FINITE ELEMENT DISCRETIZATION OF THE 'PARABOLIC' EQUATION IN AN UNDERWATER VARIABLE BOTTOM ENVIRONMENT

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Abstract. The standard 'parabolic' approximation to the Helmholtz equation is used in order to model long-range propagation of sound in the sea in the presence of cylindrical symmetry in a domain with a rigid bottom of variable topography. The rigid bottom is modeled by a homogeneous Neumann condition and a paraxial approximation thereof proposed by Abrahamsson and Kreiss. The resulting initial-boundary-value problems are transformed, by a change of variables, to equivalent ones in a horizontal strip and, subsequently, are solved numerically by fully discrete finite element-Crank-Nicolson schemes. L^2 and H^1 error estimates of optimal order are proved for the numerical schemes in the case of upsloping bottoms for the Neumann boundary condition and for general bottom topographies in the case of Abrahamsson-Kreiss conditions. Results of numerical experiments are presented in domains with various bottom topographies. Some of these provide motivation for an alternative implementation of the Neumann boundary condition that yields improved numerical approximations.

Key words. Linear Schrödinger evolution equation, parabolic approximation, underwater acoustics, finite element methods, error estimates, non-cylindrical domain, rigid bottom boundary condition, Crank-Nicolson time stepping.

AMS subject classifications. 65M60, 65M12, 65M15, 76Q05

1. Introduction

We consider the Helmholtz equation (HE) in cylindrical coordinates in the presence of cylindrical symmetry

$$\Delta p + k_0^2\,\eta^2(r,z)p = 0. \tag{HE}$$

Here $z \geq 0$ is the depth variable increasing downwards and $r \geq 0$ is the horizontal distance from a harmonic point source of frequency f_0 placed on the z axis. For simplicity we shall assume that the medium consists of a single layer of water of constant density, occupying the region, $0 \leq z \leq \ell(r)$, $r \geq 0$, between the free surface $z = 0$ and the range-dependent bottom $z = \ell(r)$ (see Fig. 1); $\ell = \ell(r)$ will be assumed to be smooth and positive. The function $p = p(r,z)$ is the acoustic pressure, $k_0 = \frac{2\pi f_0}{c_0}$ is a reference wave number, c_0 a reference sound speed, and $\eta(r,z)$ the index of refraction, defined as $\frac{c_0}{c(r,z)}$, where $c(r,z)$ is the speed of sound in the water. (HE) is supplemented by the surface 'pressure-release' condition $p(r,0) = 0$. In the case of a soft bottom the homogeneous Dirichlet boundary condition

$$p = 0 \quad \text{at} \quad z = \ell(r) \tag{D}$$

is assumed to hold. The case of a rigid bottom is modeled by a Neumann boundary condition (with $\dot{\ell} = \frac{d\ell}{dr}$)

$$p_z - \dot{\ell}(r)p_r = 0 \quad \text{at} \quad z = \ell(r). \tag{N}$$

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Applying the change of variables $p(r,z)=\psi(r,z)\,\frac{e^{\mathrm{i}k_0 r}}{\sqrt{k_0 r}}$, assuming that $|2\mathrm{i}k_0\psi_r|\ggg|\psi_{rr}|$ (narrow-angle paraxial approximation) and neglecting terms of $O(\frac{1}{r^2})$ (far-field approximation) we arrive (cf., e.g., [13], [10], [6]) at the standard Parabolic Equation (PE), which is a linear partial differential equation of Schrödinger type of the form

(PE) $$\psi_r=\frac{\mathrm{i}}{2k_0}\,\psi_{zz}+\mathrm{i}\,\frac{k_0}{2}\,(\eta^2(r,z)-1)\,\psi,$$

where $\psi=\psi(r,z)$ is a complex-valued function of the two real variables r and z. The (PE) has been widely used in underwater acoustics to model one-way, long-range sound propagation near the horizontal plane of the source, in inhomogeneous, weakly range-dependent marine environments. Its solution will be sought in the domain $0\le z\le \ell(r)$, $r\ge 0$. The (PE) will be supplemented by an initial condition $\psi(0,z)=\psi_0(z)$, $0\le z\le\ell(0)$, modeling the source at $r=0$, the surface boundary condition $\psi=0$ for $z=0$, $r\ge0$, and a bottom boundary condition obtained by transforming (D) or (N). The Dirichlet boundary condition (D) remains of the same type ($\psi=0$ at $z=\ell(r)$) while the Neumann boundary condition (N) is transformed to a condition of the form

(PN) $$\psi_z-\dot\ell(r)\,\psi_r-g_{B}(r)\,\dot\ell(r)\,\psi=0\quad\text{at}\quad z=\ell(r),$$

where $g_B(r)$ is complex-valued and is usually taken simply as $\mathrm{i}\,k_0$. The analysis of this initial-boundary-value problem is complicated in the case of the Neumann boundary condition, when $\dot\ell(r)$ is not the zero function, due to the presence of the term ψ_r in (PN). In [1] Abrahamsson and Kreiss proved existence and uniqueness of solutions, in the case of a strictly monotone bottom, i.e. when $\dot\ell(r)$ is of one sign for $r\ge0$. The same authors proposed in [2], in place of (PN), the 'paraxialized' rigid bottom boundary condition

(AK) $$\psi_z-\mathrm{i}\,k_0\,\dot\ell(r)\,\psi=0\quad\text{at}\quad z=\ell(r),$$

which has several advantages over (PN) from a theoretical and a numerical point of view.

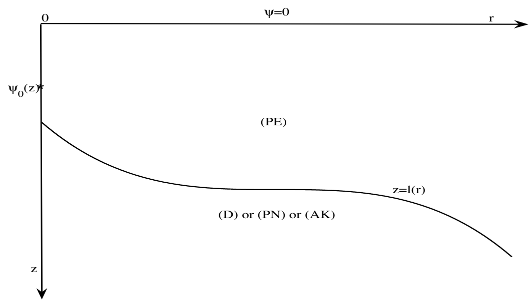


Figure 1. The domain of the initial-boundary-value problems for the (PE) in the r, z variables

We shall transform the above initial-boundary-value problems to equivalent ones posed on a horizontal strip. With this aim in mind, we first introduce non-dimensional variables as in [4], defined by $y:=\frac{z}{L}$, $t:=\frac{r}{L}$, $w:=\frac{\psi}{\psi_{\mathrm{ref}}}$, where we take $L:=\frac{1}{k_0}$ and $\psi_{\mathrm{ref}}:=\max|\psi_0|$. Then, letting $s(t):=k_0\,\ell\big(\frac{t}{k_0}\big)$, $g(t)=k_0\,g_B\big(\frac{t}{k_0}\big)$, $\gamma(t,y):=\frac12\left[\eta^2\big(\frac{t}{k_0},\frac{y}{k_0}\big)-1\right]$ we see that the (PE) becomes

(1.1) $$w_t=\frac{\mathrm{i}}{2}\,w_{yy}+\mathrm{i}\,\gamma(t,y)\,w,\quad 0\le y\le s(t),\quad t\ge0.$$

We note that the index of refraction η, and consequently the function γ, may be taken to be complex-valued in order to model attenuation of sound in the water. The initial value becomes

(1.2) $$w(0,y)=w_0(y):=\frac{1}{\psi_{\mathrm{ref}}}\,\psi_0\big(\frac{y}{k_0}\big),\quad 0\le y\le s(0).$$

The surface pressure-release boundary condition remains the same, i.e.,

(1.3) $$w(t,0)=0,\quad t\ge0,$$

while the boundary conditions $\psi = 0$ at $z = \ell(r)$, (PN), and (AK), become, respectively

$$w(t, s(t)) = 0, \quad t \geq 0, \tag{1.4}$$

$$w_y(t, s(t)) - \dot{s}(t) \left[w_t(t, s(t)) + g(t)\, w(t, s(t)) \right] = 0, \quad t \geq 0, \tag{1.5}$$

and

$$\mathrm{i}\, w_y(t, s(t)) + \dot{s}(t)\, w(t, s(t)) = 0, \quad t \geq 0. \tag{1.6}$$

We now perform the range-dependent change of depth variable $x := \frac{y}{s(t)}$, that maps the domain of the problem onto the horizontal strip $0 \leq x \leq 1$, $t \geq 0$. We also make the transformation

$$u(t, x) = \exp(-\mathrm{i}\, \delta(t) x^2)\, w(t, s(t)x), \tag{1.7}$$

which defines the new field variable $u(t, x)$ for $0 \leq x \leq 1$, $t \geq 0$. In (1.7) $\delta(t) := \frac{\dot{s}(t)s(t)}{2}$, $t \geq 0$, where a dot denotes differentiation with respect to t. In terms of the new variables (1.1) becomes

$$u_t = \mathrm{i}\, a(t)\, u_{xx} + \mathrm{i}\, \beta(t, x) u, \quad 0 \leq x \leq 1, \quad t \geq 0, \tag{1.8}$$

where, for $0 \leq x \leq 1$, $t \geq 0$

$$\begin{aligned} a(t) &= \tfrac{1}{2s^2(t)}, \quad \beta(t, x) = \beta_R(t, x) + \mathrm{i}\beta_I(t, x), \\ \beta_R(t, x) &= \mathrm{Re}[\gamma(t, xs(t))] - \tfrac{\ddot{s}(t)s(t)x^2}{2}, \\ \beta_I(t, x) &= \mathrm{Im}[\gamma(t, xs(t))] + \tfrac{\dot{s}(t)}{2\, s(t)}. \end{aligned} \tag{1.9}$$

The purpose of introducing in (1.7) the factor $e^{-\mathrm{i}\, \delta(t) x^2}$ with $\delta = \frac{\dot{s}s}{2}$ is to avoid the presence of a u_x term in the right-hand side of (1.8) and, consequently, simplify somewhat the energy estimates. Under the transformation (1.7), the initial and boundary conditions (1.2)-(1.6) change accordingly. Specifically, we have

$$u(0, x) = u_0(x) := e^{-\mathrm{i}\delta(0)x^2} w_0(xs(0)) \quad \forall\, x \in [0, 1], \tag{1.10}$$

and

$$u(t, 0) = 0, \;\; t \geq 0. \tag{1.11}$$

The homogeneous Dirichlet bottom boundary condition (1.4) is, of course, preserved

$$u(t, 1) = 0, \;\; t \geq 0, \tag{1.12}$$

while the Neumann boundary condition (1.5) becomes

$$u_x(t, 1) = s_1(t) u_t(t, 1) + m(t) u(t, 1), \;\; t \geq 0, \tag{1.13}$$

where

$$s_1(t) := \tfrac{\dot{s}(t)s(t)}{1+\dot{s}(t)^2}, \;\; m(t) := g(t)s_1(t) + \mathrm{i}(s_1(t)\dot{\delta}(t) - 2\delta(t)), \;\; t \geq 0. \tag{1.14}$$

The Abrahamsson-Kreiss condition (1.6) becomes simply

$$u_x(t, 1) = 0, \;\; t \geq 0, \tag{1.15}$$

i.e. just a homogeneous Neumann condition with respect to the new depth variable x; this is another advantage of the specific choice of $\delta(t)$ in (1.7).

The proof of well-posedness of the initial-boundary-value problem (1.8), (1.10), (1.11) for t in a finite interval $[0, T]$ with either one of the boundary conditions (1.12) and (1.15) is standard, cf. [11]. As we have already mentioned, Abrahamsson and Kreiss proved in [1] the well-posedness of the problem with the boundary condition (1.13) assuming that $\dot{s}(t)$ is either positive or negative for all t in a finite interval $[0, T]$.

Our main purpose in this paper is to construct and analyze fully discrete Galerkin-finite element methods for approximating the solutions of the above initial-boundary-value problems with bottom boundary conditions given by (1.13) or (1.15). (The case of the homogeneous Dirichlet bottom boundary condition is standard, cf. e.g. [3]. One may, in a straightforward manner, prove L^2 error estimates of optimal order for Galerkin-finite element semidiscrete and various fully discrete schemes; we shall not dwell upon it any further.)

We first consider the initial-boundary-value problem with the exact Neumann bottom boundary condition, i.e. the problem consisting of (1.8)-(1.11) and (1.13)-(1.14). We assume that the bottom is *upsloping*, i.e. that $\dot{s}(t) \leq 0$, and that the problem has a unique solution, smooth enough for the purposes of the error estimation. In Section 2 we discretize the problem in x by the standard Galerkin method and prove optimal-order L^2 and H^1 estimates for the error of the resulting semidiscretization. This is achieved by using appropriate properties of the L^2 and the elliptic projections onto the finite element subspace and a relevant H^1 superconvergence result. (The difficulty of the problem lies in the presence of the u_t term in (1.13); the condition $\dot{s}(t) \leq 0$, which implies that $s_1(t) \leq 0$, is needed to obtain a basic energy inequality for the error of the semidiscretization). Subsequently we discretize the semidiscrete problem in the t variable using a Crank-Nicolson type method with a variable step-length. Again, under, the assumption that $\dot{s}(t) \leq 0$ for $0 \leq t \leq T$, we prove L^2 and H^1 error estimates which are of optimal order in x and t. We also consider the problem with the Abrahamsson-Kreiss condition, i.e. the initial-boundary value problem (1.8)-(1.11), (1.15). Under no restriction on the sign of $\dot{s}(t)$ we prove optimal order L^2 and H^1 error estimates for the semidiscrete Galerkin scheme and its Crank-Nicolson full discretization.

In Section 3 we present results of various numerical experiments that we performed for problems on variable domains with Neumann and Abrahamsson-Kreiss bottom boundary conditions, using the fully discrete finite element methods analyzed in Section 2, and, also, other numerical schemes for comparison purposes. As predicted by the theoretical stability and convergence analysis, the finite element scheme is stable and second-order accurate when Neumann boundary conditions are considered in domains with upsloping bottoms. (It also appears to be convergent in small scale problems with downsloping bottoms and also in more realistic examples if the downsloping bottom has very small slope.). The scheme with the Abrahamsson-Kreiss condition behaved well, as predicted by the theory, in all examples of bottoms of arbitrary shape that we ran.

When we compared the results of the schemes using both boundary conditions in the case of the upsloping and downsloping rigid bottom ASA wedge (a standard test problem for long range sound propagation in underwater acoustics, [8]), we found that in the upsloping case there was very good agreement between the two schemes. In the downsloping case, the scheme implementing the Neumann boundary condition was not convergent. This is in agreement with the results of Abrahamsson and Kreiss, [1], [2], who pointed out that for some downsloping bottom profiles, one may observe instabilities in the case of the Neumann boundary condition. On the other hand, the scheme with the Abrahamsson-Kreiss condition was convergent and its results agreed well with those furnished by the finite difference code IFD, [9], [10], implemented with the rigid bottom boundary condition option. The IFD scheme uses a discretized version of the Neumann boundary condition (PN), wherein the ψ_r term is replaced by the right hand side of (PE). To explain the stability of the model discretized by the IFD, we consider, in an Appendix of the paper at hand, an initial-boundary-value problem with this new boundary condition, in which we replace the second-order derivative with respect to the depth variable with at most first-order terms by differentiating first the PE with respect to the depth variable. We prove *a priori* L^2 estimates of the solution of this problem and demonstrate that the new formulation is readily amenable to a L^2-stable finite element discretization.

A final point of interest emerging from the numerical experiments is that, for some downsloping bottom profiles $s(t)$ with an inflection point at some $t = t^*$, we observed violent growth of the L^2-norm of the numerical solution of the problem with the Neumann boundary condition for $t > t^*$. This growth (blow-up?) of the solution seems to be a feature of the problem and not an artifact of the numerical scheme.

Error estimates for a finite difference scheme of second-order of accuracy in x and t for some of the initial-boundary-value problems considered here were proved in [4]. In the case of the Neumann boundary condition

(1.13) these error estimates were shown to hold not only when $\dot{s}(t) \le 0$ but also in the strictly downsloping case $\dot{s}(t) > 0$, $t \in [0,T]$, as a result of the validity of a certain discrete H^1 estimate; this estimate mimics an analogous H^1 estimate for the continuous problem which holds provided $\dot{s}(t) \le 0$ or $\dot{s}(t) > 0$ when $t \in [0,T]$.

In [12] Sturm considers the Abrahamsson-Kreiss bottom boundary condition for the (PE) in three dimensions over a variable bottom. (In fact, a multilayered fluid medium is considered in [12] with homothetic layers, across the interfaces of which the field is continuous and where an Abrahamsson-Kreiss paraxial approximation replaces the normal derivative in the usual transmission conditions of acoustics). The problem is mapped onto one with horizontal layers via a change of the depth variable that depends on range and azimuth; the resulting initial-boundary-value problem is discretized by a standard Galerkin method with C^0 rectangles in depth and azimuth with Crank-Nicolson discretization in range with uniform range step; L^2 error estimates are obtained. When restricted to single layer problems in the presence of azimuthal symmetry, the scheme of [12] is similar to the one analyzed herein (see Section 2). We have considerably modified the analysis of [12] and obtain optimal-order estimates, since, by using the transformation (1.7), we essentially avoid an elliptic projection with time-dependent terms.

A uniform range step version of the scheme of this paper and also three-dimensional extensions thereof were analyzed in [5]. For references to underwater acoustics computations with the (PE) coupled with change-of-variable techniques and a relevant discussion on its advantages and disadvantages we refer the reader to [12] and [4].

2. Numerical Schemes and error Estimates

2.1. Preliminaries. Let $D := (0,1)$. We will denote by $L^2(D)$ the space of the Lebesgue measurable complex-valued functions which are square integrable on D, and by $\|\cdot\|$ the standard norm of $L^2(D)$, i.e., $\|f\| := \{\int_D |f(x)|^2\,dx\}^{\frac{1}{2}}$ for $f \in L^2(D)$. The inner product in $L^2(D)$ that induces the norm $\|\cdot\|$ will be denoted by $(\cdot,\cdot)$, i.e. $(f_1,f_2) := \int_D f_1(x)\,\overline{f_2(x)}\,dx$ for $f_1,\ f_2 \in L^2(D)$. Also, we will denote by $L^\infty(D)$ the space of the Lebesgue measurable functions which are bounded a.e. on D, and by $|\cdot|_\infty$ the associated norm, i.e., $|f|_\infty := \operatorname{ess\,sup}_D |f|$ for $f \in L^\infty(D)$. For $s \in \mathbb{N}_0$, we denote by $H^s(D)$ the Sobolev space of complex-valued functions having generalized derivatives up to order s in $L^2(D)$, and by $\|\cdot\|_s$ its usual norm, i.e. $\|f\|_s := \big\{\sum_{\ell=0}^s \|\partial_x^\ell f\|^2\big\}^{\frac{1}{2}}$ for $f \in H^s(D)$. In addition, we set $|v|_1 := \|v'\|$ for $v \in H^1(D)$. Also, $\mathbb{H}^1(D)$ will denote the subspace of $H^1(D)$ consisting of functions which vanish at $x=0$ in the sense of trace; we set $\mathbb{H}^s(D) = H^s(D) \cap \mathbb{H}^1(D)$ for $s \ge 2$. In addition, for $s \in \mathbb{N}_0$, we denote by $W^{s,\infty}(D)$ the Sobolev space of complex-valued functions having generalized derivatives up to order s in $L^\infty(D)$, and by $|\cdot|_{s,\infty}$ its usual norm, i.e. $|f|_{s,\infty} := \max_{0\le\ell\le s} |\partial_x^\ell f|_\infty$ for $f \in W^{s,\infty}(D)$. In what follows, C will denote a generic constant independent of the discretization parameters and having in general different values at any two different places.

For later use, we recall the well-known Poincaré-Friedrichs inequality

$$\|v\| \le C\,|v|_1 \quad \forall\, v \in \mathbb{H}^1(D), \tag{2.1}$$

the Sobolev-type inequality

$$\|v\|_\infty \le |v|_1 \quad \forall\, v \in \mathbb{H}^1(D) \tag{2.2}$$

and the trace inequality

$$|v(1)|^2 \le 2\,\|v\|\,|v|_1 \quad \forall\, v \in \mathbb{H}^1(D). \tag{2.3}$$

Let $r \in \mathbb{N}$ and S_h be a finite dimensional subspace of $\mathbb{H}^1(D)$ consisting of complex-valued functions that are polynomials of degree less or equal to r in each interval of a non-uniform partition of D with maximum length $h \in (0,h_\star]$. It is well-known [7], that the following approximation property holds:

$$\inf_{\chi\in S_h} \big\{\|v-\chi\| + h\|v-\chi\|_1\big\} \le C\,h^{s+1}\,\|v\|_{s+1}, \quad \forall\, v \in \mathbb{H}^{s+1}(D),\ \forall\, h \in (0,h_\star], \quad s = 0,\dots,r. \tag{2.4}$$

Also, we assume that the following inverse inequality holds

$$|\phi|_1 \leq C\, h^{-1}\, \|\phi\| \quad \forall\, \phi \in S_h, \quad \forall\, h \in (0, h_\star], \tag{2.5}$$

which is true when, for example, the partition of D is quasi-uniform, [7]. In addition, we define the L^2-projection operator $P_h : L^2(D) \to S_h$ by

$$(P_h v, \phi) = (v, \phi) \quad \forall\, \phi \in S_h, \quad \forall\, v \in L^2(D),$$

and the elliptic projection operator $R_h : H^1(D) \to S_h$ by

$$\mathcal{B}(R_h v, \phi) = \mathcal{B}(v, \phi) \quad \forall\, \phi \in S_h, \quad \forall\, v \in H^1(D), \tag{2.6}$$

where $\mathcal{B}$ is the sesquilinear form defined for $u,\, w \in H^1(D)$ by $\mathcal{B}(u,\, w) := (u',\, w')$. It follows, [7], [14], that there exists a positive constant C such that

$$\|R_h v - v\| + h\, \|R_h v - v\|_1 \leq C\, h^{s+1}\, \|v\|_{s+1}, \quad \forall\, v \in \mathbb{H}^{s+1}(D),\ \forall\, h \in (0, h_\star], \quad s = 0, \ldots, r. \tag{2.7}$$

Finally, for $v \in L^2(D)$, we define the discrete negative norm

$$\|v\|_{-1,h} := \sup\left\{ \tfrac{|(v,\phi)|}{|\phi|_1} : \quad \phi \in S_h \quad \text{and} \quad \phi \neq 0 \right\}, \quad \forall\, h \in (0, h_\star].$$

Lemma 2.1. *The elliptic projection operator R_h has the following property:*

$$R_h v(1) = v(1), \quad \forall\, v \in \mathbb{H}^1(D). \tag{2.8}$$

Proof. Let $v \in \mathbb{H}^1(D)$ and ω be the element of S_h given by $\omega(x) = x$ for $x \in \overline{D}$. Then (2.6) gives $R_h v(1) - v(1) = \mathcal{B}(R_h v - v, \omega) = 0$, which is the desired result. □

Lemma 2.2. *Let $\omega \in C^1(\overline{D})$. Then, there exists a constant $C > 0$ such that*

$$|P_h(\omega\phi)|_1 \ \leq\ C\, |\omega|_{1,\infty}\, |\phi|_1 \quad \forall\, \phi \in S_h, \quad \forall\, h \in (0, h_\star]. \tag{2.9}$$

Proof. Let $h \in (0, h_\star]$ and $\phi \in S_h$. Since $|P_h(\omega\phi)|_1 \leq |P_h(\omega\phi - R_h(\omega\phi))|_1 + |R_h(\omega\phi)|_1$, using (2.5) and (2.6) we arrive at $|P_h(\omega\phi)|_1 \leq C\, h^{-1}\, \|\omega\phi - R_h(\omega\phi)\| + |\omega\phi|_1$. Next, we use the estimate (2.7) for $s = 0$ to obtain $|P_h(\omega\phi)|_1 \ \leq\ C\, \big[\, |\omega|_\infty\, |\phi|_1 + |\omega'|_\infty \|\phi\|\, \big]$. Thus, the bound (2.9) follows by combining the latter inequality and (2.1). □

2.2. Semidiscretization-Neumann boundary condition. In this subsection, and the one that follows, we shall consider the PE with the Neumann boundary condition, i.e. the initial-boundary-value problem (1.8), (1.10), (1.11), (1.13). We shall write this problem in a slightly more general form, as follows. For $T > 0$ given, we seek a function $u : [0, T] \times \overline{D} \to \mathbb{C}$ satisfying

$$(\mathcal{N}) \qquad \begin{aligned} &u_t = \mathrm{i}\, a(t) u_{xx} + \mathrm{i}\, \beta(t, x)\, u + f(t, x),\ t \in\ [0, T],\ x \in\ \overline{D},\\ &u(0, x) = u_0(x),\ x \in\ \overline{D},\\ &u(t, 0) = 0,\ t \in\ [0, T],\\ &u_x(t, 1) = \mu(t)\big[S(t) u_t(t, 1) + G(t) u(t, 1)\big],\ t \in\ [0, T]. \end{aligned}$$

We shall assume that $a : [0, T] \to \mathbb{R} - \{0\}$, β, $f : [0, T] \times \overline{D} \to \mathbb{C}$, $u_0 : \overline{D} \to \mathbb{C}$, μ, $S : [0, T] \to \mathbb{R}$, $G : [0, T] \to \mathbb{C}$ are given functions. We shall assume that the solution u of $(\mathcal{N})$ exists uniquely, and that the data and the solution of $(\mathcal{N})$ are smooth enough for the purposes of the error estimates that will follow. (In the realistic numerical experiments of Section 3 we shall revert to the specific physical data in (1.9), (1.13), (1.14), and take the functions $a(t)$, $\beta(t, x)$ as in (1.9), $\mu(t) = \frac{\dot{s}(t)}{s(t)}$, $S(t) = \frac{s^2(t)}{1 + (\dot{s}(t))^2}$, $G(t) = g(t) S(t) + \mathrm{i}(S(t)\dot{\delta}(t) - s^2(t))$, where $\delta = s\dot{s}/2$.) The weak formulation of $(\mathcal{N})$, obtained by taking the $L^2(D)$ inner product of the p.d.e. in

($\mathcal{N}$) with a function in $\mathbb{H}^1(D)$, integrating by parts and using the boundary conditions, motivates defining $u_h : [0,T] \to S_h$, the semidiscrete approximation of u, by the equation

$$
\begin{aligned}
(\partial_t u_h(t,\cdot),\phi) = &\,\mathrm{i}\,a(t)\,\mu(t)\,\big[\,S(t)\,\partial_t u_h(t,1) + G(t)\,u_h(t,1)\,\big]\,\overline{\phi(1)}\\
&- \mathrm{i}\,a(t)\,\mathcal{B}(u_h(t,\cdot),\phi) + \mathrm{i}\,(\beta(t,\cdot)\,u_h(t,\cdot),\phi) + (f(t,\cdot),\phi) \quad \forall\,\phi\in S_h, \quad \forall\,t\in[0,T],
\end{aligned}
\tag{2.10}
$$

and

$$
u_h(0,\cdot) = R_h u_0(\cdot). \tag{2.11}
$$

Proposition 2.3. *The problem* (2.10)-(2.11) *admits a unique solution* $u_h \in C^1([0,T];S_h)$.

Proof. Let $\dim(S_h) = J$ and $\{\phi_j\}_{j=1}^{J}$ be a basis of S_h consisting of real-valued functions. Hence, we have $R_h u_0 = \sum_{j=1}^{J}\gamma_j^0\,\phi_j$ and $u_h(t,x) = \sum_{j=1}^{J}\gamma_j(t)\,\phi_j(x)$, where $\gamma_j : [0,T] \to \mathbb{C}$ for $j = 1,\dots,J$. Then (2.10)-(2.11) is equivalent to the following o.d.e. initial-value-problem: Find $\widetilde{G} \in C^1([0,T];\mathbb{C}^J)$ such that $\widetilde{G}(0) = \widetilde{G}^0$ and $\widetilde{A}(t)\,\widetilde{G}'(t) = \widetilde{B}(t)\,\widetilde{G}(t) + \widetilde{F}(t)$, $\forall\,t \in [0,T]$, where $\widetilde{G}(t) := (\gamma_1(t),\dots,\gamma_J(t))^T$, $\widetilde{G}^0 := (\gamma_1^0,\dots,\gamma_J^0)^T$, $\widetilde{A} : [0,T] \to \mathbb{C}^{J\times J}$ with $\widetilde{A}_{\ell j}(t) := (\phi_\ell,\phi_j) - \mathrm{i}\,a(t)\,S(t)\,\mu(t)\,\phi_\ell(1)\,\phi_j(1)$, $\widetilde{B} : [0,T] \to \mathbb{C}^{J\times J}$ with $\widetilde{B}_{\ell j}(t) := \mathrm{i}\,a(t)\,G(t)\,\mu(t)\,\phi_\ell(1)\,\phi_j(1) - \mathrm{i}\,a(t)\,\mathcal{B}(\phi_\ell,\phi_j) + \mathrm{i}\,(\beta(t,\cdot)\,\phi_\ell,\phi_j)$, and $\widetilde{F} : [0,T] \to \mathbb{C}^J$ with $\widetilde{F}(t) := ((f(t,\cdot),\phi_1),\dots,(f(t,\cdot),\phi_J))^T$. Since $\widetilde{A}$, $\widetilde{B}$, $\widetilde{F}$ are continuous maps, to ensure existence and uniqueness of the solution $\widetilde{G}$, it is sufficient to show that $\widetilde{A}(t)$ is nonsingular for $t \in [0,T]$. Indeed, letting $t \in [0,T]$ and $x \in \mathrm{Ker}(\widetilde{A}(t))$, we have $\mathrm{Re}(\overline{x}^T\widetilde{A}(t)x) = 0$, from which we conclude that $\big\|\sum_{j=1}^{J} x_j\phi_j\big\|^2 = 0$ and hence $x = 0$. □

Let us first present a H^1 superconvergence error estimate for the semidiscrete approximation u_h.

Proposition 2.4. *Let u be the solution of ($\mathcal{N}$) and u_h its semidiscrete approximation defined by* (2.10)-(2.11). *Assume that $\mu(t) \le 0$ and $S(t) > 0$ for $t \in [0,T]$. Then, there exists a constant $C > 0$ such that:*

$$
\|u_h(t,\cdot) - R_h u(t,\cdot)\|_1 \le C\,h^{r+1}\left(\int_0^t \Gamma(\tau)\,d\tau\right)^{\frac{1}{2}} \quad \forall\,t\in[0,T], \quad \forall\,h\in(0,h_\star], \tag{2.12}
$$

where $\Gamma(\tau) := \sum_{\ell=0}^{1}\|\partial_t^\ell u(\tau,\cdot)\|_{r+1}^2 + \sum_{\ell=0}^{2}\int_0^\tau \|\partial_t^\ell u(s,\cdot)\|_{r+1}^2\,ds$.

Proof. Let $h \in (0,h_\star]$, $\theta_h := u_h - R_h u$ and $\xi(t) := \frac{1}{a(t)}$. Using (2.6) we obtain

$$
\begin{aligned}
(\partial_t\theta_h(t,\cdot),\phi) = &\,\mathrm{i}\,a(t)\,\mu(t)\,\big[S(t)\,\partial_t\theta_h(t,1) + G(t)\,\theta_h(t,1)\,\big]\,\overline{\phi(1)}\\
&- \mathrm{i}\,a(t)\,\mathcal{B}(\theta_h(t,\cdot),\phi) + \mathrm{i}\,(P_h(\beta(t,\cdot)\,\theta_h(t,\cdot)),\phi)\\
&+ \big(\Psi_\star(t,\cdot),\phi\big) \quad \forall\,\phi\in S_h, \quad \forall\,t\in[0,T],
\end{aligned}
\tag{2.13}
$$

where $\Psi_\star := [\partial_t u - R_h(\partial_t u)] - \mathrm{i}\,\beta\,(u - R_h u)$. Set $\phi = \partial_t\theta_h$ in (2.13) and then take imaginary parts to obtain

$$
\begin{aligned}
\tfrac{d}{dt}|\theta_h(t,\cdot)|_1^2 \le &\;|\mu(t)|\big[-2\,S^\star\,|\partial_t\theta_h(t,1)|^2 + 2\,|G(t)|\,|\theta_h(t,1)|\,|\partial_t\theta_h(t,1)|\,\big]\\
&+ 2\,|\xi(t)|\;\|\partial_t\theta_h(t,\cdot)\|_{-1,h}\;|P_h(\beta(t,\cdot)\,\theta_h(t,\cdot))|_1\\
&+ 2\,\xi(t)\,\mathrm{Im}(\Psi_\star(t,\cdot),\partial_t\theta_h(t,\cdot)), \quad \forall\,t\in[0,T],
\end{aligned}
\tag{2.14}
$$

where $S^\star := \inf_{[0,T]} S > 0$. In order to bound properly the quantity $\|\partial_t\theta_h\|_{-1,h}$, first use (2.7) to obtain

$$
\|\Psi_\star(t,\cdot)\| \le C\,h^{r+1}\,\big[\,\|u(t,\cdot)\|_{r+1} + \|\partial_t u(t,\cdot)\|_{r+1}\,\big], \quad \forall\,t\in[0,T]. \tag{2.15}
$$

Then, use of (2.2) and (2.15) in (2.13) gives

$$\begin{aligned}\big|(\partial_t\theta_h(t,\cdot),\phi)\big| \le |a(t)|\,\Big[S(t)\,|\mu(t)|\,|\partial_t\theta_h(t,1)| + \big(\,|G(t)|\,|\mu(t)|+1\,\big)\,|\theta_h(t,\cdot)|_1\,\Big]\,|\phi|_1\\ + |\beta(t,\cdot)|_\infty\,\|\theta_h(t,\cdot)\|\,\|\phi\|\\ + C\,h^{r+1}\big(\,\|\partial_t u(t,\cdot)\|_{r+1} + \|u(t,\cdot)\|_{r+1}\,\big)\,\|\phi\|,\quad \forall\,\phi\in S_h,\quad \forall\,t\in[0,T],\end{aligned}$$

which, along with (2.1), yields that

$$\begin{aligned}2\,|\xi(t)|\,\|\partial_t\theta_h(t,\cdot)\|_{-1,h} \le\ 2\,S(t)\,|\mu(t)|\,|\partial_t\theta_h(t,1)|\\ + C\,\Big[\,|\theta_h(t,\cdot)|_1 + h^{r+1}\big(\,\|\partial_t u(t,\cdot)\|_{r+1} + \|u(t,\cdot)\|_{r+1}\,\big)\,\Big]\quad \forall\,t\in[0,T].\end{aligned}\tag{2.16}$$

Thus, combining (2.14), (2.16), (2.1), (2.2) and (2.9), we arrive at

$$\tfrac{d}{dt}|\theta_h|_1^2 \le\ C\ \big[\,|\theta_h|_1^2 + h^{2(r+1)}\,(\,\|\partial_t u\|_{r+1}^2 + \|u\|_{r+1}^2\,)\,\big] + 2\,\xi\ \mathrm{Im}(\Psi_\star,\partial_t\theta_h)\quad \text{on}\quad [0,T].$$

Since $\theta_h(0,\cdot)=0$, integrating with respect to t in the inequality above yields

$$\begin{aligned}|\theta_h(t,\cdot)|_1^2 \le C\,\Big[\int_0^t |\theta_h(s,\cdot)|_1^2\,ds + h^{2(r+1)}\int_0^t\big(\,\|\partial_t u(s,\cdot)\|_{r+1}^2 + \|u(s,\cdot)\|_{r+1}^2\,\big)\,ds\Big]\\ + \mathrm{Im}\Big\{\,2\,\xi(t)\,(\Psi_\star(t,\cdot),\theta_h(t,\cdot)) - 2\int_0^t \xi'(s)\,(\Psi_\star(s,\cdot),\theta_h(s,\cdot))\,ds\\ - 2\int_0^t \xi(s)\,(\partial_t\Psi_\star(s,\cdot),\theta_h(s,\cdot))\,ds\Big\}\quad \forall\,t\in[0,T].\end{aligned}$$

Using in the above the Cauchy-Schwarz inequality, (2.1) and (2.15), we obtain

$$|\theta_h(t,\cdot)|_1^2 \le\ C\int_0^t |\theta_h(s,\cdot)|_1^2\,ds + C\,h^{2(r+1)}\,\Gamma(t)\quad \forall\,t\in[0,T].\tag{2.17}$$

The estimate (2.12) follows from (2.17) using Gronwall's lemma and (2.1). □

A simple consequence of this superconvergence result and the approximation property (2.7) of the elliptic projection is the following convergence result:

Theorem 2.5. *Let u be the solution of $(\mathcal{N})$ and u_h its semidiscrete approximation defined by* (2.10)-(2.11). *Assume that $\mu(t)\le 0$ and $S(t)>0$ for $t\in[0,T]$. Then, there exists a constant $C>0$ such that*

$$\|u_h(t,\cdot)-u(t,\cdot)\| + h\,\|u_h(t,\cdot)-u(t,\cdot)\|_1 \le\ C\,h^{r+1}\left(\|u(t,\cdot)\|_{r+1}^2 + \int_0^t \Gamma(\tau)\ d\tau\right)^{\frac12}\quad \forall\,t\in[0,T],\tag{2.18}$$

where Γ is the function defined in the statement of Proposition 2.4. □

Therefore, taking into account the relation of a, μ and S to the function $s(t)$ describing the bottom topography, we conclude that the error estimate of Theorem 2.5 holds in the case of domains with upsloping bottom profiles, i.e. when $\dot{s}(t)\le 0$, $t\in[0,T]$.

Remark. The H^1 superconvergence estimate (2.12), (2.2), and a standard L^∞ estimate for the error of the elliptic projection yield as usual an optimal-order L^∞ error estimate for $0\le t\le T$.

2.3. A Crank-Nicolson fully discrete method-Neumann boundary condition. Let $N\in\mathbb{N}$ and $\{t^n\}_{n=0}^N$ be the nodes of the partition of $[0,T]$ defined by $t^0=0$, $t^N=T$ and $t^n<t^{n+1}$ for $n=0,\dots,N-1$. Define $k_n:=t^n-t^{n-1}$ for $n=1,\dots,N$, $t^{n+\frac12}:=\frac{t^n+t^{n+1}}{2}$ for $n=0,\dots,N-1$, and $k:=\max_{1\le n\le N}k_n$. We set $u^n:=u(t^n,\cdot)$ for $n=0,\dots,N$, where u is the solution of $(\mathcal{N})$. Finally, for functions $V^m\in S_h$, $m=1,\cdots,N$, we define $\partial V^n:=\frac{1}{k_n}(V^n-V^{n-1})$ and $\mathcal{A}V^n=\frac12(V^n+V^{n-1})$ for $n=1,\dots,N$.

For $n=0,\dots,N$, the Crank-Nicolson method constructs a fully discrete approximation $U_h^n\in S_h$ of $u(t^n,\cdot)$ as follows:

`Step 1`: Set

$$U_h^0 := R_h u_0. \tag{2.19}$$

`Step 2`: For $n = 1, \dots, N$, find $U_h^n \in S_h$ such that

$$\begin{aligned}(\partial U_h^n, \chi) = &\,\mathrm{i}\, a^{n-\frac{1}{2}}\, \mu^{n-\frac{1}{2}} \left[S^{n-\frac{1}{2}}\, \partial U_h^n(1) + G^{n-\frac{1}{2}}\, \mathcal{A} U_h^n(1) \right] \overline{\chi(1)} \\ &- \mathrm{i}\, a^{n-\frac{1}{2}}\, \mathcal{B}\big(\mathcal{A}U_h^n, \chi\big) + \mathrm{i}\, \big(\beta^{n-\frac{1}{2}}\, \mathcal{A}U_h^n, \chi\big) + \big(f^{n-\frac{1}{2}}, \chi\big), \quad \forall\, \chi \in S_h,\end{aligned} \tag{2.20}$$

where $S^{n-\frac{1}{2}} := S(t^{n-\frac{1}{2}})$, $\mu^{n-\frac{1}{2}} := \mu(t^{n-\frac{1}{2}})$, $a^{n-\frac{1}{2}} := a(t^{n-\frac{1}{2}})$, $G^{n-\frac{1}{2}} := G(t^{n-\frac{1}{2}})$, $f^{n-\frac{1}{2}} := f(t^{n-\frac{1}{2}}, \cdot)$ and $\beta^{n-\frac{1}{2}} := \beta(t^{n-\frac{1}{2}}, \cdot)$.

We first examine the problem of existence and uniqueness of the fully discrete approximation U_h^n.

Proposition 2.6. *Let $n \in \{1, \dots, N\}$ and suppose that $U_h^{n-1} \in S_h$ is well-defined. If $S^{n-\frac{1}{2}} > 0$ and $\mu^{n-\frac{1}{2}} \le 0$, then, there exists a constant C_n such that if $k_n < C_n$, then U_h^n is well-defined by* (2.20).

Proof. Since (2.20) is equivalent to a linear system of algebraic equations with unknowns the coefficients of U_h^n with respect to a basis of S_h, existence and uniqueness of U_h^n will follow if we show that if there is a $V \in S_h$ such that

$$\begin{aligned}\tfrac{1}{k_n}\,(V, \phi) = &\,\mathrm{i}\, a^{n-\frac{1}{2}}\, \mu^{n-\frac{1}{2}} \left[S^{n-\frac{1}{2}}\, \tfrac{1}{k_n}\, V(1) + G^{n-\frac{1}{2}}\, \tfrac{1}{2}\, V(1) \right] \overline{\phi(1)} \\ &- \mathrm{i}\, \tfrac{a^{n-\frac{1}{2}}}{2}\, \mathcal{B}\big(V, \phi\big) + \tfrac{\mathrm{i}}{2}\, \big(P_h\big(\beta^{n-\frac{1}{2}}\, V\big), \phi\big), \quad \forall\, \phi \in S_h,\end{aligned} \tag{2.21}$$

then, $V = 0$. Set $\phi = \frac{1}{k_n} V$ in (2.21), and then take imaginary parts and use the arithmetic-geometric mean inequality and (2.9) to obtain

$$\begin{aligned}|V|_1^2 = &\,\mu^{n-\frac{1}{2}}\, k_n \left[2\, S^{n-\frac{1}{2}} \left| \tfrac{V(1)}{k_n} \right|^2 + \tfrac{1}{k_n} \operatorname{Re}(G^{n-\frac{1}{2}})\, |V(1)|^2 \right] + \tfrac{k_n}{a^{n-\frac{1}{2}}} \operatorname{Re}(P_h(\beta^{n-\frac{1}{2}}\, V), \tfrac{V}{k_n}) \\ \le &\,|\mu^{n-\frac{1}{2}}|\, k_n \left[-S^{n-\frac{1}{2}} \left| \tfrac{V(1)}{k_n} \right|^2 + \tfrac{|G^{n-\frac{1}{2}}|^2}{4 S^{n-\frac{1}{2}}}\, |V(1)|^2 \right] + \tfrac{C\, |\beta^{n-\frac{1}{2}}|_{1,\infty}}{|a^{n-\frac{1}{2}}|}\, k_n\, |V|_1\, \| \tfrac{V}{k_n} \|_{-1,h}.\end{aligned} \tag{2.22}$$

For $\phi \in S_h$, we use (2.21), (2.2) and (2.1) to obtain

$$\begin{aligned}\big|(\tfrac{V}{k_n}, \phi)\big| \le &\,|a^{n-\frac{1}{2}}|\, |\mu^{n-\frac{1}{2}}| \left[S^{n-\frac{1}{2}} \left| \tfrac{V(1)}{k_n} \right| + |G^{n-\frac{1}{2}}|\, \tfrac{1}{2}\, |V(1)| \right] |\phi|_1 \\ &+ \tfrac{1}{2} \left[|a^{n-\frac{1}{2}}| + C\, |\beta^{n-\frac{1}{2}}|_\infty \right] |V|_1\, |\phi|_1,\end{aligned}$$

which yields

$$\| \tfrac{V}{k_n} \|_{-1,h} \le\, |a^{n-\frac{1}{2}}|\, |\mu^{n-\frac{1}{2}}|\, S^{n-\frac{1}{2}} \left| \tfrac{V(1)}{k_n} \right| + C_{E,1}\, |V|_1, \tag{2.23}$$

where $C_{E,1} := \frac{1}{2} \left[|a^{n-\frac{1}{2}}| + C\, |\beta^{n-\frac{1}{2}}|_\infty + |a^{n-\frac{1}{2}}|\, |\mu^{n-\frac{1}{2}}|\, |G^{n-\frac{1}{2}}| \right]$. Using (2.22), (2.23) and (2.2) gives

$$|V|_1^2 \left\{ 1 - k_n \left[\tfrac{|\mu^{n-\frac{1}{2}}|\, |G^{n-\frac{1}{2}}|^2}{4 S^{n-\frac{1}{2}}} + \tfrac{C\, C_{E,1}\, |\beta^{n-\frac{1}{2}}|_{1,\infty}}{|a^{n-\frac{1}{2}}|} + C\, |\mu^{n-\frac{1}{2}}|\, S^{n-\frac{1}{2}}\, |\beta^{n-\frac{1}{2}}|_{1,\infty}^2 \right] \right\} \le 0, \tag{2.24}$$

which ends the proof. □

In particular, if suppose that β is in $C([0, T], W^{1,\infty}(D))$, that a, μ, S, G are continuous functions on $[0, T]$ and that $S(t) > 0$ and $\mu(t) \le 0$ for $t \in [0, T]$, (i.e. the upsloping case), then the existence and uniqueness of the fully discrete approximation U_h^n follows if $k_n \le C$, where C is a constant independent of n. This follows from Proposition 2.6 and the fact that the quantity multiplying k_n in (2.24) may be uniformly bounded with respect to n.

In the case of general bottom topography we have:

Proposition 2.7. *Let $n \in \{1,\dots,N\}$ and suppose that $U_h^{n-1} \in S_h$ is well-defined. Then, there exist constants $C_{n,1}$ and $C_{n,2}$ such that if $\frac{k_n}{h} < C_{n,1}$ and $k_n < C_{n,2}$, then U_h^n is well-defined by* (2.20).

Proof. Let $\dim S_h = J$ and $\{\phi_j\}_{j=1}^J$ be a basis of S_h consisting of real-valued functions. It is easily seen that existence and uniqueness of U_h^n is equivalent to the invertibility of a matrix $\widetilde{M} \in \mathbb{C}^{J\times J}$ defined by $\widetilde{M}_{\ell j} := \mathcal{M}(\phi_j, \phi_\ell)$ for $j, \ell = 1,\dots,J$, where $\mathcal{M} : S_h \times S_h \to \mathbb{C}$ is given by $\mathcal{M}(\chi,\phi) := (\chi,\phi) - \mathrm{i}\, a^{n-\frac12}\, \mu^{n-\frac12}\, S^{n-\frac12}\, \chi(1)\,\overline{\phi(1)} + \frac{k_n}{2}\big[\, -\mathrm{i}\,(\beta^{n-\frac12}\chi,\phi) - \mathrm{i}\; \mu^{n-\frac12}\, a^{n-\frac12}\, G^{n-\frac12}\, \chi(1)\,\overline{\phi(1)} + \mathrm{i}\, a^{n-\frac12}\, \mathcal{B}(\chi,\phi)\,\big]$ for $\chi, \phi \in S_h$. If $x \in \mathrm{Ker}\widetilde{M}$ we have $\mathrm{Re}\big[\mathcal{M}(\phi_\star,\phi_\star)\big] = 0$ with $\phi_\star := \sum_{j=1}^J x_j\, \phi_j$. Then using (2.3) and (2.5), we get

$$\begin{aligned}\|\phi_\star\|^2 &\le \tfrac{k_n}{2}\Big[\, |\beta^{n-\frac12}|_\infty\, \|\phi_\star\|^2 + 2|\mu^{n-\frac12}|\, |a^{n-\frac12}|\, |G^{n-\frac12}|\, \|\phi_\star\|\, |\phi_\star|_1\Big]\\ &\le \tfrac{k_n}{2}\Big[\, |\beta^{n-\frac12}|_\infty + \tfrac{C}{h}\, |\mu^{n-\frac12}|\, |a^{n-\frac12}|\, |G^{n-\frac12}|\,\Big]\, \|\phi_\star\|^2,\end{aligned}$$

which, under our hypothesis, yields $x = 0$ and ends the proof. □

Hence, if $\beta \in C([0,T], L^\infty(D))$ and a, μ, G are continuous on $[0,T]$ (i.e. in the case of general bottom topography), the existence and uniqueness of U_h follows if we take $k_n \le C_1$ and $\frac{k_n}{h} \le C_2$, for some constants C_1 and C_2 independent of n.

We next establish the consistency of our fully discrete scheme in the t variable.

Proposition 2.8. *Let u be the solution of ($\mathcal{N}$). For $n = 1,\dots,N$, define $\sigma^n : \overline{D} \to \mathbb{C}$ by*

$$\frac{u^n - u^{n-1}}{k_n} = \mathrm{i}\, a^{n-\frac12}\, u_{xx}(t^{n-\frac12},\cdot) + \mathrm{i}\, \beta^{n-\frac12}\, \mathcal{A}u^n + f^{n-\frac12} + \sigma^n. \tag{2.25}$$

Then, there exists a constant C such that

$$\|\sigma^n\| \le C\, (k_n)^2\; B_1^n(u), \quad n = 1,\dots,N, \tag{2.26}$$

$$\|\sigma^{n+1} - \sigma^n\| \le C\,\big[\, (k_n)^2 + |k_{n+1} - k_n|\,\big]\, (k_n + k_{n+1})\; B_2^n(u), \quad n = 1,\dots,N-1, \tag{2.27}$$

where $B_1^n(u) := \sum_{\ell=2}^3 \max_{[t^{n-1},t^n]} \|\partial_t^\ell u\|$ and $B_2^n(u) := \sum_{\ell=2}^4 \max_{[t^{n-1},t^{n+1}]} \|\partial_t^\ell u\|$.

Proof. It follows easily by using the partial differential equation and Taylor's formula. □

We prove now that the following H^1-superconvergence estimate holds in the fully discrete case.

Proposition 2.9. *Let u be the solution of ($\mathcal{N}$) and $\{U_h^n\}_{n=0}^N$ be the fully discrete approximations that the method* (2.19)-(2.20) *produces. Assume that $\mu(t) \le 0$ and $S(t) > 0$ for $t \in [0,T]$. In addition, assume that there exists a constant $C \ge 0$ such that*

$$|k_{n+1} - k_n| \le C\, \max\big\{k_n^2, k_{n+1}^2\big\}, \quad n = 1,\dots,N-1. \tag{2.28}$$

Then, there exists a constant C_1 such that, if $\max\limits_{1\le n\le N}(k_n\, C_1) \le \frac13$, there exists a constant $C > 0$ such that

$$\max_{1\le n\le N} \|U_h^n - R_h u^n\|_1 \le C\, (k^2 + h^{r+1})\; \Xi(u), \quad \forall\, h \in (0, h_\star], \tag{2.29}$$

where

$$\Xi(u) := \sum_{\ell=0}^{2} \max_{[0,T]} \|\partial_t^\ell u\|_{r+1} + \max_{[0,T]} \|\partial_t^3 u\|_1 + \max_{[0,T]} \|\partial_t^4 u\|.$$

Proof. Let $h \in (0, h_\star]$, $\theta_h^n := U_h^n - R_h u^n$ for $n = 0,\dots,N$, $\xi := \frac{1}{a}$ and $\xi^{n-\frac12} := \xi(t^{n-\frac12})$ for $n = 1,\dots,N$. We use (2.20), (2.25) and (2.6), to obtain

$$\begin{aligned}(\partial\theta_h^n, \chi) =&\, \mathrm{i}\, a^{n-\frac12}\, \mu^{n-\frac12}\, \Big[S^{n-\frac12}\, \partial\theta_h^n(1) + G^{n-\frac12}\, \mathcal{A}\theta_h^n(1) - \mathcal{E}_3^n\,\Big]\, \overline{\chi(1)}\\ &- \mathrm{i}\, a^{n-\frac12}\, \mathcal{B}(\mathcal{A}\theta_h^n, \chi) + \mathrm{i}\, (P_h(\beta^{n-\frac12}\, \mathcal{A}\theta_h^n), \chi)\\ &+ (\mathcal{E}_1^n - \sigma^n, \chi) + \mathrm{i}\, a^{n-\frac12}\, \mathcal{B}(\mathcal{E}_2^n, \chi) \quad \forall\, \chi \in S_h, \quad n = 1,\dots,N,\end{aligned} \tag{2.30}$$

where

$$\mathcal{E}_1^n := \partial u^n - R_h(\partial u^n) - \mathrm{i}\, P_h\big[\beta^{n-\frac12}(\mathcal{A}u^n - R_h(\mathcal{A}u^n))\big],$$
$$\mathcal{E}_2^n := u(t^{n-\frac12}) - \mathcal{A}u^n,$$
$$\mathcal{E}_3^n = S^{n-\frac12}\left[\partial_t u(t^{n-\frac12},1) - \partial u^n(1)\right] + G^{n-\frac12}\left[u(t^{n-\frac12},1) - \mathcal{A}u^n(1)\right].$$

Using Taylor's formula and (2.7), we deduce the following estimates:

$$\begin{aligned}\|\mathcal{E}_1^n\| &\le C\,\tfrac{h^{r+1}}{k_n}\,\Big\|\int_{t^{n-1}}^{t^n}\partial_t u(s,\cdot)\,ds\Big\|_{r+1} + C\,|\beta^{n-\frac12}|_\infty\, h^{r+1}\,\|\mathcal{A}u^n\|_{r+1}\\ &\le C\,h^{r+1}\Big\{\max_{[t^{n-1},t^n]}\|u\|_{r+1} + \max_{[t^{n-1},t^n]}\|\partial_t u\|_{r+1}\Big\}, \quad n=1,\dots,N,\end{aligned} \tag{2.31}$$

$$|\mathcal{E}_2^n|_1 \le C\,k_n^2 \max_{[t^{n-1},t^n]}|\partial_t^2 u|_1, \quad n=1,\dots,N, \tag{2.32}$$

and

$$|\mathcal{E}_3^n| \le C\,k_n^2\Big[\max_{t\in[t^{n-1},t^n]}|\partial_t^2 u(t,1)| + \max_{t\in[t^{n-1},t^n]}|\partial_t u(t,1)|\Big], \quad n=1,\dots,N. \tag{2.33}$$

Set $\chi = \partial\theta_h^n$ in (2.30), and then take imaginary parts to obtain

$$\begin{aligned}|\theta_h^n|_1^2 \le{}& |\theta_h^{n-1}|_1^2 + 2\,k_n\,|\xi^{n-\frac12}|\;|P_h(\beta^{n-\frac12}\,\mathcal{A}\theta_h^n)|_1\;\|\partial\theta_h^n\|_{-1,h}\\ &+ k_n\,|\mu^{n-\frac12}|\,\big[-2\,S_\star\,|\partial\theta_h^n(1)|^2 + 2\,|G^{n-\frac12}|\,|\mathcal{A}\theta_h^n(1)|\,|\partial\theta_h^n(1)| + 2\,|\mathcal{E}_3^n|\,|\partial\theta_h^n(1)|\,\big]\\ &+ 2\,k_n\,\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^n,\partial\theta_h^n)] + 2\,k_n\,\xi^{n-\frac12}\,\mathrm{Im}(\mathcal{E}_1^n - \sigma^n,\partial\theta_h^n), \quad n=1,\dots,N,\end{aligned} \tag{2.34}$$

where $S_\star := \inf_{[0,T]} S$. Now let us estimate $\|\partial\theta_h^n\|_{-1,h}$. For $\varphi \in S_h$, (2.30)-(2.33), (2.26), (2.2) and (2.1) give

$$\begin{aligned}|(\partial\theta_h^n,\varphi)| \le{}& |a^{n-\frac12}|\,|\mu^{n-\frac12}|\,S^{n-\frac12}\,|\partial\theta_h^n(1)|\,|\varphi|_1\\ &+ C\,|\mathcal{A}\theta_h^n|_1\,|\varphi|_1 + C\,(h^{r+1}+k_n^2)\;|\varphi|_1\;\Xi_1(u), \quad n=1,\dots,N,\end{aligned}$$

where

$$\Xi_1(u) := \max_{[0,T]}\|u\|_{r+1} + \max_{[0,T]}\|\partial_t u\|_{r+1} + \max_{[0,T]}\|\partial_t^2 u\|_1 + \max_{[0,T]}\|\partial_t^3 u\|.$$

Hence, we conclude that

$$\begin{aligned}2\,k_n\,|\xi^{n-\frac12}|\,\|\partial\theta_h^n\|_{-1,h} \le{}& 2\,k_n\,|\mu^{n-\frac12}|\,S^{n-\frac12}\,|\partial\theta_h^n(1)|\\ &+ C\,k_n\,|\mathcal{A}\theta_h^n|_1 + C\,k_n\,(h^{r+1}+k_n^2)\;\Xi_1(u), \quad n=1,\dots,N.\end{aligned} \tag{2.35}$$

Now, combining (2.35) and (2.34) we have

$$\begin{aligned}|\theta_h^n|_1^2 \le{}& |\theta_h^{n-1}|_1^2 + C\,k_n\,|\mathcal{A}\theta_h^n|_1^2 + C\,k_n\,\big[\,(k_n)^4 + (h^{r+1}+k_n^2)\,|\mathcal{A}\theta_h^n|_1\,\big]\,\Xi_1(u)\\ &+ 2\,k_n\,\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^n,\partial\theta_h^n)] + 2\,k_n\,\xi^{n-\frac12}\,\mathrm{Im}(\mathcal{E}_1^n - \sigma^n,\partial\theta_h^n), \quad n=1,\dots,N,\end{aligned}$$

from which there follows that for some constant $C_1 \ge 0$

$$\begin{aligned}(1 - C_1\,k_n)\,|\theta_h^n|_1^2 \le{}& (1 + C_1\,k_n)\,|\theta_h^{n-1}|_1^2 + C_2\,k_n\,(h^{r+1}+k_n^2)^2\,(\Xi_1(u))^2\\ &+ 2\,k_n\,\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^n,\partial\theta_h^n)] + 2\,k_n\,\xi^{n-\frac12}\,\mathrm{Im}(\mathcal{E}_1^n - \sigma^n,\partial\theta_h^n), \quad n=1,\dots,N.\end{aligned} \tag{2.36}$$

To continue, we assume that $\max_{1\le n\le N}(C_1\,k_n) \le \frac13$, which allows us to conclude that $\frac{1+C_1\,k_n}{1-C_1\,k_n} \le e^{3C_1 k_n}$ for $n=1,\dots,N$. Hence, (2.36) yields

$$\begin{aligned}|\theta_h^n|_1^2 \le{}& e^{3C_1 k_n}\,|\theta_h^{n-1}|_1^2 + \tfrac{C_2\,k_n}{1-C_1\,k_n}\,(h^{r+1}+k_n^2)^2\,(\Xi_1(u))^2\\ &+ \tfrac{2\,k_n}{1-C_1\,k_n}\Big[\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^n,\partial\theta_h^n)] + \xi^{n-\frac12}\,\mathrm{Im}(\mathcal{E}_1^n - \sigma^n,\partial\theta_h^n)\Big], \quad n=1,\dots,N.\end{aligned}$$

Next, we define $\lambda_j^n := \frac{\exp\left(3C_1\sum_{\ell=j+1}^n k_\ell\right)}{1-C_1\,k_j}$ and use a simple induction argument to arrive at

$$
\begin{aligned}
|\theta_h^n|_1^2 \le C_2\,(\Xi_1(u))^2 \sum_{j=1}^n k_j\,\lambda_j^n\,(h^{r+1}+k_j^2)^2 \\
+ 2\sum_{j=1}^n k_j\,\lambda_j^n \Big[\,\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^j,\partial\theta_h^j)] + \xi^{j-\frac12}\,\mathrm{Im}(\mathcal{E}_1^j-\sigma^j,\partial\theta_h^j)\Big], \quad n=1,\dots,N,
\end{aligned}
$$

which yields

$$
|\theta_h^n|_1^2 \le\, C\,(h^{r+1}+k^2)^2\,(\Xi_1(u))^2 + T_A^n + T_B^n, \quad n=1,\dots,N, \tag{2.37}
$$

where

$$
\begin{aligned}
T_A^n &:= 2\sum_{j=1}^n \lambda_j^n\,\mathrm{Re}\big[\,\mathcal{B}(\mathcal{E}_2^j,\theta_h^j-\theta_h^{j-1})\,\big],\\
T_B^n &:= 2\sum_{j=1}^n \lambda_j^n\,\xi^{j-\frac12}\,\mathrm{Im}(\mathcal{E}_1^j-\sigma^j,\theta_h^j-\theta_h^{j-1}).
\end{aligned}
$$

First we observe that

$$
\begin{aligned}
T_A^n = \tfrac{2}{1-C_1\,k_n}\;\mathrm{Re}\big[\,\mathcal{B}(\mathcal{E}_2^n,\theta_h^n)\,\big] \\
+ 2\sum_{j=1}^{n-1}\lambda_j^n\;\mathrm{Re}[\mathcal{B}(\mathcal{E}_2^j-\mathcal{E}_2^{j+1},\theta_h^j)] \\
+ 2\sum_{j=1}^{n-1}\exp\left(3\,C_1\sum_{\ell=j+2}^n k_\ell\right)\left[\tfrac{\exp(3\,C_1\,k_{j+1})-1+C_1\,k_j}{1-C_1\,k_j} - \tfrac{C_1\,k_{j+1}}{1-C_1\,k_{j+1}}\right]\;\mathrm{Re}\big[\,\mathcal{B}(\mathcal{E}_2^{j+1},\theta_h^j)\,\big],
\end{aligned}
\tag{2.38}
$$

for $n=1,\dots,N$. Since

$$
|\mathcal{E}_2^j-\mathcal{E}_2^{j+1}|_1 \le\, C\,(k_j+k_{j+1})\,\big[\,(k_j)^2+|k_{j+1}-k_j|\,\big]\,\Xi_2(u), \quad j=1,\dots,N-1,
$$

with

$$
\Xi_2(u) := \max_{[0,T]}|\partial_t^2 u|_1 + \max_{[0,T]}|\partial_t^3 u|_1,
$$

we see that (2.38), (2.32) and (2.28) yield

$$
|T_A^n| \le\, C\,k^2\,\Xi_2(u)\,\max_{1\le m\le n}|\theta_h^m|_1, \quad n=1,\dots,N. \tag{2.39}
$$

In addition, we have

$$
\begin{aligned}
T_B^n = \tfrac{2}{1-C_1\,k_n}\,\xi^{n-\frac12}\,\mathrm{Im}(\mathcal{E}_1^n-\sigma^n,\theta_h^n) \\
+ 2\sum_{j=1}^{n-1}\lambda_j^n\,\xi^{j-\frac12}\,\mathrm{Im}(\mathcal{E}_1^j-\sigma^j-\mathcal{E}_1^{j+1}+\sigma^{j+1},\theta_h^j) \\
+\; 2\sum_{j=1}^{n-1}\xi^{j-\frac12}\;\exp\left(3\,C_1\sum_{\ell=j+2}^n k_\ell\right)\left[\tfrac{\exp(3\,C_1\,k_{j+1})-1+C_1\,k_j}{1-C_1\,k_j} - \tfrac{C_1\,k_{j+1}}{1-C_1\,k_{j+1}}\right]\mathrm{Im}(\mathcal{E}_1^{j+1}-\sigma^{j+1},\theta_h^j) \\
+\; 2\sum_{j=1}^{n-1}(\xi^{j-\frac12}-\xi^{j+\frac12})\,\lambda_{j+1}^n\;\mathrm{Im}(\mathcal{E}_1^{j+1}-\sigma^{j+1},\theta_h^j)
\end{aligned}
\tag{2.40}
$$

for $n=1,\dots,N$. Observing that

$$
|\mathcal{E}_1^j-\mathcal{E}_1^{j+1}|_1 \le\, C\,(k_j+k_{j+1})\,h^{r+1}\,\Xi_3(u), \quad j=1,\dots,N-1,
$$

with

$$\Xi_3(u) := \max_{[0,T]} \|\partial_t u\|_{r+1} + \max_{[0,T]} \|\partial_t^2 u\|_{r+1},$$

we see that (2.40), (2.31), (2.26)-(2.28) and (2.1) yield

$$|T_B^n| \le C\,(k^2 + h^{r+1})\,\Xi_4(u) \max_{1\le m\le n} |\theta_h^m|_1, \quad n = 1,\dots,N. \tag{2.41}$$

where

$$\Xi_4(u) := \sum_{\ell=0}^{2} \max_{[0,T]} \|\partial_t^\ell u\|_{r+1} + \sum_{\ell=3}^{4} \max_{[0,T]} \|\partial_t^\ell u\|$$

Now, from (2.37), (2.39) and (2.41) there follows that

$$|\theta_h^n|_1^2 \le C\,(h^{r+1} + k^2)^2\,(\Xi_1(u))^2 + C\,(k^2 + h^{r+1})\,\big(\Xi_2(u) + \Xi_4(u)\big) \max_{1\le m\le n} |\theta_h^m|_1, \quad n = 1,\dots,N,$$

which easily yields

$$\max_{0,\le n\le N} |\theta_h^n|_1^2 \le C\,(h^{r+1} + k^2)^2\,(\Xi_1(u) + \Xi_2(u) + \Xi_4(u))^2. \tag{2.42}$$

The desired estimate (2.29) is then a simple consequence of (2.42) and (2.1). □

Now we are ready to prove error estimates in L^2 and H^1 norms.

Theorem 2.10. *Let u be the solution of $(\mathcal{N})$ and $\{U_h^n\}_{n=0}^N$ be the fully discrete approximations that the method* (2.19)-(2.20) *produces. Assume that $\mu(t) \le 0$, $S(t) > 0$, for $t \in [0,T]$, that* (2.28) *holds and $\max_{1\le n\le N} k_n\,C_1 \le \frac{1}{3}$, where C_1 is the constant specified in Proposition 2.9. Then, there exists a constant C such that*

$$\max_{0\le n\le N} \|U_h^n - u^n\| \le C\,(k^2 + h^{r+1})\;\Xi(u), \quad \forall\, h \in (0, h_\star],$$

$$\max_{0\le n\le N} \|U_h^n - u^n\|_1 \le C\,(k^2 + h^{r})\;\Xi(u), \quad \forall\, h \in (0, h_\star],$$

where $\Xi(u)$ was specified in Proposition 2.9.

Proof. It is a simple consequence of (2.29) and (2.7). □

We conclude that in the case of upsloping bottoms, the fully discrete Galerkin-Crank-Nicolson method (2.19)-(2.20) yields fully discrete approximations U_h^n that converge to the solution u of $(\mathcal{N})$ at optimal rates in the L^2 and H^1 norms.

2.4. The case of the Abrahamsson-Kreiss boundary condition. We consider now the PE with the Abrahamsson-Kreiss bottom boundary condition, i.e. the initial-boundary value problem (1.8), (1.10), (1.11), (1.15), which we rewrite here, in slightly more general form, for the convenience of the reader. For $T > 0$ given, seek a function $u : [0,T] \times \overline{D} \to \mathbb{C}$ satisfying

$$(\mathcal{AK}) \qquad \begin{aligned} &u_t = \mathrm{i}\,a(t)u_{xx} + \mathrm{i}\,\beta(t,x)\,u + f(t,x),\ t \in [0,T],\ x \in \overline{D},\\ &u(0,x) = u_0(x),\ x \in \overline{D},\\ &u(t,0) = 0,\ t \in [0,T],\\ &u_x(t,1) = 0,\ t \in [0,T]. \end{aligned}$$

We assume again that $a : [0,T] \to \mathbb{R} - \{0\}$, $\beta,\ f : [0,T] \times \overline{D} \to \mathbb{C}$, $u_0 : \overline{D} \to \mathbb{C}$ are given functions. We shall assume that the solution of $(\mathcal{AK})$ exists uniquely and that the data and the solution of $(\mathcal{AK})$ are smooth enough for the purposes of the error estimation. We note that $(\mathcal{AK})$ may be considered as a special case of $(\mathcal{N})$ obtained by setting μ equal to zero in $(\mathcal{N})$. (This does not imply of course that we assume that $\dot{s}$ is zero. We recall that in the Abrahamsson-Kreiss formulation the effect of variable bottom enters explicitly in the definition of a and β, cf. (1.9), and in the change of variable formula (1.7).). All the error estimates for $(\mathcal{AK})$ that follow may then be considered as special cases of the analogous estimates in the two preceding subsections but with some important simplifications. For the convenience of the reader we shall restate the results but not prove them in detail; we shall just point out some differences between them and the analogous estimates for the problem $(\mathcal{N})$. It will be seen that the finite element approximations of $(\mathcal{AK})$ exist and satisfy optimal-order error estimates under no further assumptions (except smoothness) on the shape of the bottom.

Using the finite element subspace S_h and the notation established in Subsection 2.1, we define the semidiscrete approximation u_h of $(\mathcal{AK})$ as the map $u_h : [0,T] \to S_h$ satisfying

$$(\partial_t u_h(t,\cdot),\phi) = -\mathrm{i}\, a(t)\,\mathcal{B}(u_h(t,\cdot),\phi) + \mathrm{i}\,(\beta(t,\cdot)\,u_h(t,\cdot),\phi) + (f(t,\cdot),\phi) \quad \forall\,\phi\in S_h, \quad \forall\, t\in[0,T], \tag{2.43}$$

and

$$u_h(0,\cdot) = u_h^0, \tag{2.44}$$

where $u_h^0 \in S_h$ may be taken, for example, as $P_h u_0$ or $R_h u_0$.

Proposition 2.11. *The problem* (2.43)-(2.44) *admits a unique solution in* $C^1([0,T],S_h)$*. If* $f = \beta_I = 0$*, the solution preserves the* $L^2(D)$ *norm, i.e.*

$$\|u_h(t,\cdot)\| = \|u_h^0\|, \quad t\in[0,T]. \tag{2.45}$$

Proof. The first part follows from Proposition 2.3 for $\mu = 0$. The conservation of the L^2 norm follows by taking $\phi = u_h$ in (2.43) and then real parts. □

Theorem 2.12. *Let* u *be the solution of* $(\mathcal{AK})$ *and* u_h *its semidiscrete approximation defined by* (2.43)-(2.44)*. There exists a constant* $C = C(T) > 0$ *such that*

$$\begin{aligned}\|u_h(t,\cdot)-u(t,\cdot)\| \le C\Big\{&\|u_h^0-u_0\|\\ &+ h^{r+1}\big[\,\|u_0\|_{r+1} + \|u(t,\cdot)\|_{r+1} + \int_0^t (\|u(\tau,\cdot)\|_{r+1} + \|u_t(\tau,\cdot)\|_{r+1})d\tau\big]\Big\}\ \forall\, t\in[0,T],\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}\|u_h(t,\cdot)-u(t,\cdot)\|_1 \le C\Big\{&\|u_h^0-u_0\|_1 + h^r(\|u_0\|_{r+1} + \|u(t,\cdot)\|_{r+1})\\ &+ h^{r+1}\big(\int_0^t \Gamma(\tau)d\tau\big)^{1/2}\Big\}\ \forall\, t\in[0,T],\end{aligned} \tag{2.47}$$

where Γ *is the function defined in the statement of Proposition 2.4.*

Proof. Putting as usual $\theta_h := u_h - R_h u$ we obtain

$$\begin{aligned}(\partial_t\theta_h(t,\cdot),\phi) = -\mathrm{i}\,a(t)\,\mathcal{B}(\theta_h(t,\cdot),\phi) &+ \mathrm{i}\,(\beta(t,\cdot)\,\theta_h(t,\cdot),\phi)\\ &+ \big(\Psi_\star(t,\cdot),\phi\big) \quad \forall\,\phi\in S_h, \quad \forall\, t\in[0,T],\end{aligned} \tag{2.48}$$

where $\Psi_\star$ was defined in the course of the proof of Proposition 2.4. Taking $\phi = \theta_h$ in (2.48) and then real parts, we may prove (2.46) in a straightforward manner. The proof of (2.47) follows the proof of Proposition 2.4 if we take $\phi = \partial_t\theta_h$ in (2.48) and then imaginary parts. □

Hence, if u_h^0 is taken equal to $P_h u^0$ or $R_h u^0$ etc., Proposition 2.11 yields optimal-order error estimates in L^2 or H^1 of the error $u_h - u$.

We now proceed to the full discretization of $(\mathcal{AK})$ by discretizing the initial-value-problem (2.43)-(2.44) in t using the Crank-Nicolson scheme. With notation introduced in Section 2.3, we define for $n = 0,\cdots,N$ a fully discrete approximation $U_h^n \in S_h$ of $u(t^n,\cdot)$, the solution of $(\mathcal{AK})$, as follows

`Step 1`: Set

$$U_h^0 := U_0, \tag{2.49}$$

where $U^0 \in S_h$ may be taken, for example, as $P_h u_0$ or $R_h u_0$.

`Step 2`: For $n = 1, \ldots, N$, find $U_h^n \in S_h$ such that

$$(\partial U_h^n, \chi) = -\mathrm{i}\, a^{n-\frac{1}{2}}\, \mathcal{B}\big(\mathcal{A}U_h^n, \chi\big) + \mathrm{i}\,\big(\beta^{n-\frac{1}{2}}\,\mathcal{A}U_h^n, \chi\big) + \big(f^{n-\frac{1}{2}}, \chi\big) \quad \forall\, \chi \in S_h. \tag{2.50}$$

Proposition 2.13. *Let $n \in \{1, \cdots, N\}$ and suppose U_h^{n-1} is well-defined. Then, there exists a constant C independent of n such that if $k_n \leq C$, U_h^n is well-defined by* (2.50). *Moreover, if $f = \beta_I = 0$*

$$\|U_h^n\| = \|U^0\|, \; n \in \{1, \cdots, N\}. \tag{2.51}$$

Proof. Since (2.50) is equivalent to a $\dim S_h \times \dim S_h$ linear system of algebraic equations, existence and uniqueness of U_h^n will follow if we show that if there is a $V \in S_h$ such that

$$\tfrac{1}{k_n}(V, \phi) = -\mathrm{i}\tfrac{a^{n-\frac{1}{2}}}{2}\mathcal{B}(V, \phi) + \tfrac{\mathrm{i}}{2}(\beta^{n-\frac{1}{2}} V, \phi) \; \forall \phi \in S_h, \tag{2.52}$$

then $V = 0$. This fact follows easily for k_n sufficiently small, if we put $\phi = V$ in (2.52) and take real parts. The conservation property (2.51) follows from (2.50) if we select $\chi = \mathcal{A}U_h^n$ and take real parts. □

Theorem 2.14. *Let u be the solution of $(\mathcal{AK})$ and $\{U_h^n\}_{n=0}^N$ be the fully discrete approximation produced by the scheme* (2.49)-(2.50). *Then, if $\max\limits_{1 \leq n \leq N} k_n$ is sufficiently small, there exists a constant $C = C(T)$ such that*

$$\max_{0 \leq n \leq N} \|U_h^n - u^n\| \leq C \Big\{ \|U^0 - u_0\| + (k^2 + h^{r+1})\, \Xi_1(u) \Big\}, \tag{2.53}$$

$$\max_{0 \leq n \leq N} \|U_h^n - u^n\|_1 \leq C \Big\{ \|U^0 - u_0\|_1 + (k^2 + h^r)\, \Xi_2(u) \Big\}, \tag{2.54}$$

where $\Xi_1(u) := \max_{[0,T]} \|u\|_{r+1} + \max_{[0,T]} \|\partial_t u\|_{r+1} + \max_{[0,T]} \|\partial_t^2 u\|_1 + \max_{[0,T]} \|\partial_t^3 u\|$, *and* $\Xi_2(u) := \max_{[0,T]} |\partial_t^2 u|_1 + \max_{[0,T]} |\partial_t^3 u|_1$.

Proof. The proof follows from those of Propositions 2.8 and 2.9 appropriately simplified. □

3. Numerical experiments

In this section we present the results of some numerical experiments that we performed using the fully discrete Galerkin-finite element methods, defined and analyzed in the previous Section, to solve the initial-boundary value problem for the (PE) in domains of variable bottom topography with Neumann and Abrahamsson-Kreiss boundary conditions. Recall that in the case of the Neumann boundary condition, i.e. for the problem $(\mathcal{N})$, our convergence results were rigorously established in the case of upsloping bottoms, that is when $\dot{s}(t) \leq 0$ for all $t \in [0, T]$. One of our goals in this section is to study numerically the behavior of the Neumann boundary condition in the presence of downsloping bottoms and compare the solution of $(\mathcal{N})$ with that of $(\mathcal{AK})$, for which rigorous convergence results hold for any smooth $s(t)$.

The results were compared in some test cases with those of analogous computations using finite difference schemes. The finite element subspace S_h consisted of continuous, piecewise linear functions defined on a uniform mesh, while the temporal discretization was effected with uniform time step. All computations were performed using double precision Fortran.

3.1. Order of convergence. To test numerically the order of convergence of the fully discrete Crank-Nicolson-Galerkin finite element method (henceforth referred to as (FE)) in the case of the initial-boundary-problem ($\mathcal{N}$), we took $T = 1$ and considered three cases of bottom profiles, namely:

Case 1: $s(t) = -0.3t + 0.7$ (upsloping).
Case 2: $s(t) = 0.4t + 0.3$ (downsloping).
Case 3: $s(t) = 0.2\cos(4\pi t) + 0.2\sin(4\pi t) + 0.7$ (oscillatory).

In ($\mathcal{N}$) we took $a = 1/(2s^2)$, $\beta = xt + \mathrm{i}(3x + t^2)$, $u_0(x) = -x(x-1)^3$. The bottom boundary condition had the form $u_x(t,1) = \mu(t)\big[S(t)u_t(t,1) + G(t)u(t,1)\big] + f_1(t)$, where $\mu(t) = \frac{\dot{s}(t)}{s(t)}$, $S(t) = \frac{s^2(t)}{1+(\dot{s}(t))^2}$, $G(t) = \mathrm{i}(S(t)\dot{\delta}(t) - s^2(t))$, $\delta = s\dot{s}/2$. The nonhomogeneous terms f and f_1 were chosen so that the exact solution of the problem was given by $u(t,x) = -x(x-1)^3 + \sin(t)x$. To compare the exact with the numerical solution we calculated the l_2 error at the nodes x_j at $T = 1$ (taking $k = h$). Table 4.1 shows the rates of convergence of the numerical solution in the three cases. The rate is clearly two in the upsloping case (as predicted by the theory), approaches two in the downsloping and seems not to have stabilized in the oscillatory bottom case. On the other hand, as predicted by the convergence theory of the previous section, (FE) when applied to ($\mathcal{AK}$) (with $u_x(t,1) = 0 + f_2(t)$, $u(t,x) = -x(x-1)^3 + \sin(t)x$) is clearly seen to be of second order of accuracy, see Table 4.2.

h	Case 1	Case 2	Case 3
1/100	1.998	1.638	1.766
1/200	1.999	1.659	1.085
1/400	1.999	2.001	1.556
1/800	2.000	2.012	2.615

Table 4.1. Orders of convergence of (FE) for ($\mathcal{N}$) in l_2 in three cases of bottom topography

h	Case 1	Case 2	Case 3
1/100	1.999	2.001	2.001
1/200	1.999	2.001	2.002
1/400	2.000	2.000	2.000
1/800	2.000	2.000	2.000

Table 4.2. Orders of convergence of (FE) for ($\mathcal{AK}$) in l_2 in three cases of bottom topography

3.2. Comparison with finite differences. We now compare the results of the scheme (FE) with those obtained by a Crank-Nicolson type finite difference scheme of second order of accuracy, cf.[4], in the case of Neumann boundary conditions. In the original (r,z) variables (see Figure 1), we consider the (PE) equipped with the Neumann bottom boundary condition (PN), in which we take $g_B(r) = \mathrm{i}k_0$. The bottom is slightly downsloping and is given by $l(r) = 874.89994 \cdot 10^{-4}r + 30.48006$ m for $0 \le r \le 5 \cdot 10^4$ m. We assumed a constant sound speed, taking $c = c_0 = 1524.003$ m/sec, and that the sound source, emitting at a frequency $f_0 = 80$ Hz, is placed at a depth $z_s = 15.24003$m, and modeled by the initial-value profile $\psi_0(z) = \sqrt{\frac{k_0}{2}}\{\exp(-(z-z_s)^2\frac{k_0^2}{4}) - \exp(-(z+z_s)^2\frac{k_0^2}{4})\}$, $0 \le z \le l(0)$. The problem is transformed as usual by the change of variables (1.7) to an equivalent one on a (t,y) strip, where it is solved numerically using the two schemes with $h = 1/1000$, $k = T/1000$, $T = 5 \cdot 10^4$. In Figure 2 we present, after transforming back to the original r, z variables, four profiles of the numerically computed field ψ, represented, as is customary in underwater acoustics, by the transmission loss function $TL = -10\log_{10}(\frac{2}{\pi k_0 r}|\psi(z,r)|^2)$ dB, depicted as a function of r at depths $z = 37.5, 150, 500$ and 1000 m. Evidently, the results of the two schemes agree well.

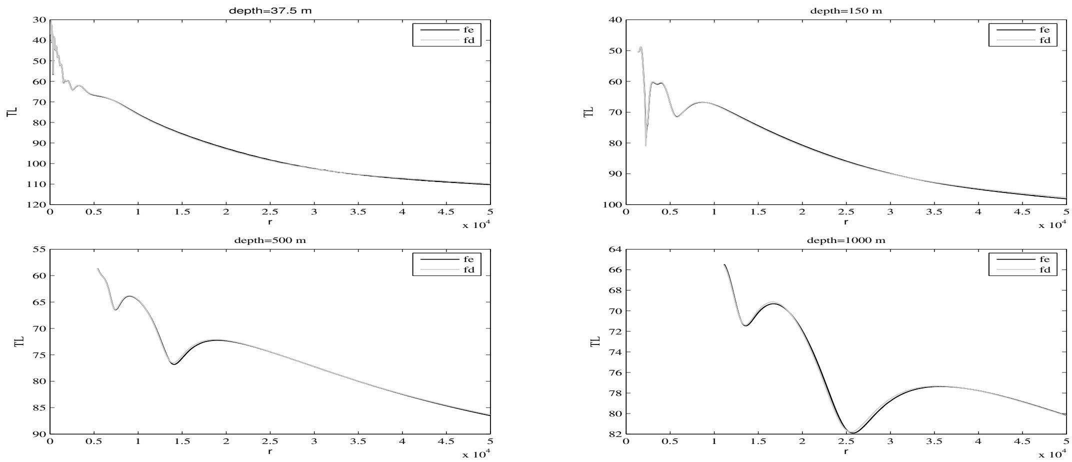


Figure 2. Transmission Loss as a function of r at various depths $z = 37.5$, 150, 500, 1000 m

3.3. Comparison of ($\mathcal{N}$) and ($\mathcal{AK}$): The upsloping and downsloping wedge. We first consider the ASA upsloping wedge underwater acoustic test problem, see [8], with rigid bottom given in the original variables r, z by the function $l(r) = 200 - 0.05r$ m for $0 \leq r \leq 3339$ m. The source, of frequency $f_0 = 25$ Hz, was placed at $z_s = 100$m and modeled by the initial value $\psi_0(z) = \sqrt{\frac{k_0}{2}}\{\exp(-(z - z_s)^2 \frac{k_0^2}{4}) - \exp(-(z + z_s)^2 \frac{k_0^2}{4})\}$. The water had constant sound speed equal to $c = c_0 = 1500$ m/sec. After transforming the problem to the t, y variables, we solved ($\mathcal{N}$) and ($\mathcal{AK}$) numerically using the (FE) scheme, taking $g_B(r) = \mathrm{i}k_0$ in (PN). We found that both models gave approximately the same acoustic field. For example, in Figure 3 we show the transmission loss curve for both models (we took $h = 1/1000$, $k = T/1000$, $T = 3339$) as a function of $r \in [0, 2200$ m$]$ at a depth of $z = 90$ m. (In this subsection we computed the transmission loss using the expression $TL = -20\log_{10}(|\psi(z,r)|) + 10\log_{10} r$.)

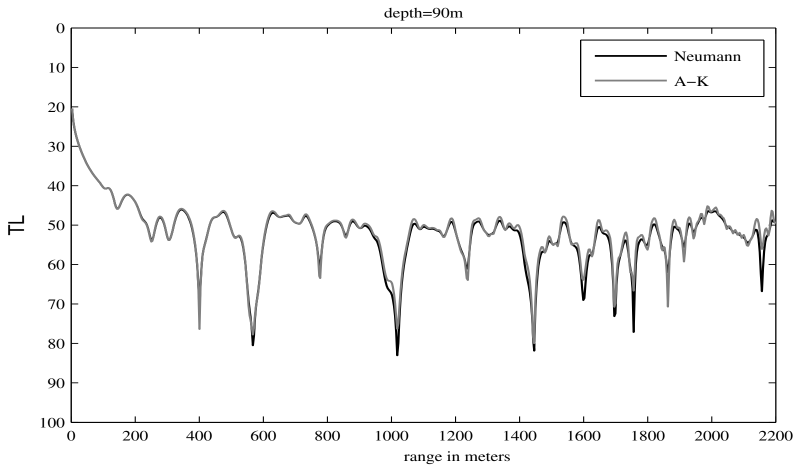
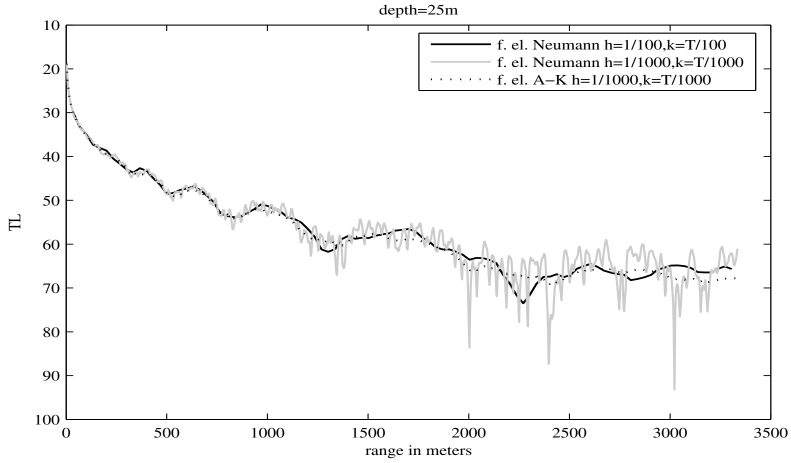


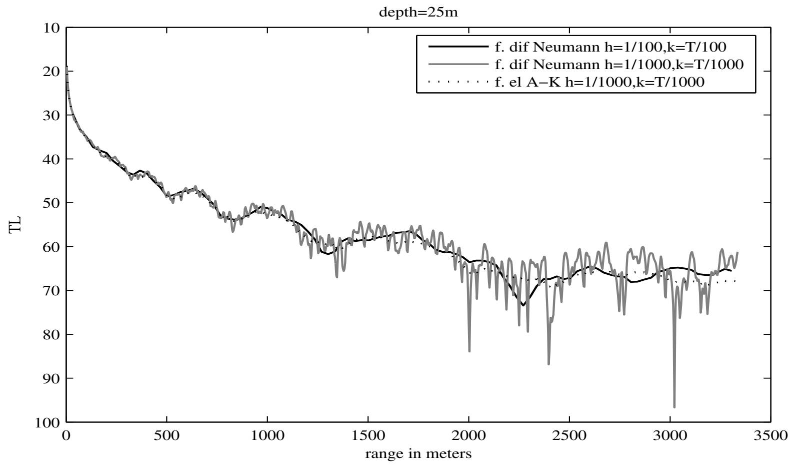
Figure 3. Upsloping ASA wedge; TL as a function of r at depth $z = 90$ m, comparison of ($\mathcal{N}$) and ($\mathcal{AK}$).

We then considered the analogous downsloping wedge given by $l(r) = 33.05 + 0.05r$ for $0 \leq r \leq 3339$ m. The source, of frequency 25 Hz, was placed at $z_s = 25$m and modeled as in the upsloping case. In this case, we found that the (FE) numerical solution of the problem ($\mathcal{N}$) apparently exhibited numerical instabilities and did not seem to converge as the discretization parameters became smaller. For example, in Figure 4(a) we superimpose the TL curves at depth $z = 25$m corresponding to the ($\mathcal{AK}$) model solved by (FE) with

$h = 1/1000$, $k = T/1000$, $T = 3339$m, with the analogous results obtained by $(\mathcal{N})$ solved by (FE) with same h and k. The $(\mathcal{AK})$ model, when discretized by (FE), yields reasonable results that converge to the solution shown in Figure 4(a) with dotted line. To make sure that the numerical method used for $(\mathcal{N})$ was not the culprit, we repeated the numerical experiment using a Crank-Nicolson finite difference scheme for $(\mathcal{N})$, and obtained the TL curves shown in Figure 4(b). The behavior of the finite difference discretization of $(\mathcal{N})$ is identical to that of the (FE). We tentatively conclude, therefore, that in this realistic downsloping bottom case, the model $(\mathcal{N})$ allows the growth of instabilities, in agreement with the remarks of Abrahamsson and Kreiss in [1] and [2].



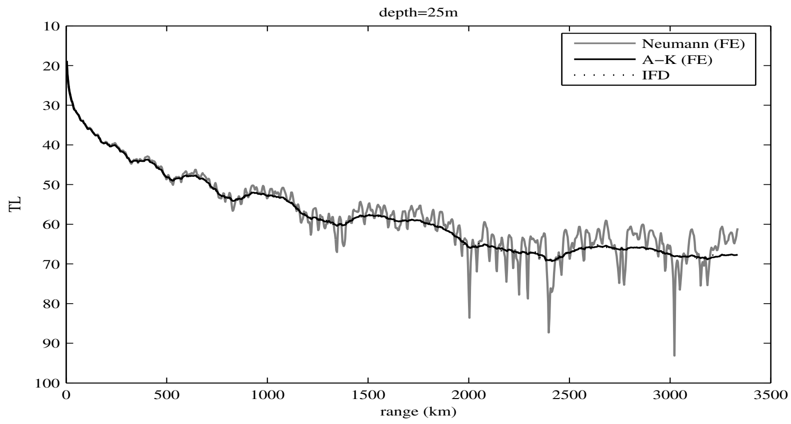
(a)



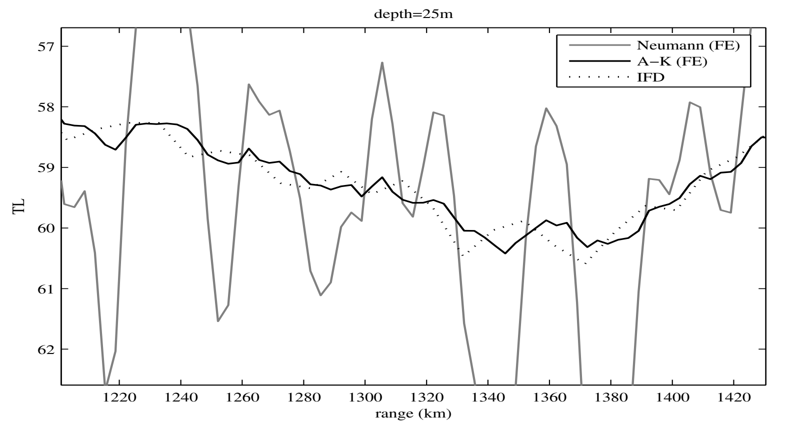
(b)

Figure 4. Downsloping ASA wedge; TL as a function of r at depth $z = 25$m. (a): Comparison of (FE) for the $(\mathcal{N})$ and $(\mathcal{AK})$ models. (b): Comparison of $(\mathcal{AK})$ solved by (FE) with $(\mathcal{N})$ solved by finite differences.

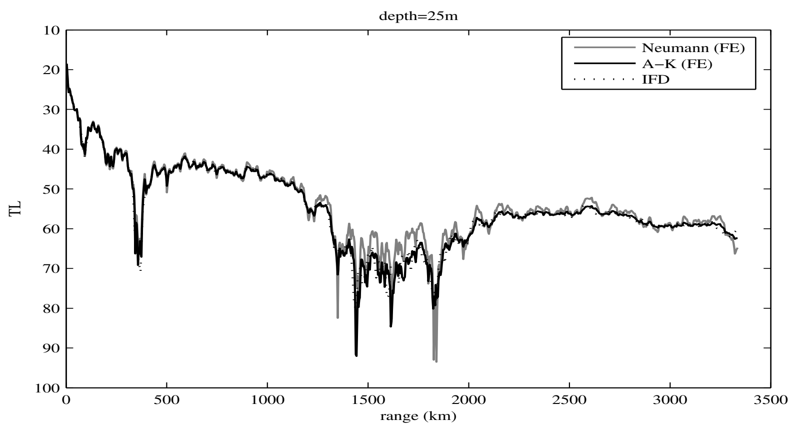
To check the validity of the $(\mathcal{AK})$ solution of this problem we compared the results of Figure 4 with those of yet another numerical method, the Crank-Nicolson type finite difference code IFD for the (PE), [9], [10], which has been widely used in underwater acoustic numerical simulations.



(a)



(b)



(c)

Figure 5. Downsloping ASA wedge; TL as a function of r at a depth $z = 25$m. Comparison of $(\mathcal{N})$ and $(\mathcal{AK})$, discretized by (FE), and IFD with rigid bottom b.c. (a): $f_0 = 25$ Hz, (b): Magnification of (a) for $r \in [1210, 1430]$, (c): $f_0 = 80$ Hz.

We chose the option of the rigid bottom boundary condition in IFD and solved the problem using $\Delta z = 3.31$ m, $\Delta r = 0.17$ m, values by which the IFD solution had converged. (The IFD code solves the problem in the original r, z wedge-shaped domain). Figure 5(a) shows the superimposed TL curves obtained at $z = 25$m by the $(\mathcal{N})$ and $(\mathcal{AK})$ models solved by (FE) with $h = 1/1000$, $k = T/1000$, $T = 3339$m (as in Figure 4(a)) and for the IFD with the rigid bottom boundary condition. The results of $(\mathcal{AK})$ and (IFD) agree well. In fact, then differ by about half a dB as inspection of a typical window of Figure 5(a), shown in Figure 5(b), reveals. (It is worthwhile to note that at a higher frequency $f_0 = 80$ Hz the results of (FE)-$(\mathcal{N})$ approach those of (FE)-$(\mathcal{AK})$ and IFD, see Figure 5(c)). To explain this result we looked closely at how IFD implements the rigid bottom boundary condition and found that it does not actually discretize (PN); instead, it uses a different boundary condition obtained by replacing the ψ_r term in (PN) by $\frac{\mathrm{i}}{2k_0}\psi_{zz} + \frac{\mathrm{i}k_0}{2}(\eta^2 - 1)\psi$ using the (PE), and then discretizing the ψ_{zz} term at the bottom with one-sided finite differences from the interior of the domain. In Appendix A we offer an explanation why this rigid bottom boundary condition yields a stable problem for any monotone bottom profile.

Our tentative conclusion, then, from this experiment is that in the case of realistic, downsloping environments, $(\mathcal{AK})$ and the rigid bottom boundary condition model implemented by IFD apparently yield correct results, while the Neumann bottom boundary condition used in $(\mathcal{N})$, which retains the term ψ_r at the bottom, allows the growth of instabilities.

3.4. Growth of solutions of $(\mathcal{N})$ for various bottom shapes. The final set of numerical experiments that we report concern the behavior of the size of the solutions of $(\mathcal{N})$, as t grows, in the presence of bottom profiles of various shapes. Recall that in [1] it was shown that $(\mathcal{N})$ is well-posed if s is strictly monotone, i.e. if $\dot{s}(t) > 0$ or $\dot{s}(t) < 0$ for $0 \le t \le T$. In addition, downsloping bottom profiles were identified for which the solution of $(\mathcal{N})$ grew exponentially with t.

The initial-boundary-value problem $(\mathcal{N})$ was solved numerically by the (FE) method up to $T = 1$, with $\beta = f = g = 0$, $u_0(x) = -x(x-1)^3$, $0 \le x \le 1$, with mesh parameters $h = k = 1/500$, in the case of the eight bottom profiles $s(t)$, $0 \le t \le 1$, labeled (a) to (h) and shown below and also in the left-hand column of Figure 6. (In all cases depth increases downwards.) The column on the right shows the corresponding numerically computed, L^2-norm of the solution of $(\mathcal{N})$ $\|u(t,\cdot)\|$ for $0 \le t \le 1$. (Note that $\|u(0,\cdot)\| = \frac{1}{6\sqrt{7}} \cong 0.062994$.). The bottom profiles were given for $0 \le t \le 1$ by the expressions:

$$\text{(a)} \quad s(t) = e^t,$$

$$\text{(b)} \quad s(t) = e^{-t},$$

$$\text{(c)} \quad s(t) = 1 + (t-0.5)^2,$$

$$\text{(d)} \quad s(t) = 1 - |t-0.5|^3,$$

$$\text{(e)} \quad s(t) = 1 - (t-0.5)^3,$$

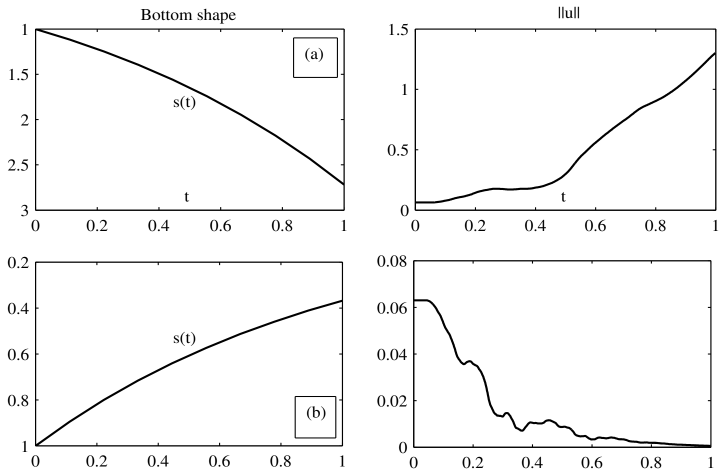
$$\text{(f)} \quad s(t) = 2 - |2t-1|,$$

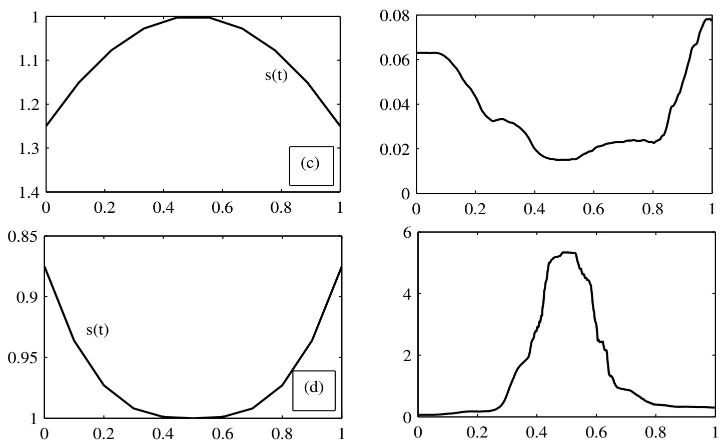
$$\text{(g)} \quad s(t) = 1 + (t-0.5)^3,$$

$$\text{(h)} \quad s(t) = 1 + t^3.$$

Only (a) and (b) correspond to strictly monotone profiles for which the theory of [1] properly applies. In cases (c), (d), (f) there is a change in monotonicity, in (e) and (g) we have that $\dot{s}(t) = \ddot{s}(t) = 0$ at $t = 1/2$, while in (h) there holds that $\dot{s}(0) = \ddot{s}(0) = 0$. (In case (f) a t-mesh node was placed at $t = 0.5$, where $\dot{s}$ fails to exist.)

We observe that the solution maintains a small L^2-norm in upsloping, like (b), or eventually upsloping bottoms, as in the cases of the trenches (d) and (f). There is a considerable growth of $\|u\|$ in the examples wherein the bottom profile is eventually downsloping, see (a), (c), (g), and (h), in agreement with the observations in [1], [2]. We note that in the case (g), an apparent singularity develops at $t = 1/2$, where the bottom curvature changes sign (with horizontal tangent) and the bottom becomes downsloping. This apparently causes the L^2-norm to grow violently for $t > 1/2$. A relatively weaker, but sizeable growth is also observed in (h), where the bottom is such that $\dot{s} = \ddot{s} = 0$ at $t = 0$ and is monotonically downsloping for $t > 0$. One cannot be of course certain about the existence of a singularity at $t = 1/2$ in the case (g), given that the (FE) code does not at present possess an adaptive refinement capability in x and t. However, when the experiment was repeated with $k = h = 1/800$, it was confirmed that the onset of rapid growth occured at about $t = 1/2$; for this mesh size, $\|u\|$ became of order $O(10^4)$ at $t = 1$.





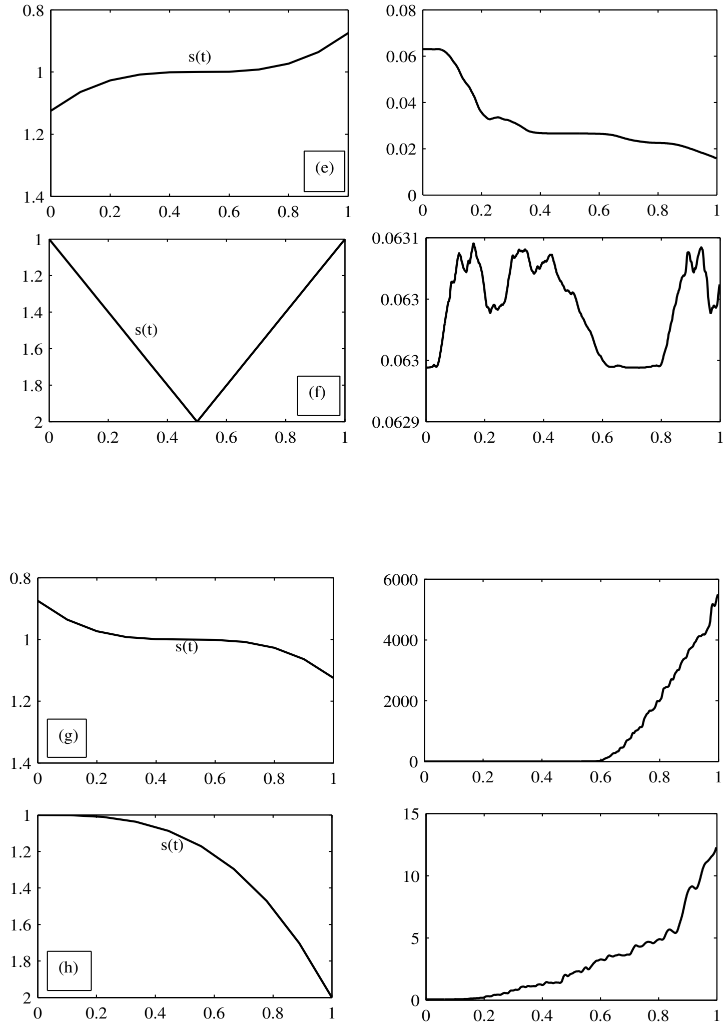


Figure 6. Behavior of the L^2-norm $\|u\|$ of the solution of $(\mathcal{N})$ as a function of t for various bottom profiles $s(t)$.

Appendix A.

Let $\omega = \omega(t, y)$ be defined for $0 \leq y \leq s(t)$, $0 \leq t \leq T$, and satisfy (1.1)-(1.3) and the Neumann bottom boundary condition (1.5). Replace the term $\omega_t(t, s(t))$ in (1.5) by its value given by PE (1.1) to obtain for $0 \leq t \leq T$

$$\omega_y(t, s(t)) - \dot{s}(t)\Big\{\frac{\mathrm{i}}{2}\omega_{yy}(t, s(t)) + [\mathrm{i}\gamma(t, s(t)) + g(t)]\omega(t, s(t))\Big\} = 0. \tag{A.1}$$

In the IFD code, the rigid bottom boundary condition used is a finite difference discretization of (A.1).

To avoid the presence of the second derivative $\omega_{yy}(t, s(t))$ in the boundary condition (A.1), we differentiate the PE (1.1) with respect to y and put $\tilde{p}(t, y) = \omega_y(t, y)$. (Note that $\omega(t, y) = \int_0^y \tilde{p}(t, \xi) d\xi$ since $\omega(t, 0) = 0$.)

Then, the initial-boundary-value problem (1.1)-(1.3), (A.1) becomes

$$\text{(A.2)}\qquad \begin{aligned}&\tilde{p}_t = \tfrac{\mathrm{i}}{2}\tilde{p}_{yy} + \mathrm{i}\gamma(t,y)\tilde{p} + \mathrm{i}\gamma_y(t,y)\omega, \quad 0 \le y \le s(t), \quad 0 \le t \le T,\\ &\tilde{p}_y(t,0) = 0, \quad 0 \le t \le T,\\ &\tilde{p}(t,s(t)) - \dot{s}(t)\Big\{\tfrac{\mathrm{i}}{2}\tilde{p}_y(t,s(t)) + [\mathrm{i}\gamma(t,s(t)) + g(t)]\omega(t,s(t))\Big\} = 0, \quad 0 \le t \le T,\\ &\tilde{p}(0,y) = \tilde{p}_0(y) := \omega_0'(y), \quad 0 \le y \le s(0).\end{aligned}$$

(Note that using the PE (1.1) at $y = 0$ and the surface boundary condition $\omega(t,0) = 0$, we obtain that $0 = \omega_{yy}(t,0) = \tilde{p}_y(t,0)$.)

In what follows, we shall obtain an *a priori* L^2 estimate for the solution of (A.2) and then propose a finite element method for solving it. With this aim in mind, we perform as usual the range-dependent change of depth variable $x := \frac{y}{s(t)}$ that maps the domain of the problem onto the horizontal strip $x \in \overline{D}$, $t \in [0,T]$, where $D = (0,1)$. Consider the transformation

$$\text{(A.3)}\qquad \tilde{p}(t,y) = \tfrac{1}{s(t)}\exp(-\zeta(t,x))\Big\{p(t,x) - \zeta_x(t,x)\int_0^x p(t,\xi)d\xi\Big\},$$

where the function ζ will be specified below. Note that the function θ, defined by $\theta(t,x) := \int_0^x p(t,\xi)d\xi$, for $(t,x) \in [0,T]\times\overline{D}$ satisfies the first-order o.d.e.

$$\theta_x(t,x) - \zeta_x(t,x)\theta(t,x) = s(t)\exp(\zeta(t,x))\tilde{p}(t,xs(t)).$$

Solving this differential equation with initial condition $\theta(t,0) = 0$ yields

$$\theta(t,x) = \int_0^x p(t,\xi)d\xi = s(t)\exp(\zeta(t,x))\int_0^x \tilde{p}(t,\xi s(t))d\xi,$$

from which we may derive the inverse of the transformation (A.3) in the form

$$\text{(A.4)}\qquad p(t,x) = s(t)\exp(\zeta(t,x))\Big\{\tilde{p}(t,xs(t)) + \zeta_x(t,x)\int_0^x \tilde{p}(t,\xi s(t))d\xi\Big\}.$$

After some calculations we also get that

$$\text{(A.5)}\qquad \theta(t,x) = \exp(\zeta(t,x))\omega(t,xs(t)), \quad (t,x) \in [0,T]\times\overline{D}.$$

Following the ideas of [4], and after analogous computations to those used in the case of similar transformations in that paper (see, in particular, (2.7) and (2.8) of [4]), we may deduce that p solves the following initial-boundary-value problem, in the case of strictly monotone bottoms, i.e. when $\dot{s}(t)$ is either positive or negative for all $t \in [0,T]$.

Define first ζ, as in [4], by the formula

$$\text{(A.6)}\qquad \zeta(t,x) = \mathrm{i}(\sigma(t)-1)\tfrac{\dot{s}(t)s(t)}{2}x^2, (t,x) \in [0,T]\times\overline{D},$$

where we define $\sigma(t) := \frac{2(1+\dot{s}(t)^2)}{\dot{s}(t)^2} + \varepsilon$, if $\dot{s}(t) > 0$, where ε is a positive constant, and $\sigma(t) := 1$, or equivalently $\zeta = 0$, if $\dot{s}(t) < 0$. Then, in the transformed domain, and expressed in terms of the new field variables p and θ, the initial-boundary-value problem (A.2) becomes

$$\text{(A.7)}\qquad \begin{aligned}&p_t = \mathrm{i}\,[1/A(t)]\,p_{xx} + B(t,x)p_x + [B_x(t,x) + G(t,x)]p + G_x(t,x)\,\theta, \quad x \in \overline{D}, \quad t \in [0,T].\\ &p_x(t,0) = 0, \quad t \in [0,T],\\ &(1 - R_1(t)B(t,1))\,p(t,1) = \mathrm{i}[R_1(t)/A(t)]p_x(t,1) + [R_1(t)G(t,1) + R_2(t)]\,\theta(t,1), \quad t \in [0,T],\\ &p(0,x) = s(0)\exp(\zeta(0,x))\Big\{\omega_0'(xs(0)) + \zeta_x(0,x)\int_0^x \omega_0'(\xi s(0))d\xi\Big\}, \quad x \in \overline{D},\end{aligned}$$

where

$$\begin{aligned}&A(t) = 2s^2(t), \;\; B(t) = x\tfrac{\dot{s}(t)}{s(t)} - (\mathrm{i}/s^2(t))\zeta_x(t,x),\\ &G(t,x) = \zeta_t(t,x) - x\tfrac{\dot{s}(t)}{s(t)}\zeta_x(t,x) + \mathrm{i}\gamma_R(t,xs(t)) - \gamma_I(t,xs(t)) + \tfrac{\mathrm{i}}{2s^2(t)}[(\zeta_x(t,x))^2 - \zeta_{xx}(t,x)],\\ &R_1(t) = \dot{s}(t)s(t)/(1+\dot{s}(t)^2), \;\; R_2(t) = [g(t) - \zeta_t(t,1)]R_1(t) + \zeta_x(t,1).\end{aligned}$$

(Recall that $\theta(t,x)=\int_0^x p(t,\xi)d\xi$. In addition, note that B is real-valued and is given by $B(t,x)=x\frac{\dot{s}(t)}{s(t)}\sigma(t)$, so that $B(t,0)=0$ and $B_x(t,x)=B(t,1)=\frac{\dot{s}(t)}{s(t)}\sigma(t)$. It is easily checked that $1-R_1(t)B(t,1)$ is not zero in $[0,T]$.)

We may now prove

Theorem A.1. *If the bottom is strictly monotone, the initial-boundary-value problem* (A.7) *is L^2-stable. Consequently,* (A.2) *is also L^2-stable.*

Proof. Multiply the p.d.e. in (A.7) by $\overline{p(t,x)}$, integrate with respect to x in $[0,1]$, use integration by parts, and take real parts to obtain, for $t\in[0,T]$

$$\frac{1}{2}\frac{d}{dt}\|p(t,\cdot)\|^2=\Big[\frac{1}{R_1(t)}-B(t,1)\Big]|p(t,1)|^2-\mathrm{Re}\Big\{\Big[G(t,1)+\frac{R_2(t)}{R_1(t)}\Big]\theta(t,1)\overline{p(t,1)}\Big\}+\frac{1}{2}B(t,1)|p(t,1)|^2$$
$$-\frac{1}{2}(B_x(t,\cdot)p(t,\cdot),p(t,\cdot))+\mathrm{Re}([B_x(t,\cdot)+\mathrm{Re}G(t,\cdot)]p(t,\cdot),p(t,\cdot))+\mathrm{Re}(G_x(t,\cdot)\theta(t,\cdot),p(t,\cdot)).$$

Using the Cauchy-Schwarz inequality, the arithmetic-geometric mean inequality, and noting that $|\theta(t,1)|\le\|p(t,\cdot)\|$, $\|\theta(t,\cdot)\|\le\|p(t,\cdot)\|$, we see from the above that for any $\xi>0$ there exists constant $C_\xi>0$ such that

$$\frac{d}{dt}\|p(t,\cdot)\|^2\le\Big[\frac{1}{R_1(t)}-\frac{1}{2}B(t,1)+\xi\Big]|p(t,1)|^2+C_\xi\|p(t,\cdot)\|^2,\quad 0\le t\le T.$$

Since $\frac{1}{R_1(t)}-\frac{1}{2}B(1,t)$ is negative for $t\in[0,T]$, we may chose ξ sufficiently small to make the first term in the right-hand side of the above negative. Hence, by Gronwall's lemma we see that for $t\in[0,T]$

$$\|p(t,\cdot)\|\le C\|p(0,\cdot)\|\text{ and }\|\theta(t,\cdot)\|_j\le C\|\theta(0,\cdot)\|_j,\quad j=0,1. \tag{A.8}$$

Hence, (A.7) is L^2 stable. If $\|\tilde{p}(t,\cdot)\|_*:=\Big(\int_0^{s(t)}|\tilde{p}(t,y)|^2dy\Big)^{\frac{1}{2}}$, relations (A.3) and (A.4) give for some positive constants c_1, c_2 and $t\in[0,T]$

$$c_1\|p(t,\cdot)\|\le\|\tilde{p}(t,\cdot)\|_*\le c_2\|p(t,\cdot)\|,$$

which yields the L^2-stability relation

$$\|\tilde{p}(t,\cdot)\|_*\le c\|\tilde{p}(0)\|_*.\ \square$$

The formulation (A.7) is suitable for a finite element discretization. We notice that it implies, for any $\phi\in H^1(0,1)$ and $t\in[0,T]$ that

$$\begin{aligned}(p_t,\phi)=&-(\mathrm{i}/A(t))(p_x,\phi_x)+(1/R_1(1))[(1-R_1(1)B(t,1))p(t,1)-(R_1(t)G(t,1)\\&+R_2(t,1))\theta(t,1)]\overline{\phi(t,1)}+(B(t,\cdot)p_x,\phi)+([B_x(t,\cdot)+G(t,\cdot)]p,\phi)+(G_x(t,\cdot)\theta,\phi).\end{aligned} \tag{A.9}$$

Let S_h be a finite-dimensional subspace of $H^1(0,1)$ consisting of piecewise polynomial functions of order at most $r-1$, and $p_h:[0,T]\to S_h$ the approximation of the solution p of problem (A.7) to be defined below. If $\theta_h(t,x):=\int_0^x p_h(t,\xi)d\xi$, then $\theta_h(t,x)$ is a piecewise polynomial function of order at most r, and, approximates $\theta(t,x)$ in (A.7). Consequently, using (A.5), we see that $\exp(-\zeta(t,x))\theta_h(t,x)$ is an approximation of the solution $\omega(t,xs(t))$ of (1.1)-(1.3), (1.5).

Following (A.9), we define the semidiscrete scheme

$$\begin{aligned}(\partial_t p_h,\phi)=&-(\mathrm{i}/A(t))(\partial_x p_h,\phi_x)+(1/R_1(1))[(1-R_1(1)B(t,1))p_h(t,1)-(R_1(t)G(1,t)\\&+R_2(t,1))\theta_h(t,1)]\overline{\phi(t,1)}(B(t,\cdot)\partial_x p_h,\phi)+([B_x(t,\cdot)+G(t,\cdot)]p_h,\phi)+(G_y(t,\cdot)\theta_h,\phi),\end{aligned} \tag{A.10}$$

for any $\phi\in S_h$ and $t\in[0,T]$, with $p_h(0,x)$ any reasonable approximation of $p(0,x)$ in S_h.

It follows, as in Theorem A.1, that the scheme is L^2-stable. In particular, we have that for $0 \le t \le T$ and some positive constant C,

$$\|p_h(t,\cdot)\| \le C\|p_h(0,\cdot)\| \text{ and } \|\theta_h(t,\cdot)\|_j \le C\|\theta_h(0,\cdot)\|_j, \text{ for } j = 0,1.$$

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