A GENERAL CONCEPT FOR SOLVING LINEAR MULTICRITERIA PROGRAMMING PROBLEMS WITH CRISP, FUZZY OR STOCHASTIC VALUES

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ABSTRACT: For modelling imprecise data the literature offers two different methods: either the use of probability distributions or the use of fuzzy sets. In our opinion these two concepts should be used parallel or combined, dependent on the real situation. Moreover, in many economic problems the well-known probabilistic or fuzzy solution procedures are not suitable methods because neither the stochastic variables have a simple classical distribution nor the fuzzy values are fuzzy numbers or fuzzy intervals. For example in investment problems, the coefficients may often be described by more complex distributions or more general fuzzy sets. In this case we propose to distinguish several scenarios and to describe the parameters of the different scenarios by fuzzy intervals. For solving such stochastic linear programs with fuzzy parameters we propose a new method, which retains the original objective functions dependent on the different states of nature and which is based on the integrated chance constrained program introduced by Klein Haneveld [3] and the interactive solution process FULPAL (FUzy Linear Programming based on Aspiration Levels), see [6, 7, 8, 9, 10].

Keywords: Fuzzy optimization; stochastic optimization; interactive decision process; investment problems

1 INTRODUCTION

Using linear programming models

\[ z(x, o) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \rightarrow \text{Max} \]

subject to

\[
\begin{align*}
    a_{11} x_1 + a_{12} x_2 + \cdots + a_{in} x_n & \leq b_1 & i = 1, 2, \ldots, m_1 \\
    a_{11} x_1 + a_{12} x_2 + \cdots + a_{in} x_n & = b_1 & i = 1, 2, \ldots, m \\
    x_j & \geq 0 & j = 1, \ldots, n
\end{align*}
\]

for solving real decision problems, we often encounter the difficulty that not all of the parameters \( c_j, a_{ij}, b_i \) are known exactly.

In this situation the literature offers two different ways for getting a better model of the real problem:

i. Imprecision of some data is modelled by probability distributions.

Then we get the stochastic linear program (SLP)

\[
\begin{align*}
    & z(x, o) = c_1 (o) \cdot x_1 + c_2 (o) \cdot x_2 + \cdots + c_n (o) \cdot x_n \rightarrow \text{Max} \\
    & \text{subject to} \\
    & a_{11} (o) \cdot x_1 + a_{12} (o) \cdot x_2 + \cdots + a_{in} (o) \cdot x_n \leq b_1 (o) & i = 1, 2, \ldots, m_1 \\
    & a_{11} (o) \cdot x_1 + a_{12} (o) \cdot x_2 + \cdots + a_{in} (o) \cdot x_n = b_1 (o) & i = 1, 2, \ldots, m \\
    & x_j \geq 0 & j = 1, \ldots, n
\end{align*}
\]

where \( c_j (o), a_{ij} (o), b_i (o) \) are random variables on a probability space.

Well-known procedures for solving stochastic linear programs are

A. concerning the constraints
A.1 the Fat solution [6]
A.2 the Chance Constrained Programming [1]
A.3 the Stochastic Programming with Recourse [4]
A.4 the Integrated Chance Constrained Program [3]

B. concerning the objectives
B.1 the Optimization of the Mean Value
\[
\max \mathbb{E}(z(x, o)) \quad x
\]
B.2 the Minimization of the Variance
\[
\min \text{Var}(z(x, o)) \quad x
\]
B.3 the Minimum Risk Problem
\[
\max P(o \mid z(x, o) \geq \gamma) \quad x
\]
where \( \gamma \) is a certain aspiration level.

But only for particular distributions a specific combination of situations A.1 - A.4 and B.1 - B.3 define an equivalent deterministic model, which may be solved easily, see [4, 11].
ii. Imprecision of some data is modelled by fuzzy sets.

In this case we have to solve the fuzzy linear program (FLP)

\[ \tilde{Z}(x) = \tilde{C}_1 x_1 + \tilde{C}_2 x_2 + \cdots + \tilde{C}_n x_n \rightarrow \tilde{M} \tilde{a} \]

subject to

\[ \tilde{A}_{i1} x_1 + \tilde{A}_{i2} x_2 + \cdots + \tilde{A}_{in} x_n \leq \tilde{B}_i \quad i = 1,2,\ldots,m \]
\[ \tilde{A}_{i1} x_1 + \tilde{A}_{i2} x_2 + \cdots + \tilde{A}_{in} x_n = \tilde{B}_i \quad i = 1,2,\ldots,m \]
\[ x_j \geq 0 \quad j = 1,\ldots,n \]

where \( \tilde{C}_j, \tilde{A}_{ij}, \tilde{B}_i \) are fuzzy sets on \( \mathbb{R} \).

If for each constraint the fuzzy values are flat fuzzy numbers of the same L-R-type, several procedures are proposed in literature for solving the FLP (3), see [2, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Rouben; Teghem [11] and Yazenin [14] do comparisons between the methodologies for SLP and FLP.

As each real number \( d \) can be modelled as a fuzzy number \( \mu_D(x) \) for each constraint, the fuzzy system (3) includes the special cases, that

1. some or all parameters of the objective function are crisp, i.e.
   \[ Z(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \rightarrow \text{Max} \]  
2. some or all constraints are crisp, i.e.
   \[ a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i \]
3. some or all constraints have the soft form
   \[ a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq \tilde{B}_i \]

Moreover, as the methods for solving stochastic linear programs lead to deterministic LP-systems it is possible to introduce these constraints and/or the objectives in the fuzzy system (3) as additional constraints and as extra objective functions. In our opinion, the probability concept and the fuzzy set theory are not in opposition but they complete each other. Therefore, we propose to use probability distributions as well as fuzzy sets for modelling imprecision of data. This can be done parallel or combined, dependent on the real situation.

2 STOCHASTIC LINEAR PROGRAMS WITH FUZZY PARAMETERS

In many economic problems, for example in investment problems, both procedures, described above, are not suitable methods because neither the stochastic variables have a classical distribution (Gaussian, exponential, uniformly,...) nor the fuzzy values are fuzzy numbers or fuzzy intervals. In investment problems, coefficients \( c_i \) or \( a_{ij} \) may often be described by distributions that are more complex or more general fuzzy sets, see the examples in Figure 1 and Figure 2.

The classical way of solving investment problems is to distinguish several scenarios (states of nature) and to attach to each parameter an unequivocal value \( c_j(\omega_k), a_{ij}(\omega_k), b_i(\omega_k) \) dependent on the states of nature \( \omega_k \).

In doing so, we get a stochastic linear program with discrete random coefficients. It has the form of the stochastic LP-system (2), where

\[ \omega \in \{\omega_1, \omega_2, \ldots, \omega_K\}, K \in \mathbb{N} \]

and

\[ \sum_{k=1}^{K} p(\omega_k) = 1. \]

In literature, this model is mostly solved by optimizing the mean value of the objective functions combined with the fat solution concerning the constraints. In doing so, we have to solve the simple classical linear programming problem:

\[ E(z(x)) = \sum_{j=1}^{n} \left[ \sum_{k=1}^{K} c_j(\omega_k) \cdot p(\omega_k) \right] \cdot x_j \rightarrow \text{Max} \]

subject to

\[ a_{i1}(\omega_k) \cdot x_1 + a_{i2}(\omega_k) \cdot x_2 + \cdots + a_{in}(\omega_k) \cdot x_n \leq b_i(\omega_k) \]
\[ a_{i1}(\omega_k) \cdot x_1 + a_{i2}(\omega_k) \cdot x_2 + \cdots + a_{in}(\omega_k) \cdot x_n = b_i(\omega_k) \]
\[ x_j \geq 0 \quad j = 1,\ldots,n \]

This proceeding has several essential disadvantages. At first, the various objectives are aggregated to a objective function, where the coefficients are expected values of the corresponding coefficients of the given objective functions. Such a “compromise” objective function has no concrete meaning to the decision maker, the values of this function serve solely as orientation criterion for cal-
calculating an solution. Secondly, the set of feasible solutions of (7) are a subset of the feasible solutions of (2). Due to the fat solution approach, this subset is normally relative small and the greater part of the feasible solutions of (2) is excluded. Thirdly, in practise not all parameter \( c_j(\omega_k), a_{ij}(\omega_k), b_i(\omega_k) \) are known exactly, but they are frequently imprecise.

We propose to describe the imprecise parameters by fuzzy sets. If the scenarios are suitably selected, it is sufficient to use fuzzy numbers or fuzzy intervals for modelling these parameters.

In doing so, we get for each state of nature \( k \in \{1,...,K\} \) a fuzzy linear program of the form

\[
\tilde{Z}_k(x) = \tilde{C}_1(\omega_k) \cdot x_1 + \cdots + \tilde{C}_n(\omega_k) \cdot x_n \rightarrow \tilde{M}ax
\]

subject to

\[
\tilde{A}_{il}(\omega_k) \cdot x_1 + \cdots + \tilde{A}_{in}(\omega_k) \cdot x_n \leq \tilde{B}_i(\omega_k)
\]

\( i = 1,2,\ldots,m_1 \)

\[
\tilde{A}_{il}(\omega_k) \cdot x_1 + \cdots + \tilde{A}_{in}(\omega_k) \cdot x_n = \tilde{B}_i(\omega_k)
\]

\( i = 1,2,\ldots,m \)

\( x_j \geq 0 \quad j = 1,\ldots,n \) \hspace{1cm} (8)

For solving the multiobjective problem, consisting of \( K \) fuzzy linear programs of type (8) and known probabilities \( p(\omega_k) \), an easy method is to combine the optimization of the mean value of the objective functions with the fat solution.

In doing so, we get the fuzzy linear program

\[
E(\tilde{Z}(x)) = \sum_{j=1}^{n} \sum_{k=1}^{K} \tilde{C}_j(\omega_k) \cdot p(\omega_k) \cdot x_j \rightarrow \tilde{M}ax
\]

subject to

\[
\tilde{A}_{il}(\omega_k) \cdot x_1 + \cdots + \tilde{A}_{in}(\omega_k) \cdot x_n \leq \tilde{B}_i(\omega_k)
\]

\( i = 1,2,\ldots,m_1 \quad k = 1,2,\ldots,K \)

\[
\tilde{A}_{il}(\omega_k) \cdot x_1 + \cdots + \tilde{A}_{in}(\omega_k) \cdot x_n = \tilde{B}_i(\omega_k)
\]

\( i = 1,2,\ldots,m \quad k = 1,2,\ldots,K \)

\( x_j \geq 0 \quad j = 1,\ldots,n \) \hspace{1cm} (9)

A compromise solution of (9) may be calculated with one of the solution methods for FLP, for example with the interactive process FULPAL (FUzy Linear Programming based on Aspiration Levels), see [5, 6, 7, 8, 9].

Due to the fat solution, the restrictions of all scenarios are equally present in the constraint system of (9), independently of their various probabilities \( p(\omega_k) \). The consequence is that each scenario restricts the set of the feasible solutions of (9) independent of the height of \( p(\omega_k) \).

3 A NEW SOLUTION METHOD FOR SOLVING STOCHASTIC LINEAR PROGRAMS WITH FUZZY DATA

To avoid this disadvantage of the fat solution, we propose a new solution method that follows the pattern of the integrated chance constrained program of Klein Haneveld [3]. It consists of three modifications of the system (9) and is orientated to the solution process FULPAL.

An essential characteristic of FULPAL is, that a constraint

\[
\tilde{A}_{il}(\omega_k) \cdot x_1 + \cdots + \tilde{A}_{in}(\omega_k) : x_n \leq \tilde{B}_i(\omega_k)
\]

where \( \tilde{A}_{il}(\omega_k) = (\tilde{a}_{ij}, \tilde{a}_{ij}^E, \tilde{a}_{ij}^e) \) \( a_{ij}^E, \alpha_{ij}^E, \beta_{ij}^E \) \( LR \) and \( \tilde{B}_i(\omega_k) = (\tilde{b}_i, 0, \tilde{b}_i^E) \) \( L \) is replaced by the crisp constraint

\[
\sum (\tilde{a}_{ij}^e + \tilde{a}_{ij}) \cdot x_j \leq \tilde{b}_i + \tilde{b}_i^E = \tilde{b}_i^E
\]

and the fuzzy objective function

\[
\mu_{D_{ik}} \left( \sum_{j=1}^{n} \tilde{a}_{ij} \cdot x_j \right) \rightarrow \text{Max}
\]

where \( \mu_{D_{ik}} (y) = \begin{cases} 1 & \text{if } y < b_{ik} \\ \mu_{B_{ik}} & \text{if } b_{ik} \leq y \leq b_{ik}^E \\ 0 & \text{if } b_{ik}^E < y \end{cases} \) \hspace{1cm} (10)

Figure 3: Membership function of \( \tilde{A}_{ij} \)
I. Reducing the aspiration levels dependent on the probabilities \( p(\omega_k) \)

For explaining this procedure, we assume that for the right side \( B_i(\omega_k) \) the decision maker has specified the interval of possible maximal values as \([b_{ik}, \overline{b}_{ik}]\) and has chosen the aspiration level \( \overline{\lambda}_{ik} \), see Figure 4.

Because the scenarios will not realise with certainty, the fat solution is to restrictive. Therefore, we propose to the decision maker to increase the aspiration levels \( \lambda_{ik} \) dependent on the probabilities \( p(\omega_k) \).

Therefore, we propose the revision formula

\[
\overline{b}_{ik}(p) = \overline{b}_{ik}^\lambda + (\overline{b}_{ik}^\lambda - \overline{b}_{ik}) \cdot (1 - p(\omega_k)) = \overline{b}_{ik} - (\overline{b}_{ik}^\lambda - \overline{b}_{ik}) \cdot p(\omega_k) \tag{12}
\]

which have the following characteristics:

- In case of certainty, i.e. \( p(\omega_k) = 1 \), \( \overline{b}_{ik}(1) \) is equal to the fixed aspiration level \( \overline{b}_{ik}^\lambda \).
- If the probabilities \( p(\omega_k) \) get smaller, the aspiration level \( \overline{b}_{ik}(p) \) will increase uniformly towards \( \overline{b}_{ik}^\lambda \), see Figure 4.
- In any case, the constraints

\[
\sum_j (\overline{a}_{ijk} + \overline{e}_{ijk}) \cdot x_j \leq \overline{b}_{ik}^e
\]

will be fulfilled.

II. Increasing the margins \( \overline{b}_{ik}^e \) dependent on \( p(\omega_k) \)

We next assume that the decision maker takes a risk and accepts a violation of the crisp constraints (10). In analogy to the integrated chance constrained program we use the mean shortage

\[
E[r_i(x)] = \sum_{k=1}^{K} r_i(x, \omega_k) \cdot p(\omega_k)
\]

where

\[
r_i(x, \omega_k) = \max \left[ 0, \sum_{k=1}^{K} (\overline{a}_{ijk} + \overline{e}_{ijk}) \cdot x_j - \overline{b}_{ik}^e \right]
\]

is a measure for risk.

With a risk aversion parameter \( d_i^e \in \mathbb{R}_0^+ \), which has to be chosen in advance and which may differ for the particular constraints, we demand

\[
E[r_i(x)] \leq d_i^e \tag{13}
\]

Obviously, the feasibility set

\[
X(d_i^e) = \{ x \in \mathbb{R}^n \mid E[r_i(x)] \leq d_i^e \}
\]

is a non-decreasing function of the risk aversion parameter \( d_i^e \).

Using the risk definition (13) as an additional constraint, the crisp constraints (10) may be weakened to

\[
\sum_j (\overline{a}_{ijk} + \overline{e}_{ijk}) \cdot x_j \leq \overline{b}_{ik}^e + d_i^e \cdot p(\omega_k)
\]

In doing so, the decision maker has the possibility to increase the set of feasible solutions of (9). Naturally, the specifying of the risk aversion parameters \( d_i^e \), \( i = 1, 2, \ldots, m \), is not trivial and require experience. As the solution of a multi-criteria fuzzy linear programming problem should be calculated by means of an interactive solution process, the decision maker should start with \( d_i^e = 0 \).

On the basis of the available solution, the expected values \( E[r_i(x)] = \sum_{k=1}^{K} r_i(x, \omega_k) \cdot p(\omega_k) \) may be calculated as orientation for the choice of \( d_i^e \).

III. Retaining the objective functions for all states of nature instead of the mean value

Using the FULPAL for getting a compromise solution of system (9), the decision maker has to specify an aspiration level, for the mean value

\[
E(\tilde{Z}(x)) = \sum_{j=1}^{n} \sum_{k=1}^{K} \tilde{C}_{ij}(\omega_k) \cdot p(\omega_k) \cdot x_j \tag{15}
\]

But, it is very difficult to do this in an intelligent manner, because this fixing don't allow an inference on the values for the original objective functions.

Instead of maximizing the mean value (15) we propose to use the original objective functions of the systems of type (8). Then, using the solution process FULPAL, the decision maker has to specify for each state of nature and for each objective function \( z_k(x) \) an aspiration level \( \lambda_k^* \). Analogous to the modification for the constraints, these aspiration levels should also be reduced according to the probabilities \( p(\omega_k) \).

\[
\lambda_k^*(p) = \lambda_k^* - (\lambda_k^* - z_k) \cdot (1 - p(\omega_k)) = z_k + (\lambda_k^* - z_k) \cdot p(\omega_k)
\]

where \( z_k \) is the smallest value, the decision maker is willing to accept for the objective function \( z_k(x) \) on the membership level 1.
4. Conclusions

The new approach is a general interactive solution process for solving multi-criteria linear programming systems with crisp, fuzzy or stochastic data. As special cases, it includes classical LP-systems, stochastic linear systems and fuzzy linear systems. In any case, the process should be done interactively by changing stepwise the aspiration levels and/or the risk aversion parameter. Moreover, FULPAL offers the possibility to use more flexible extended additions of the left-hand side of the constraints, see [8].

5. References