Spreading Speed and Traveling Wave Solutions of a Partially Sedentary Population

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Abstract

In this paper, we extend the population genetics model of [5] to the case where a fraction of the population does not migrate after the selection process. Mathematically, we study the asymptotic behavior of solutions to the recursion $u_{n+1} = Q_g[u_n]$ where

$$Q_g[u](x) = (1-g)\int_{\mathbf{R}^d} K(x-y)f(u(y))dy + gf(u(x)), \quad 0 \le g \le 1\,.$$

In the above definition of Q_g, K is a probability density function and f behaves qualitatively like the Beverton-Holt function. Under some appropriate conditions on K and f, we show that for each unit vector $\xi \in \mathbf{R}^d$, there exists $c_g^*(\xi)$ which has an explicit formula and is the spreading speed of Q_g in the direction ξ. We also show that for each $c \ge c_g^*(\xi)$, there exists a traveling wave solution in the direction ξ which is continuous if $gf'(0) \le 1$.

Keywords: monostable, spreading speed, wave speed, traveling wave solutions, order-preserving.

1 Introduction

Consider a population living in the infinite habitat $\mathbf{R}^d$ whose adult density in the $n-$generation at location x is given by $u_n(x)$. The population reproduces and its offspring grows according to some law f. Thereafter, a fraction of the population begins to migrate according to the probability density function K. Let the fraction of the population that does not migrate be denoted by the constant $g \in [0,1]$. Then the density of the adult population in the next generation is given by

$$u_{n+1}(x) = Q_g[u_n](x) \equiv (1-g)\int_{\mathbf{R}^d} K(x-y)f(u_n(y))dy + gf(u_n(x)). \tag{1}$$

A law that is commonly used to describe the growth of population in fisheries is the Beverton-Holt function

$$f(u) = \frac{rMu}{M+(r-1)u} \tag{2}$$

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where $r > 1$ is the intrinsic growth rate and $M > 0$ is the carrying capacity of the population. Throughout this paper, instead of (2), we assume that f has the following properties:

(H1) $f \in C^1[0, M]$.
(H2) $f(0) = 0,\ f(M) = M$.
(H3) $f(u) > u$ for $u \in (0, M)$.
(H4) $f'(u) \geq 0$ and $f'(0) = r > 1$.
(H5) $f(u) \leq ru$ for $u \in [0, M]$.

Let $\mathcal{C}_M$ to be the set of all bounded functions defined on $\mathbf{R}^d$ with values in $[0, M]$. Then Q_g has the following properties:

(i) $Q_g(0) = 0$, $Q_g(M) = M$.
(ii) Q_g commutes with translations in $\mathbf{R}^d$.
(iii) Q_g is order-preserving; i.e. if $u, v \in \mathcal{C}_M$ and $u \leq v$, then $Q_g[u] \leq Q_g[v]$.
(iv) If u_n converges to u as $n \to \infty$ uniformly on every compact subset of $\mathbf{R}^d$, then $Q_g[u_n](x) \to Q_g[u](x)$ as $n \to \infty$ uniformly on every compact subset of $\mathbf{R}^d$.
(v) $Q_g[\alpha] > \alpha$ for $\alpha \in (0, M)$.

Conditions (i) and (iii) imply that $Q_g : \mathcal{C}_M \to \mathcal{C}_M$. Condition (ii) implies that Q_g maps constant functions to constant functions and condition (v) follows from hypothesis (H3). Also condition (v) implies that $Q_g^n[\alpha] \to M$ as $n \to \infty$ for any $0 < \alpha < M$. Thus M is a stable fixed point and 0 is an unstable fixed point for the operator Q_g, which has no other constant function fixed point between 0 and M. This case is often called the monostable case. The monostable case corresponds to the well studied Fisher's equation, $u_t = du_{xx} + u(1-u)$, [4], [1]. Fisher used this equation to model the spatial spread of an advantageous gene in a population living in a homogeneous one-dimensional habitat. Fisher's model being in continuous time is of doubtful validity. In 1978, Weinberger proposed a more realistic model in which time occurs in discrete steps designed to simulate synchronous generations. The model is given by (1) with $g = 0$ and $M = 1$. In 1982, Weinberger generalized the results in [5] to solutions of a recursion of the form

$$u_{n+1} = Q[u_n] \tag{3}$$

where Q satisfies hypotheses (i) to (v). Under these assumptions, Weinberger was able to define a number $c^*(\xi)$ for each unit vector $\xi \in \mathbf{R}^d$ which he called the wave speed.[1] He proved that if u_0 has bounded support and is positive on a set with positive measure, then $c^*(\xi)$ is the spreading speed of u_n defined by the recursion (3). Loosely speaking, $c^*(\xi)$ is the spreading speed of u_n means that if one runs at a speed greater than $c^*(\xi)$ in the direction ξ, then for large n, u_n is near 0 and if one runs at a speed less than $c^*(\xi)$ in the direction ξ, then u_n is near M. The precise statements are given in [6, Thms. 6.1 and 6.2] which we reproduce below.

Theorem 1.1. *Let*

$$\mathcal{S} = \{x \in \boldsymbol{R}^d \,|\, x \cdot \xi \leq c_g^*(\xi) \ \ \forall \ \ |\xi| = 1\}.$$

(i) Let $\mathcal{S}'$ be an open set in $\boldsymbol{R}^d$ containing $\mathcal{S}$ in its interior and let $u_0 \in [0, M]$ have compact support

[1]The definition of $c^*(\xi)$ is quite involved and will not be given here.

in $\boldsymbol{R}^d$. Then the solutions of the recursion $u_{n+1} = Q[u_n]$ satisfy

$$\lim_{n\to\infty} \max_{x\notin n\mathcal{S}'} u_n(x) = 0. \tag{4}$$

(ii) Let the interior of $\mathcal{S}$ be nonempty and let $\mathcal{S}''$ be a compact subset of $\mathcal{S}$. Suppose $u_0 > 0$ on an open subset of $\boldsymbol{R}^d$ and $u_0 \in [0, M]$. Then

$$\lim_{n\to\infty} \min_{x\in n\mathcal{S}''} u_n(x) = M. \tag{5}$$

In [6], under some additional conditions on Q, Weinberger also found an explicit formula for $c^*(\xi)$. We shall state these conditions and the formula for $c^*(\xi)$ in the next section. Finally, if Q satisfies the compactness condition

(vi) for every sequence $\{v_n\}$ in $\mathcal{C}_M$, there exists a subsequence $\{v_{n_i}\}$ such that $Q[v_{n_i}]$ converges uniformly on every bounded subset of $\mathbf{R}^d$,

then Weinberger showed that in the monostable case, Q has plane waves solutions for all $c \geq c^*(\xi)$. This means that for every $c \geq c^*(\xi)$, there exists a non-increasing function w_c defined on $\mathbf{R}$ such that $w_c(-\infty) = M$, $w_c(\infty) = 0$ and $u_n(x) = w_c(x \cdot \xi - nc)$ satisfies the recursion (3) for all n. It is clear that if w_c is a plane wave solution, then so is any of its translates. What makes plane wave solutions interesting is that the spreading speed $c^*(\xi)$ is precisely the smallest c for which plane wave solutions exist. Plane waves are often called traveling waves although this is not quite correct since plane wave solution is only one kind of traveling wave solutions [2]. However we shall use the terminology traveling wave solutions or traveling waves in this paper. Note that from part (ii) of Theorem 1.1, traveling wave solutions cannot exist for $c < c^*(\xi)$.

Returning to our model (1), since Q_g satisfies conditions (i) to (v), Theorem 1.1 is valid for solutions of the recursion $u_{n+1} = Q_g[u_n]$ and the set $\mathcal{S}$ is actually nonempty. Let the spreading speed of Q_g be denoted by $c_g^*(\xi)$. If $g \neq 0$, the operator Q_g does not satisfy the compactness property (vi) so that the results in [6] concerning the existence of traveling wave solutions cannot be applied. One of the main results in this paper is to show that under certain conditions on K, f and g, traveling wave solutions indeed exist for Q_g if $c \geq c_g^*(\xi)$. When $g = 0$, the existence of traveling waves was proved in [5] for the operator Q_0 using a method that does not rely on (vi). This method can be generalized to Q_g for $0 < g < 1$. For the case $c > c_g^*(\xi)$, the conditions that K is rotationally symmetric and has compact support assumed in [5] are not needed. For the case $c = c_g^*(\xi)$, we need to make additional assumptions on K and g. We either need to assume that K vanishes on the set $\{x \,|\, x \cdot \xi \geq B\}$ for some $B > 0$ or that $\nabla K \in L^1(R^d)$ and $gr < 1$ to prove the existence of traveling waves. Finally, we show that if K is continuous and $gr \leq 1$, then traveling wave solutions are necessarily continuous. All these results are stated in the next section and their proofs are given in the subsequent sections. In the last section, we briefly discuss the case when g is allowed to depend on the population density.

2 Formula for $c_g^*(\xi)$ and Statement of Results

We first consider the case when the population does not migrate.

Example 1. Let $g = 1$ so that $Q_1[u] = f(u)$. From the definition of $c_g^*(\xi)$ given in [6], we have $c_g^*(\xi) = 0$. To construct a traveling wave solution for $c > 0$, take any non-increasing function v defined on the interval $[0, c]$ such that $0 < v(c) < M$ and $v(0) = f(v(c))$. Let n be a positive integer. Then define v on the interval $[nc, (n+1)c]$ by induction by setting $v(s) = f^{-1}(v(s-c))$. Similarly, define v on the interval $[-nc, -(n-1)c]$ by induction by setting $v(s) = f(v(s+c))$. It is clear that if we let $u_n(x) = v(x \cdot \xi - nc)$, then $u_{n+1}(x) = v(x \cdot \xi - (n+1)c) = f(v(x \cdot \xi - nc)) = Q_1[u_n](x)$ for all n. Hence v is a traveling wave solution and v is continuous in $\mathbf{R}$ if and only if v is continuous in $[0, c]$. This example shows that there can be infinitely many traveling wave solutions when $g = 1$. Traveling wave solutions for $c = 0$ are any translates of $MH(-s)$ where H is the Heaviside function.

Because of example 1 and because the case $g = 0$ has been treated in [5], we shall assume that $0 < g < 1$ for the rest of this paper. We also assume that K satisfies the following conditions:

(K1) K is a probability density function defined in $\mathbf{R}^d$.

(K2) $\displaystyle\int_{\mathbf{R}^d} e^{\mu x \cdot \xi} K(x)dx < \infty$ for all $\mu \in \mathbf{R}$ and $|\xi| = 1$.

In practical applications, it is desirable to have a formula for $c_g^*(\xi)$ or at least for it to be computable. We now state Weinberger's results on the formula for $c^*(\xi)$ and then apply them to the operator Q_g. In his 1982 paper, Weinberger proved that if there exists a bounded nonnegative measure $m(x, dx)$ on $\mathbf{R}^d$ such that $Q[u](x) \leq \int_{\mathbf{R}^d} u(x-y)\, m(y, dy)$ for $u \in [0, M]$. Then

$$c^*(\xi) \leq \inf_{\mu > 0} \frac{1}{\mu} \log \int_{\mathbf{R}^d} e^{\mu x \cdot \xi}\, m(x, dx)\,. \tag{6}$$

If there exists a bounded nonnegative measure $\ell(x, dx)$ with the properties that $\int_{\mathbf{R}^d} \ell(x, dx) > 1$, and there exists $\epsilon > 0$ such that $Q[u](x) \geq \int u(x-y)\, \ell(y, dy)$ for all $u \in [0, \epsilon]$. Then the set $\mathcal{S}$ defined in Theorem 1.1 is non-empty and

$$c^*(\xi) \geq \inf_{\mu > 0} \frac{1}{\mu} \log \int_{\mathbf{R}^d} e^{\mu x \cdot \xi}\, \ell(x, dx)\,. \tag{7}$$

From hypothesis (H5), we see that $Q_g[u](x) \leq (1-g)r \int_{\mathbf{R}^d} K(x-y)u(y)dy + gru(x)$ for all $u \in [0, M]$. If we let $m(x, dx) = (1-g)rK(x) + rg\delta_0$ where δ_0 is the Dirac delta measure concentrated at the origin, then (6) holds. On the other hand, for any $1 < r_1 < r$, there exists $\epsilon > 0$ such that $f(u) \geq r_1 u$ for $u \in [0, \epsilon]$. Therefore for such u, $Q_g[u](x) \geq (1-g)r_1 \int_{\mathbf{R}^d} K(x-y)u(y)dy + gr_1 u(x)$. Let $\ell(x, dx) = (1-g)r_1 K(x) + gr_1\delta_0$. Then $\int_{\mathbf{R}^d} \ell(x, dx) = r_1 > 1$ so that (7) holds. Since r_1 can be chosen arbitrarily close to r, the spreading speed for Q_g is given by

$$c_g^*(\xi) = \inf_{\mu > 0} \frac{1}{\mu} \log[(1-g)r \int_{\mathbf{R}^d} e^{\mu x \cdot \xi} K(x)dx + rg]\,. \tag{8}$$

For the rest of this paper, we set

$$\Phi_{\xi,g}(\mu) = \frac{1}{\mu} \log[(1-g)r \int_{\mathbf{R}^d} e^{\mu x \cdot \xi} K(x)dx + rg]\,, \qquad \mu \neq 0\,. \tag{9}$$

From conditions (K1) and (K2), $\Phi_{\xi,g}$ is well defined for $\mu \neq 0$. Also, if we translate the operator Q_g by $\eta \in \mathbf{R}^d$, then δ_0 above is replaced by δ_η so that $\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}\delta_\eta\, dx = e^{\mu\eta\cdot\xi}$. The spreading speed in this case is $c_g^*(\xi) + \eta\cdot\xi$. We now give some examples to illustrate the use of (8).

Example 2. Let $K(x) = (\sigma/\pi)^{n/2}\, e^{-\sigma|x|^2}$. Then K is a probability density function and

$$\Phi_{\xi,0}(\mu) = \frac{1}{\mu}\log(r\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}K(x)dx) = \frac{\log r}{\mu} + \frac{\mu}{4\sigma}\,.$$

Thus $c_0^*(\xi) = \sqrt{\log r/\sigma}$ and $\mathcal{S}_0 = \{x \mid |x| \leq \sqrt{\log r/\sigma}\,\}$. Let $K(x) = (\sigma/\pi)^{n/2}\ e^{-\sigma(x-b)^2}$. Then $c_0^*(\xi) = b\cdot\xi + \sqrt{\log r/\sigma}$. Note that given a direction ξ, by choosing b properly, $c_g^*(\xi)$ can take on any value between $-\infty$ and ∞.

Example 3. Let K be as in example 2. Then

$$\Phi_{\xi,g}(\mu) = \frac{1}{\mu}\,\log\left[r(1-g)e^{\,\mu^2/4\sigma} + rg\right].$$

Since $e^{\mu^2/4\sigma} > 1$, $\Phi_{\xi,g}$ is decreasing in g for each fixed μ. Formula (8) implies that $c_g^*(\xi) < c_{g'}^*(\xi)$ if $g > g'$. Note that this result is valid for K with the property $\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}K(x)dx > 1$ for all $\mu > 0$.

Example 4. Using the fact that the logarithm is a concave function and $r > 1$, we have

$$\begin{aligned}\frac{1}{\mu}\log((1-g)r\int_{\mathbf{R}^d} K(x)e^{\mu x\cdot\xi}dx + gr) \ &\geq\ \frac{1}{\mu}[(1-g)\,\log(r\int_{\mathbf{R}^d} K(x)e^{\mu x\cdot\xi}dx) + g\log r] \\ &\geq\ (1-g)c_0^*(\xi) + \frac{1}{\mu}g\log r\,.\end{aligned}$$

Thus, $c_g^*(\xi) \geq (1-g)c_0^*(\xi)$.

The following lemma is important for the proof of the existence of traveling waves. It gives criteria when the infimum of $\Phi_{\xi,g}$ is achieved at a unique point $\mu_g^*(\xi) > 0$.

Lemma 2.1. *Let the unit vector $\xi \in \boldsymbol{R}^d$ be given. Then the following properties of $\Phi_{\xi,g}$ hold: (i) $\Phi_{\xi,g}$ has no local maximum on $\boldsymbol{R}^+$. (ii) If $\int_U K(x)dx > 0$ for some open set $U \subset P = \{x \mid x\cdot\xi \geq 0\}$, or if $K(x) = 0$ almost everywhere on the half space P and $0 < rg < 1$, then the infimum of $\Phi_{\xi,g}$ is achieved at a unique point $\mu_g^*(\xi)$ and $\Phi_{\xi,g}(\mu_g^*(\xi)) = c_g^*(\xi)$. Furthermore, $\Phi_{\xi,g}$ is strictly decreasing on the interval $(0, \mu_g^*(\xi))$. (iii) If $K(x) = 0$ on P and $rg \geq 1$, then $\Phi_{\xi,g}$ is strictly decreasing on $\boldsymbol{R}^+$ and $\lim_{\mu\to\infty}\Phi_{\xi,g}(\mu) = c_g^*(\xi) = 0$. We define $\mu_g^*(\xi) = \infty$ in this case.*

Proof. We first show that $\Phi_{\xi,g}$ has no local maximum on $\mathbf{R}^+$. Set

$$\Psi_{\xi,g}(\mu) = \frac{r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}\, x\cdot\xi\, K(x)dx}{r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}K(x)dx + rg}\,. \tag{10}$$

Then

$$\Phi'_{\xi,g}(\mu) = \frac{1}{\mu}(-\Phi_{\xi,g}(\mu) + \Psi_{\xi,g}(\mu)) \tag{11}$$

which implies that

$$(\mu^2 \Phi'_{\xi,g}(\mu))' = \mu \Psi'_{\xi,g}(\mu)\,. \tag{12}$$

Multiplying (10) by its denominator and differentiating, we have

$$\begin{aligned}\left[r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi} K(x)dx + rg\right]\Psi'_{\xi,g}(\mu) \quad &+ \quad r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}\, x\cdot\xi\, K(x)dx\ \Psi_{\xi,g}(\mu) \\ &= \quad r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}\,(x\cdot\xi)^2\, K(x)dx\,.\end{aligned}$$

Rearranging, we obtain

$$\left[r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi} K(x)dx + rg\right]\Psi'_{\xi,g}(\mu) = r(1-g)\int_{\mathbf{R}^d} e^{\mu x}\left[(x\cdot\xi)^2 - (x\cdot\xi)\Psi_{\xi,g}(\mu)\right] K(x)dx.$$

From (10),

$$r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}\Psi^2_{\xi,g}(\mu)\, K(x)dx + rg\,\Psi^2_{\xi,g}(\mu) - r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}(x\cdot\xi)\Psi_{\xi,g}(\mu)K(x)dx = 0\,.$$

Adding these two equations, we have

$$\begin{aligned}\left[r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi} K(x)dx + rg\right]\Psi'_{\xi,g}(\mu) \quad &= \quad r(1-g)\int_{\mathbf{R}^d} e^{\mu x\cdot\xi}[x\cdot\xi - \Psi_g(\mu)]^2\, K(x)\, dx \\ &+ \quad rg\Psi^2_{\xi,g}(\mu)\,.\end{aligned}$$

This implies that $\Psi'_{\xi,g} > 0$ on $\mathbf{R}^+$. From (12),

$$(\mu^2 \Phi'_{\xi,g}(\mu))' > 0 \tag{13}$$

which implies that $\Phi''_{\xi,g}(\mu_0) > 0$ at any critical point μ_0 of $\Phi_{\xi,g}$. Therefore $\Phi_{\xi,g}$ has no local maximum on $\mathbf{R}^+$. By Lebesgue's theorem, we notice that

$$\lim_{\mu\to 0^+} \Phi_{\xi,g}(\mu) = \lim_{\mu\to 0^+} \frac{\log r}{\mu} = \infty. \tag{14}$$

Since $\Phi_{\xi,g}$ is real analytic, to complete the proof, it suffices to determine whether the infimum of $\Phi_{\xi,g}$ is achieved. In case (iii), it is clear from (9) that $\Phi_{\xi,g}(\mu) > 0$ and $\lim_{\mu\to\infty}\Phi_{\xi,g}(\mu) = 0$ so $\mu^*_g(\xi) = \infty$. For the second half of case (ii), it is clear that the term inside the square bracket in (9) is less than 1 for sufficiently large μ and $\lim_{\mu\to\infty}\Phi_{\xi,g}(\mu) = 0$. Hence $\mu^*_g(\xi) < \infty$ and $c^*_g(\xi) < 0$. We now focus on the first half of case (ii) when K does not vanish on P.

First assume that there exists $B_\xi > 0$ such that $\int_{\{x\,|\,x\cdot\xi\geq B_\xi\}} K(x)dx = 0$ and we can choose B_ξ to be minimal for that condition. Then

$$\begin{aligned}\Phi_{\xi,g}(\mu) \quad &= \quad \frac{1}{\mu}\log[r(1-g)\int_{\{x\,|\,x\cdot\xi\leq B_\xi\}} e^{\mu x\cdot\xi} K(x)dx + rg] \\ &\leq \quad \frac{1}{\mu}\log[r(1-g)e^{\mu B_\xi} + rg] \leq B_\xi + \frac{\log r}{\mu}\,.\end{aligned}$$

Now for $0 < \epsilon < B_\xi/2$, there exists an interval $[u,v] \subset (B_\xi - \epsilon, B_\xi)$ and $\alpha > 0$ such that $\int_{\{x\,|\,u \le x\cdot\xi \le v\}} K(x)dx \ge \alpha$. Thus

$$\begin{aligned}
\Phi_{\xi,g}(\mu) &\ge \frac{1}{\mu}\log\left[r(1-g)\int_{\{x\,|\,u\le x\cdot\xi\le v\}} e^{\mu x\cdot\xi}K(x)dx + rg\right] \\
&\ge \frac{1}{\mu}\log\left[r(1-g)\,\alpha\, e^{\mu u} + rg\right] \\
&\ge u + \frac{1}{\mu}\log\left[r(1-g)\alpha + rge^{-\mu u}\right] \\
&\ge B_\xi - \epsilon
\end{aligned}$$

for sufficiently large μ. Thus $\lim_{\mu\to\infty}\Phi_{\xi,g}(\mu) = B_\xi$. We now show that there exists $\mu > 0$ such that $\Phi_{\xi,g}(\mu) < B_\xi$. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\int_{\{x\,|\,B_\xi-\delta\,\le\, x\cdot\xi\,\le\, B_\xi\}} K(x)\,dx < \epsilon\,.$$

Then

$$\begin{aligned}
\Phi_{\xi,g}(\mu) &\le \frac{1}{\mu}\log\left[r(1-g)\int_{\{x\,|\,x\cdot\xi\,\le\, B_\xi-\delta\}} e^{\mu x\cdot\xi}K(x)dx + r(1-g)\epsilon e^{\mu B_\xi} + rg\right] \\
&\le \frac{1}{\mu}\log\left[r(1-g)e^{\mu(B_\xi-\delta)} + r(1-g)\epsilon e^{\mu B_\xi} + rg\right] \\
&= B_\xi + \frac{1}{\mu}\log\left[r(1-g)e^{-\mu\delta} + r(1-g)\epsilon + rge^{-B_\xi\mu}\right].
\end{aligned}$$

Let $\epsilon = 1/(2r(1-g))$. Then the expression inside the square bracket above converges to $1/2$ as $\mu\to\infty$. Therefore, $\Phi_{\xi,g}(\mu) < B_\xi$ for sufficiently large μ. This completes the proof for the case K vanishes on the set $\{x\,|\,x\cdot\xi > B_\xi\}$.

Let us assume instead that for any positive integer n, $\int_{\{x\,|\,x\cdot\xi\ge n\}} K(x) > 0$. Then

$$\begin{aligned}
\Phi_{\xi,g}(\mu) &\ge \frac{1}{\mu}\log\left[(1-g)\,r\int_{\{x\,|\,x\cdot\xi\ge n\}} e^{\mu x\cdot\xi}K(x)dx\right] \\
&\ge \frac{1}{\mu}\log\left[(1-g)\,re^{\mu n}\int_{\{x\,|\,x\cdot\xi\ge n\}} K(x)dx\right] \\
&= n + \frac{1}{\mu}\log\left[(1-g)\,r\int_{\{x\,|\,x\cdot\xi\ge n\}} K(x)dx\right].
\end{aligned}$$

Thus $\liminf_{\mu\to\infty}\Phi_{\xi,g}(\mu) \ge n$. Since n is arbitrary, $\lim_{\mu\to\infty}\Phi_{\xi,g}(\mu) = \infty$. We conclude that the infimum of $\Phi_{\xi,g}$ can not be achieved at infinity. □

Example 5. From (9) we have

$$\Phi_{-\xi,g}(\mu) = \frac{1}{\mu}\log\left[r(1-g)\int_{\mathbf{R}^d} e^{-\mu x\cdot\xi}K(x)dx + rg\right] = -\Phi_{\xi,g}(-\mu)\,.$$

Therefore, the graph of $\Phi_{\xi,g}$ on $\mathbf{R}^-$ is a reflection of the graph of $\Phi_{-\xi,g}$ on $\mathbf{R}^+$ about the origin. Suppose the minimum of $\Phi_{-\xi,g}$ is achieved on $\mathbf{R}^+$. Then from Lemma 2.1, there exists a $\mu^*_- < 0$ such that $\Phi_{\xi,g}$ is increasing on $(-\infty, \mu^*_-)$, decreasing on $(\mu^*_-, 0)$, and $\lim_{\mu\to 0-} \Phi_{\xi,g}(\mu) = -\infty$. From the above formula, $c^*_g(-\xi) = -\Phi_{\xi,g}(\mu^*_-)$. Note that (11) is valid on $(-\infty, 0) \cup (0, \infty)$ and $\Psi'_{\xi,g} > 0$ on $\mathbf{R}$. It follows that

$$-c^*_g(-\xi) = \Phi_{\xi,g}(\mu^*_-) = \Psi_{\xi,g}(\mu^*_-) < \Psi_{\xi,g}(\mu^*_g(\xi)) = \Phi_{\xi,g}(\mu^*_g(\xi)) = c^*_g(\xi)\,.$$

Thus $c^*_g(\xi) + c^*_g(-\xi) > 0$.

We now state the main results of this paper which concern the existence of traveling wave solutions. Recall that a traveling wave solution with speed c is a non-increasing function w_c defined on $\mathbf{R}$ such that $w_c(-\infty) = M$, $w_c(\infty) = 0$ and the sequence of functions $u_n(x) = w_c(x\cdot\xi - nc)$, $n \geq 0$ satisfy (1). If we substitute u_n defined this way into (1), then w_c satisfies the integral equation

$$w_c(x\cdot\xi) = (1-g)\int_{\mathbf{R}^d} K(x + c\xi - y) f(w_c(y\cdot\xi))dy + g f(w_c(x\cdot\xi + c))\,. \qquad (15)$$

From part (ii) of Theorem 1.1, there is no traveling wave solution with speed $c < c^*_g(\xi)$. In the following theorems $\mu^*_g(\xi) = \infty$ is allowed in which case $c^*_g(\xi) = \lim_{\mu\to\infty} \Phi_{\xi,g}(\mu)$. Note that in [5], Weinberger assumed that K is rotationally symmetric. Hence K cannot vanish on the set $P = \{x \,|\, x\cdot\xi \geq 0\}$ and Lemma 2.1 implies that $\mu^*_g(\xi) < 0$. If K is allowed to vanish on P, then Theorem 2.2 says that we may have standing wave; i.e. traveling wave with zero wave speed.

Theorem 2.1. *Let K satisfy conditions (K1) and (K2) and let f satisfy conditions (H1) to (H5). Suppose also that $f''(0)$ exists. Then for each $c > c^*_g(\xi)$, there exists a traveling wave solution w_c with speed c. Furthermore, let $\mu_c \in (0, \mu^*_g(\xi))$ be defined by $\Phi_{\xi,g}(\mu_c) = c$. Then $w_c(s) \sim Me^{-\mu_c s}$ as $s \to \infty$.*

Theorem 2.2. *Let K satisfy conditions (K1) and (K2) and suppose there exists a number B such that $K(x) = 0$ almost everywhere for $x\cdot\xi \geq B$. Let f satisfy conditions (H1) to (H5) and suppose that $f''(0)$ exists. If $\mu^*_g(\xi) < \infty$, then there exists a traveling wave solution w_c with speed $c = c^*_g(\xi)$. Furthermore, $w_{c^*_g(\xi)}(s) \sim Mse^{-\mu^*_g(\xi)s}$ as $s \to \infty$. If $\mu^*_g(\xi) = \infty$ and $gr > 1$, then there exists a standing wave (i.e. traveling wave with $c^*_g(\xi) = 0$) and $w_0(s) = 0$ for $s \geq 0$.*

The following theorem offers another set of conditions when traveling wave solutions exist for $c \geq c^*_g(\xi)$.

Theorem 2.3. *Let K satisfy conditions (K1) and (K2) and assume that K is differentiable in $\boldsymbol{R}^d$ and $\nabla K \in L^1(\boldsymbol{R}^d)$. Suppose f satisfies conditions (H1) to (H5) and $f'(u) \leq r$ for $u \in [0, M]$. Let $0 < gr < 1$. Then there exists a traveling wave solution w_c with speed $c \geq c^*_g(\xi)$.*

Theorem 2.4. *Let K satisfy conditions (K1) and (K2) and assume that K is continuous in $\boldsymbol{R}^d$. Also assume that $0 < gr \leq 1$ and $f'(u) < r$ on $(0, M)$. Then any traveling wave solution of Q_g is continuous on $\boldsymbol{R}$.*

Regarding the uniqueness of traveling wave solutions, the case $g = 0$ and $c > c^*_0(1)$ was studied in [3]. The case $c = c^*_0(1)$ was studied in [7] under the additional assumption that $f(u)/u$ is non-increasing. We shall not attempt to prove these results for the operator Q_g in this paper.

3 Proof of Theorem 2.1

The proof is similar to [5, Thm. 5] except for the modifications to allow K to have unbounded support and not be rotationally symmetric. For $c > c_g^*(\xi)$, let $\mu_c \in (0, \mu_g^*(\xi))$ satisfy $\Phi_{\xi,g}(\mu_c) = c$. Let

$$z_0(x) = \begin{cases} M & \text{if } x \cdot \xi < 0 \\ Me^{-\mu_c x\cdot\xi} & \text{if } x \cdot \xi \geq 0 \,. \end{cases}$$

From hypothesis (H5), we have

$$\begin{aligned} Q_g[z_0](x) &\leq (1-g)r \int_{\mathbf{R}^d} K(y) z_0(x-y)dy + grz_0(x) \\ &\leq (1-g)rM \int_{\mathbf{R}^d} K(y) e^{-\mu_c(x-y)\cdot\xi} dy + grMe^{-\mu_c x\cdot\xi} \\ &= Me^{-\mu_c(x-c\xi)\cdot\xi} \,. \end{aligned}$$

Let T_g be defined by $T_g[u](x) = Q_g[u](x + c\xi)$ for $u \in \mathcal{C}_M$. Since $z_0 \leq M$, $Q_g[z_0] \leq Q_g[M] = M$. Therefore $T_g[z_0] \leq z_0$. Let $z_{n+1} = T_g[z_n]$ for $n \geq 0$. Then since T_g is order preserving, z_n is non-increasing in n and bounded below. Therefore $\lim_{n\to\infty} z_n(x) = u(x)$ for each x and $T_g[u] = u$. It remains to show that u is not uniformly zero. Let λ satisfy $\mu_c < \lambda \leq \min(\mu_g^*(\xi), 2\mu_c)$ and let

$$v(x) = \begin{cases} 0 & \text{if } x \cdot \xi < 0 \\ \tau M(e^{-\mu_c x\cdot\xi} - e^{-\lambda x\cdot\xi}) & \text{if } x \cdot \xi \geq 0 \,. \end{cases}$$

One can then verify that

$$\begin{aligned} (1-g)r \int_{\mathbf{R}^d} K(y) v(x-y)dy + rgv(x) &\geq \tau M \left[e^{-\mu_c(x\cdot\xi - c)} - e^{-\lambda(x\cdot\xi - \Phi_{\xi,g}(\lambda))} \right] \\ &= v(x-c\xi) + \tau Me^{-\lambda(x\cdot\xi - c)} \left[1 - e^{-\lambda(c - \Phi_{\xi,g}(\lambda))} \right] \,. \end{aligned} \tag{16}$$

Since $f''(0)$ exists, the function $(f(u) - ru)/u^2$ is continuous on $[0, M]$ which implies that there exists $D > 0$ such that $f(u) \geq r[u - Du^2]$ for $u \in [0, M]$. For $x \cdot \xi \geq c$,

$$\begin{aligned} Q_g[v](x) &\geq v(x-c\xi) + \tau Me^{-\lambda(x\cdot\xi - c)} \left[1 - e^{-\lambda(c - \Phi_{\xi,g}(\lambda))} \right] \\ &- \tau^2 M^2 De^{-\lambda(x\cdot\xi - c)} \left[e^{-(2\mu_c - \lambda)x\cdot\xi + 2\mu_c\Phi_{\xi,g}(2\mu_c) - \lambda c} \right] \,. \end{aligned} \tag{17}$$

The last term of (17) comes from the fact that $v(x) \leq \tau Me^{-\mu_c x\cdot\xi}$ if $x \cdot \xi \geq 0$ and evaluating the integral $-D\tau^2 M^2 r \int_{R^d} K(x-y) e^{-2\mu_c x\cdot\xi} dy$. Since $\mu_c < \lambda$, we have $c > \Phi_{\xi,g}(\lambda)$ so that for sufficiently small $\tau > 0$, the last two terms in (17) together is positive and $Q_g[v](x) \geq v(x - c\xi)$ if $x \cdot \xi \geq c$. If $x \cdot \xi < c$, then $v(x - c\xi) = 0$ so that $T_g[v] \geq v$. Since $v(x + (\log \tau/\mu_c)\xi)) \leq z_0(x)$ and T_g is order preserving, we have $v(x + (\log \tau/\mu_c)\xi) \leq z_n(x)$ for all n. Hence u is not the identically zero function.

The next step is to define the traveling wave solution w_c. Note that z_0 has the property that $z_0(x) = z_0(x')$ if $x\cdot\xi = x'\cdot\xi$. By induction this is also true for all z_n and u so that $u(x) = u((x\cdot\xi)\xi)$. Let $w_c(s) = u(s\,\xi)$. Then $w_c(-\infty) = M$, $w_c(\infty) = 0$ and from the fact that $T_g[u](x) = u(x)$, w_c satisfies the integral equation (15). The asymptotic of w_c at ∞ follows from the definitions of v and z_0, and the inequalities $v(x + (\log\tau/\mu_c)\,\xi) \le u(x) \le z_0(x)$. The proof of Theorem 2.1 is complete.

4 Proof of Theorem 2.2

The proof is similar to [5, Thm. 5] except that we provide more details here. The idea of the proof is to construct a sequence $\{z_n\}$ such that $z_{n+1} = T_g[z_n] \le z_n$ where $T_g[u](x) = Q_g(x + c^*\,\xi)$ and then show that z_n has a limit u which is not the identically zero function and u depends only on $x\cdot\xi$. To simplify notations, we let $c^* = c^*_g(\xi)$ and $\mu^* = \mu^*_g(\xi)$.

We first consider the case when $\mu^*_g(\xi) < \infty$. Let $E > B$ be chosen such that

$$e^{\mu^* E} > \max\{\frac{1}{\mu^*}, 1 + E, B\}\,.$$

Let

$$z_0(x) = \begin{cases} M & \text{if } x\cdot\xi < E \\ Me^{-\mu^* x\cdot\xi}(x\cdot\xi - E + e^{\mu^* E}) & \text{if } x\cdot\xi \ge E \end{cases}$$

and let $h_1(s) = Me^{-\mu^* s}(s - E + e^{\mu^* E})$. Then $h_1(0) > M$, $h_1(E) = M$ and h_1 is first increasing then decreasing in $\mathbf{R}$. Hence $h_1 > M$ in $(0, E)$. If $x\cdot\xi \ge E$ and $y\cdot\xi \le 0$, then $K(x-y) = 0$. Therefore for $x\cdot\xi \ge E$,

$$\begin{aligned} Q_g[z_0](x) &\le (1-g)rM\int_{\mathbf{R}^d} K(x-y)(y\cdot\xi - E + e^{\mu^* E})e^{-\mu^* y\cdot\xi}dy + grM(x\cdot\xi - E + e^{\mu^* E})e^{-\mu^* x\cdot\xi} \\ &= (1-g)rM\int_{\mathbf{R}^d} K(y)(x\cdot\xi - y\cdot\xi - E + e^{\mu^* E})e^{-\mu^*(x-y)\cdot\xi}dy + grM(x\cdot\xi - E + e^{\mu^* E})e^{-\mu^* x}\,. \end{aligned} \tag{18}$$

The expression in (18) may be simplified. From (11), we infer that $\Phi_{\xi,g}(\mu^*) = \Psi_{\xi,g}(\mu^*)$. From the definition of $\Psi_{\xi,g}$ and $\Phi_{\xi,g}(\mu^*) = c^*$, we have

$$\begin{aligned} \left[(1-g)r\int_{\mathbf{R}^d} K(y)\,y\cdot\xi\, e^{\mu^* y\cdot\xi}dy\right]e^{-\mu^* x\cdot\xi} &= c^*\left[(1-g)r\int_{\mathbf{R}^d} K(y)e^{\mu^* y\cdot\xi}dy + gr\right]e^{-\mu^* x\cdot\xi} \\ &= c^* e^{-\mu^*(x - c^*\,\xi)\cdot\xi}\,. \end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned} &(1-g)rM\textstyle\int_{\mathbf{R}^d} K(y)\,e^{\mu^* y\cdot\xi}dy\,(x\cdot\xi - E + e^{-\mu^* E})e^{-\mu x\cdot\xi} + grM(x\cdot\xi - E + e^{\mu^* E})e^{-\mu^* x\cdot\xi} \\ &= M(x\cdot\xi - E + e^{-\mu^* E})e^{-\mu^*(x - c^*\,\xi)\cdot\xi}\,. \end{aligned} \tag{20}$$

From (19) and (20), inequality (18) simplifies to $Q_g[z_0](x + c^*\,\xi) \le z_0(x)$ if $x\cdot\xi \ge E$. If $x\cdot\xi < E$, then $z_0 = M$ so $Q_g[z_0](x + c^*\,\xi) \le z_0(x)$ for $x \in \mathbf{R}^d$ or that $T_g[z_0] \le z_0$. Let $z_{n+1} = T_g[z_n]$ for $n \ge 0$.

Since T_g is order preserving, we have $z_{n+1} \le z_n$ for all $n \ge 0$ and z_n converges monotonically to a limiting function u. We now show that u is not the identically zero function.

Let $\mu > \mu^*$ and let

$$v(x) = \begin{cases} \tau M \left(x\cdot\xi e^{-\mu^* x\cdot\xi} - B \dfrac{e^{-\mu^* x\cdot\xi} - e^{-\mu x\cdot\xi}}{e^{(\mu-\mu^*)B} - 1} \right) & \text{if} \quad x\cdot\xi > 0 \\ 0 & \text{if} \quad x\cdot\xi \le 0\,. \end{cases}$$

Let

$$h_2(s) = M\left(s - B\,\frac{1 - e^{-(\mu-\mu^*)s}}{e^{(\mu-\mu^*)B} - 1} \right).$$

Then $h_2(-B) = 0$, $h_2(0) = 0$ and h_2 is first decreasing then increasing in $\mathbf{R}$. Hence

$$h_2(s) < 0 \quad \text{for} \quad s \in (-B, 0). \tag{21}$$

Note that $v(x) = \tau h_2(x\cdot\xi)e^{-\mu^* x\cdot\xi}$ if $x\cdot\xi > 0$. Since $f''(0)$ exists, there exists $D > 0$ such that $f(u) \ge r[u - Du^2]$ for $u \in [0, M]$. If $x\cdot\xi \ge 0$ and $y\cdot\xi \le -B$, then $K(x-y) = 0$. Thus for $x\cdot\xi > 0$

$$\begin{aligned} Q_g[v](x) \ge \tau r(1-g) &\int_{\{y\,|\,y\cdot\xi\ge -B\}} K(x-y)h_2(y\cdot\xi)e^{-\mu^* y\cdot\xi}dy \;+\; \tau rg\, h_2(x\cdot\xi)e^{-\mu^* x\cdot\xi} \\ -\tau^2 Dr(1-g) &\int_{\{y\,|\,y\cdot\xi\ge -B\}} K(x-y)h_2^2(y\cdot\xi)e^{-2\mu^* y\cdot\xi}dy - \tau^2 Drg\, h_2^2(x\cdot\xi)e^{-2\mu^* x\cdot\xi}. \end{aligned} \tag{22}$$

The first term on the right of (22) may be written as

$$\tau(1-g)r \int_{\mathbf{R}^d} K(y)e^{-\mu^*(x-y)\cdot\xi}h_2((x-y)\cdot\xi)dy + \tau gr\, h_2(x\cdot\xi)e^{-\mu^* x\cdot\xi}\,. \tag{23}$$

We now substitute in the definition of $h_2((x-y)\cdot\xi)$ and simplify the resulting integral. Using the definition of $\Psi_{\xi,g}$, the fact that $\Psi_{\xi,g}(\mu^*) = \Phi_{\xi,g}(\mu^*) = c^*$, and the relation

$$r(1-g)\int_{\mathbf{R}^d} K(y)e^{\mu y\cdot\xi}dy + rg = e^{\mu\Phi(\mu)}\,,$$

one can show that (23) equals

$$\begin{aligned} \tau M &\left[(x - c^*\xi)\cdot\xi\, e^{-\mu^*(x-c^*\xi)\cdot\xi} - B\, \frac{e^{-\mu^*(x-c^*\xi)\cdot\xi} - e^{-\mu(x-\Phi(\mu)\xi)\cdot\xi}}{e^{(\mu-\mu^*)B} - 1} \right] \\ &= v(x - c^*\xi) + \tau M B e^{-\mu x\cdot\xi}\left[\frac{e^{\mu\Phi(\mu)} - e^{\mu c^*}}{e^{(\mu-\mu^*)B} - 1} \right]. \end{aligned}$$

Since $\mu > \mu^*$, we have $\Phi(\mu) > c^*$ so that the last square bracket above is positive. From (22), we see that for sufficiently small τ, we have $Q_g[v](x + c^*\xi) \ge v(x)$ if $x\cdot\xi > 0$. The same inequality is certainly true for $x\cdot\xi \le 0$ since $v(x) = 0$. Therefore $T_g[v] \ge v$. Since $v(x + (\log\tau/\mu^*)\xi) \le z_0(x)$, we have $v(x + (\log \tau/\mu^*)\xi) \leq u(x) \leq z_0(x)$ so u is not the identically zero function. Since z_0 depends only on $x \cdot \xi$, by induction, the same is true for all z_n and u. Let $w_{c^*}(s) = u(s\,\xi)$. Then it is clear that w_{c^*} is a traveling wave solution of Q_g with speed c^*, $w_{c^*}(-\infty) = M, w_{c^*}(\infty) = 0$. The asymptotic behavior of $w_{c^*}(s)$ as $s \to \infty$ follows from the fact that $w_{c^*}(s)$ bounded between $z_0(s\,\xi)$ and $v((s + \log/\mu^*)\,\xi)$. The completes the proof for the case $\mu_g^*(\xi) < \infty$.

Next we consider the case $\mu_g^*(\xi) = \infty$. From Lemma 2.1, K vanishes on the set $\{x \,|\, x \cdot \xi \geq 0\}$, $\Phi_{\xi,g}$ is strictly decreasing on $(0, \infty)$, and $c_g^*(\xi) = 0$. Let

$$z_0(x) = \begin{cases} M & \text{if} \quad x \cdot \xi \leq 0 \\ 0 & \text{if} \quad x \cdot \xi > 0 \end{cases}$$

and let $z_{n+1} = Q_g[z_n]$ for $n \geq 0$. If $x \cdot \xi > 0$, we have

$$Q_g[z_0](x) = (1-g) \int_{\{y \,|\, y \cdot \xi \leq 0\}} K(y) f(z_0(x-y)) dy + g f(z_0(x)) = 0\,.$$

If $x \cdot \xi \leq 0$, we have $Q_g[z_0](x) \leq Q_g[M] = M$. Therefore, $z_1 \leq z_0$ and we find by induction that $0 \leq z_{n+1} \leq z_n$. To find a lower bound for z_n, choose any $\mu > 0$ and let

$$v_0(x) = \begin{cases} \tau M(1 - e^{\mu x \cdot \xi}) & \text{if} \quad x \cdot \xi \leq 0 \\ 0 & \text{if} \quad x \cdot \xi > 0\,. \end{cases}$$

From our hypothesis on $f''(0)$, there exists $D > 0$ such that $f(u) \geq r[u - Du^2]$. For $x \cdot \xi < 0$,

$$\begin{aligned} Q_g[v_0](x) &\geq g f(\tau M(1 - e^{\mu x \cdot \xi})) \\ &\geq g r \tau M(1 - e^{\mu x \cdot \xi}) - g r D \tau^2 M^2 (1 - e^{\mu x \cdot \xi})^2\,. \end{aligned}$$

Thus,

$$Q_g[v_0](x) - v_0(x) \;\;\geq\;\; \tau M(1 - e^{\mu x \cdot \xi}) \left[g r - 1 - g r D \tau M (1 - e^{\mu x \cdot \xi}) \right].$$

Since $gr > 1, \mu > 0$ and $x \cdot \xi \leq 0$, for sufficiently small $\tau < 1$, the term inside the square bracket in the above inequality is positive. Since $v_0(x) = 0$ for $x \cdot \xi \geq 0$, we conclude that $Q_g[v_0] \geq v_0$ on $\mathbf{R}^d$. From the definitions, $z_0 \geq v_0$. This implies that the sequence z_n converges to a limit w_0 as $n \to \infty$ and $v_0 \leq w_0 \leq z_0$. Also, w_0 is a traveling wave solution with speed 0 and vanishes on $\mathbf{R}^+$. The proof of Theorem 2.2 is complete.

5 Proof of Theorem 2.3

We only show the existence of traveling wave solutions in the case $c = c_g^*(\xi)$. Let c_n be a sequence of real numbers such that $c_n > c_g^*(\xi)$ and $\lim_{n\to\infty} c_n = c_g^*(\xi)$. According to Theorem 2.1, there exists a sequence of non-increasing functions $\{u_n\}$ such that

$$\lim_{s \to -\infty} u_n(s) = M, \qquad \lim_{s \to \infty} u_n(s) = 0\,, \quad \text{and}$$
$$u_n(s) = (1-g) \int_{\mathbf{R}^d} K(s\xi + c_n\xi - y) f(u_n(y \cdot \xi)) dy + g f(u_n(s + c_n))\,. \tag{24}$$

By translating u_n, we may assume that $u_n(0) = M/2$. We first show that u_n is uniformly Lipschitz continuous. Let

$$v_n(s) = \ (1-g)\int_{\mathbf{R}^d} K(s\xi + c_n\xi - y) f(u_n(y\cdot\xi))dy\,.$$

Then

$$|v_n'(s)| = |\ (1-g)\int_{\mathbf{R}^d} \nabla K(s\xi + c_n\xi - y)\cdot\xi f(u_n(y\cdot\xi))dy| \le (1-g)M\|\nabla K\|_{L^1(\mathbf{R}^d)}$$

which implies that

$$|v_n(s_1) - v_n(s_2)| \le (1-g)M\|\nabla K\|_{L^1(\mathbf{R}^d)}|s_1 - s_2| \quad \text{for all } \ s_1, s_2 \in \mathbf{R}\,.$$

From (24), we have

$$|u_n(s_1) - u_n(s_2)| \le (1-g)M\|\nabla K\|_{L^1(\mathbf{R}^d)}|s_1 - s_2| + gr|u_n(s_1 + c_n) - u_n(s_2 + c_n)|\,. \tag{25}$$

Let $\beta = M\dfrac{1-g}{1-gr}\|\nabla K\|_{L^1(\mathbf{R}^d)}$. If $c_n = 0$ it is clear from (25) that

$$\sup_{s_1,s_2\in\mathbf{R},s_1\neq s_2} \frac{|u_n(s_1) - u_n(s_2)|}{|s_1 - s_2|} \le \beta. \tag{26}$$

Otherwise, suppose there exist $s_1 \neq s_2$ such that $\beta < |u_n(s_1) - u_n(s_2)|/|s_1 - s_2|$. We first claim that

$$\beta < \frac{|u_n(s_1 + lc_n) - u_n(s_2 + lc_n)|}{|s_1 - s_2|} \tag{27}$$

for any positive integer l. This inequality holds when $l = 0$ by assumption. Suppose it holds for l. Then replacing s_1, s_2 in (25) by $s_1 + lc_n, s_2 + lc_n$, respectively, we see that (27) also holds for $l+1$ and the claim is valid. But this produces a contradiction since $\lim_{l\to\infty}|u_n(s_1 + lc_n) - u_n(s_2 + lc_n)| = 0$. Thus (26) is still valid. From Arzelá-Ascoli theorem, a subsequence, still labelled as $\{u_n\}$, converges uniformly on every compact subsets of $\mathbf{R}$ to a function u as $n \to \infty$. Since

$$\begin{aligned} |u_n(s + c_n) - u(s + c_g^*(\xi))| &\le |u_n(s + c_n) - u_n(s + c_g^*(\xi))| + |u_n(s + c_g^*(\xi)) - u(s + c_g^*(\xi))| \\ &\le \beta|c_n - c_g^*(\xi)| + |u_n(s + c_g^*(\xi)) - u(s + c_g^*(\xi))|\,, \end{aligned}$$

we have

$$\lim_{n\to\infty} u_n(s + c_n) = u(s + c_g^*(\xi)).$$

Letting $n \to \infty$ in the second line of (24), we obtain

$$u(s) = (1-g)\int_{\mathbf{R}^d} K(s\,\xi + c_g^*(\xi)\,\xi - y) f(u(y\cdot\xi))dy + gf(u(s + c_g^*(\xi))).$$

Let $s \to -\infty$ in the above equation, we have $u(-\infty) = f(u(-\infty))$. Since $u(0) = M/2$, we have $u(-\infty) = M$. Similarly, $u(\infty) = 0$. Thus u is a traveling wave solution with speed $c_g^*(\xi)$. The proof of Theorem 2.3 is complete.

6 Proof of Theorem 2.4

Let u be any non-increasing function satisfying

$$u(s) = (1-g)\int_{\mathbf{R}^d} K(s\xi + c\xi - y)f(u(y\cdot\xi))dy + gf(u(s+c)) \tag{28}$$

such that $\lim_{s\to-\infty} u(s) = M$ and $\lim_{s\to\infty} u(s) = 0$. If $g = 0$, as K is continuous, it is clear by Lebesgue's theorem that u is continuous on $\mathbf{R}$. Otherwise, as u is non-increasing, u can only have jump discontinuities. Let $s_0 \in \mathbf{R}$ be such that

$$\lim_{s\to s_0-} u(s) = l_0 > m_0 = \lim_{s\to s_0+} u(s).$$

From (28), the function $\varphi(s) = u(s) - gf(u(s+c))$ is continuous in $\mathbf{R}$. Let

$$l_n = \lim_{s\to s_0-} u(s+nc) \qquad \text{and} \qquad m_n = \lim_{s\to s_0+} u(s+nc).$$

By continuity of φ and f, we infer that $l_n - gf(l_{n+1}) = m_n - gf(m_{n+1})$. By the mean value theorem $l_n - m_n = gf'(\zeta_{n+1})(l_{n+1} - m_{n+1})$ for some $\zeta_{n+1} \in (m_{n+1}, l_{n+1})$. From our hypotheses, we infer that $l_n - m_n \le l_{n+1} - m_{n+1}$ and thus $0 \le l_0 - m_0 \le l_n - m_n$ for all n. Three cases arise depending on the sign of c. In the first case $c > 0$. If we let $n \to \infty$, then $l_n \to 0$ and $m_n \to 0$ and we obtain $l_0 = m_0$ which is a contradiction. In the second case $c < 0$. If we let $n \to \infty$, then $l_n \to M$ and $m_n \to M$ and we obtain $l_0 = m_0$ which is again a contradiction. If $c = 0$, then the continuity of φ implies that $l_0 - gf(l_0) = m_0 - gf(m_0)$. From the mean value theorem, there is a $\zeta \in (l_0, m_0)$ such that $1 = f'(\zeta)g$. This contradicts the hypotheses that $f'(u) < r$ for $u \in (0, M]$ and $rg \le 1$. The proof of the Theorem 2.4 is complete.

7 Discussion

An interesting and important extension to our model (1) is to allow the fraction of the population that does not disperse to depend on the local density of the population. The model will then become

$$u_{n+1} = \int_{\mathbf{R}^d} (1 - g(u_n(y))K(x-y)f(u_n(u))dy + g(u_n(x))f(u_n(x))\,.$$

It is reasonable to assume that $g(0) = 1$ and g is a non-increasing function defined on $[0, M]$. Properties (i) through (v) in section 1 still hold with the exception of (iii). It is possible to construct examples to show that Q_g is not order-preserving. Very little is known in the mathematical literature on problems like this. This problem is currently under investigation.

Another interesting problem is to allow allee effect in the model. Thus, we let f to have an intermediate fixed point $\alpha \in (0, M)$, $f(u) < u$ on $(0, \alpha)$ and $f(u) > u$ on (α, M). If f is increasing on $[0, M]$, then conditions (i) to (v) still hold, $0, M$ are stable fixed points and α is an unstable fixed point of Q_g. This is called the bistable case. It is expected that spreading speed $c^*_g(\xi)$ can be defined and Theorem 1.1 holds but then traveling wave solutions exist only when $c = c^*_g(\xi)$.

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