexes, and let $V_* \to W_*$ be a map of inverse systems. Let $V = \lim V_*$ and $W = \lim W_*$.*

Suppose that

(1) *The maps $f_i : V_i \to V_{i-1}$, $g_i : W_i \to W_{i-1}$ are fibrations of differentiable cochain complexes.*
(2) *For each k, the inverse systems $V_* / F^k V_*$ and $W_* / F^k W_*$ are eventually constant, as in lemma A.3.1.3.*
(3) *The maps $\operatorname{Ker} f_i \to \operatorname{Ker} g_i$ are quasi-isomorphisms of differentiable cochain complexes.*

Then, the map $V \to W$ is a quasi-isomorphism of differentiable pro-cochain complexes.

PROOF. This follows immediately from lemma A.3.1.3. □

A.5.1. Differentiable pro-cochain complexes form a multicategory, just like differentiable cochain complexes.

A.5.1.1 Definition. *Let $V_1, \dots, V_k, W$ be differentiable pro-cochain complexes. In the multicategory of differentiable pro-cochain complexes, an element of $\operatorname{Hom}(V_1, \dots, V_k; W)$ is a smooth multilinear cochain map*

$$\Phi : V_1 \times \cdots \times V_k \to W = \lim W / F^i W$$

which preserves filtrations: if $v_i \in F^{r_i}(V_i)$, then

$$\Phi(v_1, \dots, v_k) \in F^{r_1 + \cdots + r_k} W.$$

Factorization algebras valued in differentiable pro-cochain complexes are defined as before.

A.6. Differentiable cochain complexes over a differentiable dg ring

The category of differentiable cochain complexes is a differential graded multi-category. Thus, we can talk about commutative differentiable dg algebras R. This is just a commutative dg algebra R, with the structure of a differentiable vector space, such that all the structure maps are smooth. Similarly, a commutative differentiable pro-algebra is a commutative dg algebra in the multi-category of differentiable pro-cochain complexes.

In either context, we can define an R-module M to be a differentiable (pro-)cochain complex equipped with an action of the commutative differentiable (pro-)algebra R, in the obvious way. We say a map $M \to M'$ is a weak equivalence if it is a weak equivalence (as defined above) in the category of differentiable (pro-)cochain complex.

In either context, we say a sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if it is exact in the category of differentiable (pro-)cochain complexes. A map $M_1 \to M_2$ is a cofibration if it can be extended to an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$; it is a fibration if it can be extended to an exact sequence $0 \to M_3 \to M_1 \to M_2 \to 0$.

The category of modules over a differentiable (pro-)dg algebra R is, as above, multicategory. In either case, the multi-maps

$$\mathrm{Hom}_R(M_1, \ldots, M_n; N)$$

are the multi-maps in the category of differentiable (pro-)cochain complexes whose underlying multilinear map $M_1 \times \cdots \times M_n \to N$ are R-multilinear.

A.7. Classes of functions on the space of sections of a vector bundle

Let M be a manifold and E a graded vector bundle on M. Let $U \subset M$ be an open subset. In this section we will introduce some notation for various classes of functionals on sections $\mathscr{E}(U)$ of E on U. These spaces of functionals will all be graded differentiable pro-vector spaces.

A.7.1. We are interested in symmetric algebras on vector spaces of the form $\overline{\mathscr{E}}_c^!(U)$, $\mathscr{E}_c^!(U)$, etc. These symmetric algebras can be defined in two ways: either using the completed projective tensor product of topological vector spaces, or in terms of sections of bundles on U^n. We will explain both points of view.

Thus, let us first define $(\mathscr{E}(U))^{\otimes n}$ to be the tensor power defined using the completed projective tensor product on the topological vector space $\mathscr{E}(U)$. Then a more concrete description of this space is as follows. Let $E^{\boxtimes n}$ denotes the vector bundle on M^n obtained as the external tensor product, so

$$(\mathscr{E}(U))^{\otimes n} = \Gamma(U^n, E^{\boxtimes n})$$

is the space of smooth sections of $E^{\boxtimes n}$ on U^n. Similarly, we can identify

$$\begin{aligned}(\mathscr{E}_c(U))^{\otimes n} &= \Gamma_c(U^n, E^{\boxtimes n})\\ (\overline{\mathscr{E}}_c(U))^{\otimes n} &= \overline{\Gamma}_c(U^n, E^{\boxtimes n})\\ (\overline{\mathscr{E}}(U))^{\otimes n} &= \overline{\Gamma}(U^n, E^{\boxtimes n})\end{aligned}$$

where $\overline{\Gamma}$ indicates the space of distributional sections and the subscript c indicates compactly supported distributional sections.

We have already seen how to equip the various kinds of spaces of sections of a vector bundle with the structure of a differentiable vector space. Since the spaces listed above are expressed as sections of various kinds of a vector bundle on U^n, they all have the structure of differentiable vector spaces.

Symmetric (or exterior) powers of the spaces $\mathscr{E}_c(U)$, $\overline{\mathscr{E}}_c(U)$, $\mathscr{E}(U)$, $\overline{\mathscr{E}}(U)$ are defined by taking coinvariants of the tensor powers defined above with respect to the action of the symmetric group. These symmetric powers inherit the structure of differentiable vector space.

Thus, we can define, for example, the completed symmetric algebra

$$\begin{aligned}\widehat{\operatorname{Sym}}^* \mathscr{E}_c^!(U) &= \prod_n \operatorname{Sym}^n \mathscr{E}_c^!(U)\\ \widehat{\operatorname{Sym}}^* \overline{\mathscr{E}}_c^!(U) &= \prod_n \operatorname{Sym}^n \overline{\mathscr{E}}_c^!(U)\end{aligned}$$

Note that since $\overline{\mathscr{E}}_c^!(U)$ is dual to $\mathscr{E}(U)$, we can view $\widehat{\mathrm{Sym}}\,\overline{\mathscr{E}}_c^!(U)$ as the algebra of formal power series on $\mathscr{E}(U)$. Thus, we often write

$$\widehat{\mathrm{Sym}}\,\overline{\mathscr{E}}_c^!(U) = \mathscr{O}(\mathscr{E}(U)).$$

Similarly, $\widehat{\mathrm{Sym}}\,\mathscr{E}_c^!(U)$ is the algebra of formal power series on $\overline{\mathscr{E}}(U)$.

In a similar way, we can construct

$$\mathscr{O}(\overline{\mathscr{E}}(U)) = \prod_n \mathrm{Sym}^n(\mathscr{E}_c^!(U))$$
$$\mathscr{O}(\overline{\mathscr{E}}_c(U)) = \prod_n \mathrm{Sym}^n(\mathscr{E}^!(U)).$$

These spaces of functionals are all products of the differentiable vector spaces of symmetric powers, and so they are themselves differentiable vector spaces. We will equip all of these spaces of functionals with the structure of a differentiable pro-vector space, induced by the filtration

$$F^i\mathscr{O}(\mathscr{E}(U)) = \prod_{n\geq i} \mathrm{Sym}^i\,\overline{\mathscr{E}}_c^!(U)$$

(and similarly for $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$).

The natural product $\mathscr{O}(\mathscr{E}(U))$ is compatible with the differentiable structure, making $\mathscr{O}(\mathscr{E}(U))$ into a commutative algebra in the multicategory of differentiable graded pro-vector spaces. The same holds for the spaces of functionals $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

A.7.2. One-forms. Recall that if V is a vector space, we can define the space of one-forms on V (treated as formal scheme) as

$$\Omega^1(V) = \mathscr{O}(V) \otimes V^\vee.$$

Similarly, we can define

$$\Omega^1(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}_c^!(U),$$

where $\otimes$ denotes the completed projective tensor product.

In concrete terms,

$$\Omega^1(\mathscr{E}(U)) = \prod_n \mathrm{Sym}^n(\overline{\mathscr{E}}_c^!(U)) \otimes \overline{\mathscr{E}}_c^!(U)$$

and we can identify the space

$$\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!_c(U)) \subset \overline{\mathscr{E}}^!_c(U)^{\otimes n+1} = \overline{\Gamma}_c(U^{n+1}, (E^!)^{\boxtimes n+1})$$

as the space of compactly supported distributional sections of $(E^!)^{\boxtimes n+1}$ that are symmetric in the first n variables.

In this way, $\Omega^1(\mathscr{E}(U))$ becomes a differentiable pro-cochain complex, where the filtration is defined by

$$F^i\Omega^1(\mathscr{E}(U)) = \prod_{n \geq i-1} \mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!_c(U).$$

Further, $\Omega^1(\mathscr{E}(U))$ is a module for the commutative algebra $\mathscr{O}(\mathscr{E}(U))$, where the module structure is defined in the multicategory of differentiable pro-vector spaces.

If V is a finite-dimensional vector space, the exterior derivative map

$$\mathrm{d} : \mathscr{O}(V) \to \mathscr{O}(V) \otimes V^\vee$$

is, in components, just the composition

$$\mathrm{Sym}^{n+1} V^\vee \to (V^\vee)^{\otimes n+1} \to \mathrm{Sym}^n(V^\vee) \otimes V^\vee$$

where the maps are the inclusion followed by the natural projection (up to an overall combinatorial constant).

We can, in a similar way, define the exterior derivative

$$\mathrm{d} : \mathscr{O}(\mathscr{E}(U)) \to \Omega^1(\mathscr{E}(U))$$

by saying that on components it is given by the same formula as in the finite-dimensional case.

A.7.3. Other classes of sections of a vector bundle. Before we introduce our next class of functionals — those with proper support — we need to introduce some further notation concerning classes of sections of a vector bundle.

Let M be a manifold, and let $f : M \to N$ be a fibration. Let E be a vector bundle on M. We say a section $s \in \Gamma(M, E)$ has *relative compact support* if the map

$$f : \mathrm{Supp}(s) \to N$$

is proper. We let $\Gamma_{c/f}(M, E)$ denote the space of sections with relative compact support. This is a differentiable vector space: if X is an auxiliary manifold, a smooth map

$X \to \Gamma_{c/f}(M, E)$ is a section of the bundle $\pi_M^* E$ on $X \times M$ which has relative compact support relative to the map

$$M \times X \to N \times X.$$

(It is straightforward to write down a flat connection on $C^\infty(X, \Gamma_{c/f}(M, E))$, using arguments of the type described in section A.2.)

Next, we need to consider spaces of the form $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$, where M and N are manifolds and E, F are vector bundles on M and N respectively. Of course, we can give an abstract definition using the projective tensor product, but we want a more geometric interpretation.

There are several ways to identify this space geometrically. We will view $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ as a subspace

$$\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N) \subset \overline{\mathscr{E}}(M) \otimes \overline{\mathcal{F}}(N).$$

It consists of those elements D with the property that, if $\phi \in \mathscr{E}_c^!(M)$, then map

$$\begin{array}{rccc} D(\phi): & \mathcal{F}_c^!(N) & \to & \mathbb{R} \\ & \psi & \mapsto & D(\phi \otimes \psi) \end{array}$$

comes from an element of $\mathcal{F}(N)$.

Alternatively, $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ is the space of continuous linear maps from $\mathscr{E}_c^!(M)$ to $\mathcal{F}(N)$.

We can similarly define $\overline{\mathscr{E}}_c(M) \otimes \mathcal{F}(N)$ as the subspace of those elements of $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ that have compact support relative to the projection $M \times N \to N$.

These spaces form differentiable vector spaces in a natural way: a smooth map from an auxiliary manifold X to $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N)$ is an element of $\overline{\mathscr{E}}(N) \otimes \mathcal{F}(N) \otimes C^\infty(X)$. Similarly, a smooth map to $\overline{\mathscr{E}}_c(M) \otimes \mathcal{F}(N)$ is an element of $\overline{\mathscr{E}}(M) \otimes \mathcal{F}(N) \otimes C^\infty(X)$ whose support is compact relative to the map $M \times N \times X \to N \times X$.

A.7.4. Functions with proper support. Recall that

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U).$$

We can thus define a subspace

$$\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}^!(U) \subset \Omega^1(\mathscr{E}_c(U)).$$

The Taylor components of elements of this subspace are in the space

$$\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \overline{\mathscr{E}}^!(U),$$

which in concrete terms is the S_n invariants of

$$\overline{\mathscr{E}}^!_c(U)^{\otimes n} \otimes \overline{\mathscr{E}}^!(U).$$

A.7.4.1 Definition. *A function $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has* proper support *if*

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}^!(U) \subset \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U)^\vee.$$

The reason for the terminology is as follows. Let $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ and let

$$\Phi_n \in \mathrm{Hom}(\mathscr{E}_c(U)^{\otimes n}, \mathbb{R})$$

be the nth term in the Taylor expansion of Φ.

Then, Φ has proper support if and only if, for all n, the composition with a projection map

$$\mathrm{Supp}(\Phi_n) \subset U^n \to U^{n-1}$$

is proper.

We will let

$$\mathscr{O}^P(\mathscr{E}_c(U)) \subset \mathscr{O}(\mathscr{E}_c(U))$$

be the subspace of functions with proper support. Note that functions with proper support are *not* a subalgebra.

Because $\mathscr{O}^P(\mathscr{E}_c(U))$ fits into a fiber square

$$\begin{array}{ccc} \mathscr{O}^P(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c(U)^\vee \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}_c(U)^\vee \end{array}$$

it has a natural structure of a differentiable pro-vector space.

A.7.5. Functions with smooth first derivative.

A.7.5.1 Definition. *A function $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has* smooth first derivative *if* $\mathrm{d}\Phi$*, which is* a priori *an element of*

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U)$$

is an element of the subspace

$$\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U).$$

Note that we can identify, concretely, $\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U)$ with the space

$$\prod_n \operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \otimes \mathscr{E}^!(U)$$

and

$$\operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \otimes \mathscr{E}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\otimes n} \otimes \mathscr{E}^!(U).$$

(Spaces of the form $\mathscr{E}(U) \otimes \overline{\mathscr{E}}(U)$ were described concretely above.)

Thus $\mathscr{O}(\mathscr{E}_c(U)) \otimes \mathscr{E}^!(U)$ is a differentiable pro-vector space. It follows that the space of functionals with smooth first derivative is a differentiable pro-vector space, since it is defined by a fiber diagram of such objects.

An even more concrete description of the space $\mathscr{O}^{sm}(\mathscr{E}_c(U))$ of functionals with smooth first derivative is as follows.

A.7.5.2 Lemma. *A functional $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has smooth first derivative if each of its Taylor components*

$$D_n\Phi \in \operatorname{Sym}^n \overline{\mathscr{E}}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\otimes n}$$

lies in the intersection of all the subspaces

$$\overline{\mathscr{E}}^!(U)^{\otimes k} \otimes \mathscr{E}^!(U) \otimes \overline{\mathscr{E}}^!(U)^{\otimes n-k-1}$$

for $0 \leq k \leq n-1$.

PROOF. The proof is a simple calculation. □

Note that the space of functions with smooth first derivative is a subalgebra of $\mathscr{O}(\mathscr{E}_c(U))$. We will denote this subalgebra by $\mathscr{O}^{sm}(\mathscr{E}_c(U))$. Again, the space of functions with smooth first derivative is a differentiable pro-vector space, as it is defined as a fiber product.

We can also define the space of functions on $\mathscr{E}(U)$ with smooth first derivative, by requiring that the exterior derivative lies in

$$\mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U) \subset \Omega^1(\mathscr{E}(U)).$$

A.7.6. Functions with smooth first derivative and proper support. We are particularly interested in those functions which have both smooth first derivative and proper support. We will refer to this subspace as $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$. The differentiable structure on $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$ is, again, given by viewing it as defined by the fiber diagram

$$\begin{array}{ccc} \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}^!(U) \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U)) \otimes \overline{\mathscr{E}}^!(U) \end{array}$$

We have inclusions

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) \subset \mathscr{O}^{sm}(\mathscr{E}_c(U)),$$

where each inclusion has dense image.

A.8. Derivations

As before, let M be a manifold, E a graded vector bundle on M, and U an open subset of M. In this section we will define derivations of algebras of functions on $\mathscr{E}(U)$.

To start with, recall that, for V a finite dimensional vector space (which we treat as a formal scheme) and $\mathscr{O}(V) = \prod \mathrm{Sym}^n V^\vee$ the algebra of formal power series on V, we identify the space of continuous derivations of $\mathscr{O}(V)$ with $\mathscr{O}(V) \otimes V$. We view these derivations as the space of vector fields on V and use the notation $\mathrm{Vect}(V)$.

In a similar way, we can define the space of vector fields $\mathrm{Vect}(\mathscr{E}(U))$ of vector fields on $\mathscr{E}(U)$ as

$$\mathrm{Vect}(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}(U) = \prod_n \left(\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \otimes \mathscr{E}(U) \right),$$

using the completed projective tensor product. We have already seen (section A.7) how to define the structure of diffeological pro-vector space on spaces of this nature.

In concrete terms, the Taylor expansion of an element of $X \in \mathrm{Vect}(\mathscr{E}(U))$ is given by a sequence of continuous symmetric multilinear maps

$$D_n X : \mathscr{E}(U) \times \cdots \times \mathscr{E}(U) \to \mathscr{E}(U).$$

More generally, if M is a smooth manifold and if $X : M \to \mathrm{Vect}(\mathscr{E}(U))$ is a smooth map, then the Taylor expansion of X is a sequence of continuous symmetric multilinear maps
$$\mathscr{E}(U) \times \cdots \times \mathscr{E}(U) \to \mathscr{E}(U) \otimes C^\infty(M) = \Gamma(U \times M, E|_U).$$

In this section we will show the following.

A.8.0.1 Proposition. $\mathrm{Vect}(\mathscr{E}(U))$ *has a natural structure of Lie algebra in the multicategory of diffeological pro-vector spaces. Further,* $\mathscr{O}(\mathscr{E}(U))$ *has an action of the Lie algebra* $\mathrm{Vect}(\mathscr{E}(U))$ *by derivations, where the structure map* $\mathrm{Vect}(\mathscr{E}(U)) \times \mathscr{O}(\mathscr{E}(U)) \to \mathscr{O}(\mathscr{E}(U))$ *is smooth.*

PROOF. To start with, let's look at the case of a finite-dimensional vector space V, to get an explicit formula for the Lie bracket on $\mathrm{Vect}(V)$, and the action of $\mathrm{Vect}(V)$ on $\mathscr{O}(V)$. Then, we will see that these formulae make sense when $V = \mathscr{E}(U)$.

Let $X \in \mathrm{Vect}(X)$, and let us consider the Taylor components $D_n X$, which are multilinear maps
$$V \times \cdots \times V \to V.$$
Our conventions are such that
$$D_n(X)(v_1, \dots, v_n) = \left(\frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_n} X \right)(0) \in V$$
Here, we are differentiating vector fields on V using the trivialization of the tangent bundle to this formal scheme arising from the linear structure.

Thus, we can view $D_n X$ as in the endomorphism operad of the vector space V.

If $A : V^{\times n} \to V$ and $B : V^{\times m} \to V$, let us define
$$A \circ_i B(v_1, \dots, v_{n+m-1}) = A(v_1, \dots, v_{i-1}, B(v_i, \dots, v_{i+m-1}), v_{i+m}, \dots, v_{n+m-1}).$$
If A, B are symmetric (under S_n and S_m, respectively), then define
$$A \circ B = \sum_{i=1}^{n} A \circ_i B.$$
Then, if X, Y are vector fields, the Taylor components of $[X, Y]$ satisfy
$$D_n([X,Y]) = \sum_{k+l=n+1} c_{k,l} \left(D_k X \circ D_l Y - D_l Y \circ D_k X \right)$$
where $c_{k,l}$ are combinatorial constants which are irrelevant for our purposes.

Similarly, if $f \in \mathscr{O}(V)$, the Taylor components of f are multilinear maps

$$D_n f : V^{\times n} \to \mathbb{C}.$$

In a similar way, if X is a vector field, we have

$$D_n(Xf) = \sum_{k+l=n+1} c'_{k,l} D_k(X) \circ D_k(f).$$

Thus, we see that in order to define the Lie bracket on $\mathrm{Vect}(\mathscr{E}(U))$, we need to give maps of diffeological vector spaces

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\otimes n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}, \mathscr{E}(U)) \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes (n+m-1)}, \mathscr{E}(U))$$

where here Hom indicates the space of continuous linear maps, treated as a diffeological vector space. Similarly, to define the action of $\mathrm{Vect}(\mathscr{E}(U))$ on $\mathscr{O}(\mathscr{E}(U))$, we need to define a composition map

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\otimes n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}) \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes n+m-1}).$$

We will treat the first case; the second is similar.

Now, if X is an auxiliary manifold, a smooth map

$$X \to \mathrm{Hom}(\mathscr{E}(U)^{\otimes m}, \mathscr{E}(U))$$

is the same as a continuous multilinear map

$$\mathscr{E}(U)^{\times m} \to \mathscr{E}(U) \otimes C^\infty(X).$$

Here, "continuous" means for the product topology.

This is the same thing as a continuous $C^\infty(X)$-multilinear map

$$\Phi : (\mathscr{E}(U) \otimes C^\infty(X))^{\times m} \to \mathscr{E}(U) \otimes C^\infty(X).$$

If

$$\Psi : (\mathscr{E}(U) \otimes C^\infty(X))^{\times n} \to \mathscr{E}(U) \otimes C^\infty(X).$$

is another such map, then it is easy to define $\Phi \circ_i \Psi$ by the usual formula:

$$\Phi \circ_i \Psi(v_1, \dots, v_{n+m-1}) = \Phi(v_1, \dots v_{i-1}, \Psi_i(v_i, \dots, v_{m+i-1}), \dots, v_{n+m-1})$$

if $v_i \in \mathscr{E}(U) \otimes C^\infty(X)$. This map is $C^\infty(X)$ linear. □

Remark: It is not hard to show that $\mathrm{Vect}(\mathscr{E}(U))$, as defined above, is the space of all continuous derivations of the topological algebra $\mathscr{O}(\mathscr{E}(U))$; we will not need this fact.

A.9. The Atiyah-Bott lemma

In [AB67], Atiyah and Bott showed that for an elliptic complex $(\mathscr{E}, d)$ on a compact closed manifold M, with $\mathscr{E}$ the smooth sections of a $\mathbb{Z}$-graded vector bundle, there is a homotopy equivalence $(\mathscr{E}, d) \hookrightarrow (\overline{\mathscr{E}}, d)$ into the elliptic complex of distributional sections. The argument follows from the existence of parametrices for elliptic operators. This result was generalized by N.N. Tarkhanov to the non-compact case [Tar87].

Let M be a smooth manifold (which, in general, will not be compact).

A.9.0.2 Definition. *An* elliptic complex *on M is a graded vector bundle E, whose space of smooth sections we denote by $\mathscr{E}$, together with a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 possessing the following property, known as ellipticity. Let $\pi^* E$ denote the pullback bundle along the projection map for the cotangent bundle $\pi : T^*M \to M$. The symbol $\sigma(Q)$ of Q is a cohomological degree 1 endomorphism of the vector bundle $\pi^* E$. We require that the complex of vector bundles $(\pi^* E, \sigma(Q))$ on T^*M is exact away from the zero section.*

Let $\overline{\mathscr{E}}$ denote the complex of distributional sections of $\mathscr{E}$. We will endow both $\mathscr{E}$ and $\overline{\mathscr{E}}$ with their natural topologies.

A.9.0.3 Lemma (Tarkhanov, [Tar87]). *There is a continuous homotopy inverse to the natural inclusion*

$$(\mathscr{E}, Q) \hookrightarrow (\overline{\mathscr{E}}, Q).$$

This is Lemma 1.7 of [Tar87]. The continuous homotopy inverse

$$\Phi : \overline{\mathscr{E}} \to \mathscr{E}$$

is given by a kernel $K_\Phi \in \mathscr{E}^! \otimes \mathscr{E}$ with proper support. The homotopy $S : \overline{\mathscr{E}} \to \overline{\mathscr{E}}$ is a continuous linear map with

$$[\mathrm{d}, S] = \Phi - \mathrm{Id}\,.$$

The kernel K_S for S is a distribution, that is, an element of $\overline{\mathscr{E}}^! \otimes \overline{\mathscr{E}}$, with proper support.

PROOF. We will reproduce the proof in [Tar87]. Choose a metric on E (Hermitian, if E is a complex vector bundle) and a volume form on M. Let Q^* be the adjoint to Q. We form the graded commutator $D = [Q, Q^*]$. This is an elliptic operator on each space $\mathscr{E}^i$, the cohomological degree i part of $\mathscr{E}$. Thus, by standard results in the theory of pseudodifferential operators, there is a parametrix P for D. The kernel K_P is an element of $\overline{\mathscr{E}}^! \otimes \overline{\mathscr{E}}$, and the corresponding operator $P : \mathscr{E}_c \to \overline{\mathscr{E}}$ is an inverse for D up to smoothing operators.

By multiplying K_P by a smooth function on $M \times M$ which is 1 in a neighborhood of the diagonal, we can assume that K_P has proper support. This means that P will extend to a map $\mathscr{E} \to \overline{\mathscr{E}}$ and will still be a parametrix: thus $P \circ D$ and $D \circ P$ both differ from the identity by smoothing operators.

The homotopy S is now defined by

$$S = Q^* P.$$

□

Note that the homotopy inverse $\Phi : \overline{\mathscr{E}} \to \mathscr{E}$ and the homotopy $S : \overline{\mathscr{E}} \to \overline{\mathscr{E}}$ we have constructed are maps of differentiable vector spaces (as well as being continuous).

APPENDIX B

Extending a factorization algebra from a basis

In this appendix we will prove the following technical proposition, stated in section 3.6. The reader should refer to that section for the notation.

B.0.0.4 Proposition. *Let $\mathfrak{U}$ be a factorizing basis of a space X, closed under finite intersection. Let $\mathcal{F}$ be a $\mathfrak{U}$-factorization algebra, as defined in section 3.6. Let $i_*^{\mathfrak{U}}(\mathcal{F})$ be the prefactorization algebra which sends an open subset $V \subset V$ to*

$$i_*^{\mathfrak{U}}(\mathcal{F})(V) = \check{C}(\mathfrak{U}_V, \mathcal{F}$$

where $\mathfrak{U}_V$ is the factorizing cover of V consisting of sets in $\mathfrak{U}$ which are subsets of V.

Then, $i_*^{\mathfrak{U}}(\mathcal{F})$ is a factorization algebra whose restriction to open sets in the cover $\mathfrak{U}$ is quasi-isomorphic to $\mathcal{F}$.

PROOF. We need to check that if $\mathfrak{W}$ is a factorizing cover of $V \subset X$, then

$$i_*^{\mathfrak{U}}(\mathcal{F})(V) \simeq \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})).$$

Before we prove this, we need a lemma. Let $\mathfrak{U}_{\mathfrak{W}}$ be the cover of V consisting of open sets in $\mathfrak{U}$ which are subordinate to $\mathfrak{W}$.

B.0.0.5 Lemma. *For any $\mathfrak{U}$-prefactorization algebra $\mathcal{F}$, the natural map*

$$\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})) \to \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$$

is a quasi-isomorphism.

PROOF. Before we check this, let us recall the notation we used when discussing Čech complexes. Let $P\mathfrak{U}$ denote the set of subsets $\alpha \subset \mathfrak{U}$, where for each distinct $i, j \in \alpha$, U_i and U_j are disjoint. If $\alpha \in P\mathfrak{U}$ we will let

$$U_\alpha = \amalg_{i \in \alpha} U_i.$$

If $\alpha_1, \dots, \alpha_k \in P\mathfrak{U}$, we will let

$$\mathcal{F}(\alpha_1, \dots, \alpha_k) = \oplus_{i_1 \in \alpha_1, \dots, i_k \in \alpha_k} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_k}).$$

With this notation, if $W \subset M$, then

$$i_*^{\mathfrak{U}}(\mathcal{F})(W) = \oplus_{\alpha_1, \dots, \alpha_r \in \mathfrak{U}_W} \mathcal{F}(\alpha_1, \dots, \alpha_r)[r-1]$$

where $\mathfrak{U}_W$ refers to the cover of W consisting of open sets in $\mathfrak{U}$ which lie in W.

Let us define a filtration on $i_*^{\mathfrak{U}}(\mathcal{F})$ by saying that

$$F^i i_*^{\mathfrak{U}}(\mathcal{F}) = \oplus_{r \leq i} \oplus_{\alpha_1, \dots, \alpha_r \in \mathfrak{U}_W} \mathcal{F}(\alpha_1, \dots, \alpha_r)[r-1].$$

This filters $i_*^{\mathfrak{U}}(\mathcal{F})$ as a prefactorization algebra.

There is a natural map

$$\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})) \to \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}).$$

Let us filter $\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F}))$ by the filtration coming from $i_*^{\mathfrak{U}}(\mathcal{F})$. Let us filter $\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$ in the same way that we filtered $i_*^{\mathfrak{U}}(\mathcal{F})$. The map preserves the filtration.

Thus, to prove that this map is a quasi-isomorphism, it suffices to show that it is on the associated graded.

The complex $\mathrm{Gr}^n \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F}))$ breaks up as a direct sum of pieces corresponding to tuples $\alpha_1, \dots, \alpha_n \in P\mathfrak{U}_{\mathfrak{W}}$, as follows. If $\beta \in P\mathfrak{W}$ and $\alpha \in P\mathfrak{U}_{\mathfrak{W}}$, say $\alpha \subset \beta$ if $U_\alpha \subset U'_\beta$. Then,

$$\mathrm{Gr}^n \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})) = \bigoplus_{\alpha_1, \dots, \alpha_n \in P\mathfrak{U}} \mathcal{F}(\alpha_1, \dots, \alpha_n)[n-1] \otimes \left(\bigoplus_{\substack{\beta_1, \dots, \beta_m \in P\mathfrak{W} \\ \alpha_i \subset \beta_j \text{ all } i,j}} \mathbb{C} \cdot (\beta_1, \dots, \beta_k) \right).$$

Here $(\beta_1, \dots, \beta_k)$ denotes a vector in degree $-k$. This is a direct sum decomposition of cochain complexes.

On the other hand,

$$\mathrm{Gr}^n \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}) = \oplus_{\alpha_1, \dots, \alpha_n \in P\mathfrak{U}_{\mathfrak{W}}} \mathcal{F}(\alpha_1, \dots, \alpha_n)$$

Thus, to prove the lemma, we need to verify that the complex

$$\oplus_{\substack{\beta_1,\ldots,\beta_m \in P\mathfrak{W} \\ \alpha_i \subset \beta_j \text{ all } i,j}} \mathbb{C} \cdot (\beta_1, \ldots, \beta_k)$$

has homology $\mathbb{C}$ if all $\alpha_i \in P\mathfrak{U}_{\mathfrak{W}}$, and zero otherwise.

It is clear that the complex is zero if all α_i are not in $P\mathfrak{U}_{\mathfrak{W}}$. So let us assume that all α_i are in $P\mathfrak{U}_{\mathfrak{W}}$. Then, the complex is simply the simplicial chain complex on the infinite simplex with vertices $\beta \in P\mathfrak{U}$ such that $\cup U_{\alpha_i} \subset U_\beta$. This is of course contractible.

□

It remains to shows that the natural map

$$\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}) \to \check{C}(\mathfrak{U}_V, \mathcal{F})$$

is a quasi-isomorphism. (Here, as before, $\mathfrak{U}_V$ refers to the cover of V consisting of sets in $\mathfrak{U}$ which lie in V).

To see that this map is a quasi-isomorphism, observe that by another application of the sub-lemma there is a quasi-isomorphism

$$\check{C}(\mathfrak{U}, i_*^{\mathfrak{U}_{\mathfrak{W}}}(\mathcal{F})) \simeq \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}).$$

Here $i_*^{\mathfrak{U}_{\mathfrak{W}}}$ refers to the prefactorization algebra on V obtained by extending $\mathcal{F}$, as before, but now considered as a $\mathfrak{U}_{\mathfrak{W}}$-factorization algebra.

Now the fact that $\mathcal{F}$ is a $\mathfrak{U}$-factorization algebra implies that, for all $U \in \mathfrak{U}$, the natural map

$$\check{C}(\mathfrak{U}_{\mathfrak{W}} \cap \mathfrak{U}_U, \mathcal{F}) \to \mathcal{F}(U)$$

is a quasi-isomorphism.

It follows that the natural map

$$\check{C}(\mathfrak{U}, i_*^{\mathfrak{U}_{\mathfrak{W}}}(\mathcal{F})) \to \check{C}(\mathfrak{U}, \mathcal{F})$$

is a quasi-isomorphism, as desired.

□

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