Non-linear dynamics of a continuous spring–block model of earthquake faults

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Abstract

The continuous one-dimensional Burridge–Knopoff model is generalized by introducing plastic creep in addition to rigid sliding. The resulting equations, for an order parameter (sliding rate) and a control parameter (driving force), exhibit a velocity-strengthening and a velocity-softening instability. In the former regime, considered to be the analog of self-organized criticality in continuum systems, anomalous diffusion is described by a non-linear diffusion equation. The latter regime, characteristic of deterministic chaos, is described by a time-dependent Ginzburg–Landau equation.

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The dynamics of non-trivial spring-block models of earthquake faults has regained attention since Bak and Tang [1] argued that the earth crust may be considered to be in a self-organized critical state. They showed that if stick–slip dynamics of a slowly driven system is computed by a cellular-automaton algorithm the system evolves into a statistical steady state with power law correlations in time and in space. These correlations compare favorably to experimentally observed power law correlations for earthquakes (for example, to the Gutenberg–Richter law for the frequency distribution of energy release, or to the Omori law for the distribution of aftershocks [2]). The statistical steady state that exemplifies scaling properties similar to those at a critical point but does not require careful tuning of an external variable has been identified as self-organized criticality (SOC) [3–5].

A complementary approach to obtaining power law scaling has been based on continuous, deterministically chaotic models. In particular, the properties of continuous velocity-softening models, whose simplest form is known as the Burridge–Knopoff model [6], have been studied extensively; see, for example Refs. [7,8]. These investigations showed that power law scaling is reproduced even in the absence of an explicit stochastic element in the continuous differential equation: randomness is introduced only via the initial conditions.

Even though both approaches reproduce power-law scaling, the connection between continuum and cellular automata models has not been fully elucidated, nor has the origin of SOC been unequivocally identified. SOC in continuum systems has been attributed to singular diffusion [9] and to the presence of conservation laws that lead to a non-linear diffusion equation [10]. Recently, an attempt has been made to reconcile discrete (cellular automata) and continuum SOC versions by Gil and Sornette [11]. They proposed a Landau–Ginzburg theory of self-organized criticality based on the coupled dynamics of a control and an order parameter. SOC was identified with an *uphill diffusion* of the control parameter close to a critical point (spinodal point). However, they did not explicitly construct a physical model that exhibits the proposed feedback mechanism, and they found that external noise is essential in obtaining scaling.

We present a generalization of the continuous, one-dimensional uniform Burridge–

Knopoff (BK) model and we analyze its stability properties. The generalization is motivated by physical arguments. Since the model is general enough it is suggested that it may become a vehicle for the identification of similarities and differences between SOC and deterministic chaos. We show that depending on driving conditions the system either evolves into a self-organized critical state or its time evolution becomes deterministically chaotic.

The standard BK model of earthquake faults describes the dynamics of displacements u of a slowly driven spring–block chain of masses m in the presence of a non-linear dynamic friction Φ. The masses (of characteristic extension ξ) are longitudinally coupled by coil springs of stiffness $k_{\rm l}$ and they are transversally coupled to the driving interface by leaf springs of stiffness $k_{\rm t}$. The essential feature of the model is the choice of the friction force that generates a velocity-softening instability.

The generalization we propose is to account for fault creep via the introduction of an internal variable. Specifically, under the imposed driving the fault responds by a combination of rigid translation $u_{\rm s}$ (sliding) and plastic displacement $u_{\rm p}$ (irreversible deformation by creep of some boundary layer of the fault). The corresponding equations of motion become (primes denote spatial derivatives and dots time derivatives)

$$m\ddot{u}_{\rm s} = F - \Phi(\dot{u}_{\rm s} + \dot{u}_{\rm p}) \quad , \tag{1a}$$

$$\dot{F} = k_{\rm l}\xi^2 \dot{u}_{\rm s}'' - k_{\rm t}(\dot{u}_{\rm s} - \bar{v}) - \frac{F - F_{\rm y}}{\tau_{\rm p}} \quad . \tag{1b}$$

The first equation describes balance of forces (Newton's law) and the second the time evolution of the driving shear force F. The external loading rate is $k_{\rm t}\bar{v}$ where $\bar{v}$ is the tectonic drift velocity. The first term on the RHS of Eq. 1b arises from longitudinal compression or tension, the second from elastic shear, and the last gives the force relaxation due to plastic deformation under the assumption that the plastic displacement rate depends linearly on the driving force,

$$\dot{u}_{\rm p} = \frac{F - F_{\rm y}}{k_{\rm t}\tau_{\rm p}} \quad . \tag{2}$$

The critical force corresponding to plastic yielding is denoted by $F_{\rm y}$ and $\tau_{\rm p}$ is the characteristic time of plastic relaxation of the shear forces. In what follows we neglect $F_{\rm y}$, without loss of generality.

In analogy to critical phenomena the two coupled differential equations may be interpreted as a system of equations that describe the dynamics of a *control parameter* F (or the plastic displacement rate $\dot{u}_{\rm p}$) and an *order parameter* $\dot{u}_{\rm s}$ (the sliding rate). This compares with the proposed feedback mechanism of Gil and Sornette [11]. Moreover, such a description of the dynamics of the fault is reminiscent of previous analyses of laboratory friction data [12,13] where a set of two coupled differential equations for the friction stress, a parameter that characterizes the evolving state of the surface, and a constitutive equation have been proposed. For the constitutive relation of Ref. [14] and in the absence of inhomogeneities ($\dot{u}_{\rm s}'' = 0$) these equation may be re-written in a form similar to ours: the equation for the driving force 1b becomes identical, whereas the equation corresponding to Eq. 1a contains additional non-linearities whose physical origin is difficult to justify [15].

A schematic diagram of the friction force we will be considering is shown in Fig. 1: for small $\dot{u} = \dot{u}_{\rm s} + \dot{u}_{\rm p}$ slow, stable creep is described whereas at higher values velocity-softening is obtained. Note that plastic deformation introduces memory-dependent friction which is related to the aging of the fault by plastic accommodation of its interface, while previous analyses of the BK model have usually considered only the velocity-softening part of the friction force. The importance of using a more realistic friction force that allows for viscous creep has already been noted by Carlson and Langer [16].

Time and space are rendered dimensionless by defining $\tilde{t} = t/\tau_{\rm p}$ and $\tilde{x} = (k_{\rm t}/k_{\rm l})^{1/2} x/\xi$, while the dimensionless sliding rate is defined by $e = \dot{u}_{\rm s}\tau_{\rm p}/l_{\rm p}$, the dimensionless shear force by $f = F/(k_{\rm t} l_{\rm p})$, and the dimensionless friction force by $\phi = \Phi/(k_{\rm t} l_{\rm p})$. By this scaling we have introduced a characteristic length scale $l_{\rm p} = \tau_{\rm p}\bar{v}$. In terms of the scaled fields Eqs. 1 become

$$\epsilon^2 \dot{e} = f - \phi[\bar{v}(e+f)] \quad , \tag{3a}$$

$$\dot{f} = e'' - e - f + 1 \quad . \tag{3b}$$

Besides the set of constants defining the friction force ϕ Eqs. 3 depend on two parameters, the drift velocity $\bar{v}$ and a new parameter

$$\epsilon = \frac{1}{\tau_{\rm p}} \left(\frac{m}{k_{\rm t}} \right)^{1/2} \quad , \tag{4}$$

which is the ratio of the natural frequency of transversal oscillations of individual blocks to the plastic relaxation time. It is tempting to identify the $\epsilon \to 0$ ($\tau_{\rm p} \to \infty$) limit as the standard, homogeneous BK model. However, inspection of the scaling relations and Eq. 2 shows that the BK limit corresponds to $\tau_{\rm p} \to \infty$ with $\dot{u}_{\rm p} \to 0$, namely creep is suppressed while the friction force remains constant. The natural limit of Eqs. 3 is $\epsilon \to 0$ keeping the plastic displacement rate $\dot{u}_{\rm p}$ fixed since the plastic relaxation time $\tau_{\rm p}$ and the characteristic length scale $l_{\rm p}$ go to infinity at the same rate (for fixed $\bar{v}$). The physical interpretation of this limit is that an increase of the plastic relaxation time is associated with an increase of the driving force F since force relaxation by creep becomes less efficient. Consequently, the limit that gives the standard BK model is not physical reasonable in our framework since the shear force and the plastic relaxation time are interrelated (cf. Eq. 2) [15].

The limit $\epsilon \to 0$ corresponds to an adiabatic elimination of the fast variable e (the order parameter $\dot{u}_{\rm s}$). This yields the evolution equation for the slow variable f (control parameter F)

$$\dot{f} = [D(f) f']' - \frac{1}{\bar{v}} \zeta(f) + 1 \quad , \tag{5a}$$

$$D = \frac{1}{\bar{v}} \partial_f \zeta(f) - 1 \quad , \tag{5b}$$

where we have defined ζ to be the inverse function of ϕ, $\zeta(f) \equiv \phi^{-1}(f)$. Since the friction force is not injective ζ is defined only in the part of the aging regime where the slope of the friction force is positive (*cf.* Fig. 1). Equation 5b shows that for $\partial_f \zeta / \bar{v} < 1$ (friction-force slope greater than unity) the diffusion coefficient becomes negative: $D < 0$. In this regime the system is driven towards the friction-force maximum via *uphill diffusion*. The requirement that the friction-force slope be greater than unity implies that the short-term response of the system be smaller than the steady-state response. Whether this condition is physically realizable (even in a system with memory) on a macroscopic scale is questionable [15]. Moreover, as the friction-force maximum is approached D becomes singular ($\partial_f \zeta \to \infty$). However, close to this maximum the adiabatic approximation is expected to break down as the time scale associated to diffusive relaxation of the control parameter vanishes.

It is apparent that this *diffusion-like instability*, which is not present in the original BK model, contains many of the features previously attributed to SOC. It is characterized by an uphill diffusion and, in the adiabatic approximation, the diffusion constant becomes singular at the friction-force maximum. Either criterion has been identified as characteristic of SOC in continuum systems [9–11].

The nature of instabilities in Eqs. 3 is elucidated by performing a linear stability analysis of the uniform, steady-state solution

$$e_0 = 1 - \phi(\bar{v}) \quad , \qquad f_0 = \phi(\bar{v}) \quad , \tag{6}$$

to small perturbations $[\delta e(0), \delta f(0)] \exp(\omega t + \mathrm{i}kx)$. The roots of the characteristic polynomial are

$$\omega_\pm = \frac{1}{\epsilon^2} \left(\mu \pm \{\mu^2 - \epsilon^2 [1 + (1 - \phi_0^{(1)}) k^2]\}^{1/2} \right) \quad , \tag{7a}$$

where

$$\mu = -(\epsilon^2 + \phi_0^{(1)})/2 \quad \text{with} \quad \phi_0^{(n)} \equiv \bar{v}^n \partial_{\bar{v}}^n \phi(\bar{v}) \quad . \tag{7b}$$

We identify two kinds of instabilities depending on the sign of μ. For $\mu \geq 0$ a transition occurs from a stable focus to an unstable focus via a *Hopf bifurcation*. The corresponding purely imaginary eigenvalues are

$$\omega_\pm = \pm \frac{\mathrm{i}}{\epsilon} [1 + (1 + \epsilon^2) k^2]^{1/2} \quad \text{for} \quad \mu = 0 \quad . \tag{8}$$

This velocity-softening instability has been identified previously [16] in the standard BK model, except that the introduction of plastic deformation shifts the instability to a finite negative value of the stead-state velocity sensitivity ($\phi_0^{(1)} < -\epsilon^2$), *cf.* Fig. 1. This is a consequence of the stabilizing influence of the plastic deformation. Furthermore, by an appropriate physical interpretation of the relaxation time $\tau_{\rm p}$ [15], the Hopf bifurcation criterion can be mapped onto the criterion for stick–slip instabilities proposed by Ruina [12].

The dynamics of the system close to the oscillatory instability (in the post-bifurcation regime) and the nature of the Hopf bifurcation are investigated in terms of a time-dependent Ginzburg–Landau equation (TDGL) for a complex order parameter A [17]. This equation is derived by introducing two new variables

$$g = e + f - 1 \quad , \qquad h = f - f_0 \quad , \tag{9}$$

and by expanding the friction force ϕ about the steady state solution ($g = 0$). If terms up to third order are kept the expansion yields

$$\dot{g} = g'' - h'' - \left(1 + \frac{\phi_0^{(1)}}{\epsilon^2}\right) g - \frac{\phi_0^{(2)}}{2\epsilon^2} g^2 - \frac{\phi_0^{(3)}}{6\epsilon^2} g^3 + \frac{1}{\epsilon^2} h \quad ,$$

$$\dot{h} = g'' - h'' - g \quad . \tag{10}$$

This set of coupled evolution equations may be viewed as generalized reaction–diffusion equations [18]. Close to the bifurcation point and under the assumption of time scale separation reduced dynamical equations may be derived. The slow modes which govern the dynamics of the system are identified by scaling space and time according to $x \to x/\epsilon$ and $t \to t/\epsilon^2$, where ϵ is assumed to be small. The reduced evolution equations derive in a standard way [15,17] by performing an expansion of the solution vector in terms of the complex eigenvectors of the linear evolution operator followed by an expansion in a small parameter that measures the distance to the bifurcation point.

An appropriate choice of scaling factors results in a TDGL equation which, expressed in a standard form, is

$$\partial_T A = -{\rm i}\, \partial_X^2 A + A - (1 + {\rm i}c)|A|^2 A \quad , \tag{11}$$

and which depends on a single dimensionless parameter

$$c = \frac{2(\phi_0^{(2)})^2}{3\epsilon\phi_0^{(3)}} > 0 \quad , \quad c \quad \text{real} \quad . \tag{12}$$

Here T and X are rescaled time and space variables that evolve on the slow scales [15]. As the amplitude A scales like $1/\sqrt{\mu}$ it follows that the Hopf bifurcation is supercritical (continuous transition), in agreement with experimental results on dry-friction dynamics [19]. The constant c is positive ($\phi_0^{(3)} > 0$) because we may assume that the inflection point in ϕ occurs at velocities larger than those that define the bifurcation point (which is close to the maximum of ϕ).

It is well-known that Eq. 11 possesses a spatially uniform oscillating in time solution $A(T) = \exp(-\mathrm{i}cT)$ that describes the motion on the limit cycle enclosing the bifurcation point in phase space. This uniform motion is unstable with respect to finite wavelength perturbations (sideband instability or modulational instability [20,21]). This is seen [22] by writing the solution of Eq. 11 as

$$\begin{aligned} A(X,T) = \rho(T)\exp[-i(cT+\theta)] + a(T)\exp(i\Omega T + iQX) \\ + b(T)\exp[-2i(cT+\theta) - i\Omega T - iQX] \quad . \end{aligned} \tag{13}$$

Substitution of this *ansatz* in Eq. 11 and neglect of higher harmonics results in a set of six, coupled, non-linear ordinary differential equations for the complex amplitudes a and b, the real amplitude ρ, and the phase θ [23]. These are Lorenz-like equations for the amplitudes associated to the original TDGL equation. The stability of the linearized equations (about $\rho = 1$, $a = b = \theta = 0$) is determined by the real part of the eigenvalues of the associated matrix, that are

$$\lambda_\pm = -1 \pm (1 + 2cQ^2 - Q^4)^{1/2} \quad . \tag{14}$$

For $c > 0$ [24] the uniform state is unstable (λ_+ positive) in the range $0 < |Q| < \sqrt{2c}$ with maximum growth occurring at $|Q| = \sqrt{c}$ (see, also, Ref. [15]).

The linearized amplitude equations have been analyzed to study the transition to chaos in the TDGL equation which is associated with the existence of a strange attractor [21]. For the purposes of this work suffices to comment that the transition to chaos depends sensitively on the friction force law (in particular on the sign of the third derivative). A deeper understanding of this transition may have implications for the dynamics of aftershocks.

In addition to the oscillatory instability the original equations exhibit another instability occurring for $\mu < 0$. For $\phi_0^{(1)} > 1$ the second term under the square root in Eq. 7 becomes positive, and an instability associated to small wavelengths

$$k^2 \geq (\phi_0^{(1)} - 1)^{-1} \quad . \tag{15}$$

appears. This instability has no analogue in the standard BK model and it is closely related to the memory friction introduced in our generalized model. We identify it as the *diffusion-like instability* that appeared in the analysis of the adiabatic elimination of the fast variable. As remarked earlier, condition 15 imposes restrictions on the relative magnitudes of the short-term and the steady-state response of the system.

The generalization of the BK model presented herein consisted in (i) the use of a more realistic friction-force law and (ii) the introduction of a control-parameter-like internal variable that allows modeling the aging of the fault by plastic accommodation of its interface (fault creep). Two regimes were identified. In the velocity-strengthening regime a finite-wavelength instability leads (in an adiabatic elimination) to a state with a singular diffusion coefficient. This regime has been previously identified as characteristic of SOC in continuum systems. The other regime, in the velocity-softening region, is characterized by an oscillatory instability similar to the one found in the standard BK mode (Hopf bifurcation). We analyzed the behaviour close to the bifurcation point by deriving a TDGL equation. Hence, as a consequence of memory friction the generalized model shows complex behaviour that can be used to analyze material aspects of fault friction, the origin of scaling, and the question of earthquake predictability. Numerical simulations of the model and the various approximate equations in the two regimes are in progress.

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FIGURES

FIG. 1. The non-linear friction force assumed in the generalized BK model. Note the velocity-strengthening and velocity-softening region, separated by the friction-force maximum $\hat{\Phi}$ at $\hat{u}$. Typical positions of the diffusion-like and the oscillatory instabilities are shown. The aging and the inertia regimes are presented for completeness.