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Independent Non-split Domination in the Join and Corona of Graphs

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Abstract

In this paper, we explore the concept of independent non-split domination in graphs. In particular, we characterized the independent non-split dominating sets of the join and corona of graphs and obtain their independent non-split domination numbers. Also, a connected graph with a given order, independent domination number, and independent non-split domination number is constructed.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = \bigcup_{v \in X} N[v]$.

The subgraph $\langle C \rangle$ of G *induced* by C is the graph having vertex-set C and whose edge set consists of those edges of G incident with two elements of C. A graph is called *connected* if every two vertices are joined by a path; otherwise, it is *disconnected*.

A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set $S \subseteq V(G)$ is called an *independent dominating set* of G if for all $u, v \in S$, $uv \notin E(G)$. The *independent domination number* of G, denoted by $i(G)$, is the smallest cardinality of an independent dominating set of G.

A dominating set $S \subseteq V(G)$ is a *non-split dominating set* of G if $\langle V(G)\backslash S\rangle$ is connected. The *non-split domination number* of G, denoted by $\gamma_{ns}(G)$, is the smallest cardinality of a non-split dominating set of G. An independent dominating set $S \subseteq V(G)$ is an *independent non-split dominating set* of G if $\langle V(G)\backslash S\rangle$ is connected. The *independent non-split domination number* of G, denoted by $i_{ns}(G)$, is the smallest cadinality of an independent non-split dominating set of G.

The concept of non-split domination was introduced by V.R. Kulli and B. Janakiram [2]. They obtained bounds on $\gamma_{ns}(G)$ and investigated relationship with other parameters. In [1], the inverse non-split domination in graphs was introduced and discussed; and in [3], the non-split dominating sets in the join and corona are characterized and the non-split and inverse non-split domination numbers of these graphs were determined. In this paper, the concept of independent nonsplit domination in graphs is revisited. In particular, the independent non-split dominating sets in the join and corona are characterized and their independent non-split domination numbers are obtained.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H.

Remark 1.1 *Let G be a graph and let S be an independent non-split dominating set of G.*

(i) If v is a cut-vertex of G, then $v \notin S$.

(ii) If v is a leaf of G, then $v \in S$.

Remark 1.2 *Let G be a graph. If $i_{ns}(G)$ exists, then $i(G) \leq i_{ns}(G)$.*

Theorem 1.3 *Given positive integers a, b, n with $3 \leq a \leq b$ and $n \geq a + b$, there exists a connected graph G such that $i(G) = a$, $i_{ns}(G) = b$, and $|V(G)| = n$.*

Proof: Consider the path $P_a = [u_1, u_2, ..., u_a]$. Let G be a graph obtained from P_a by adding the edges u_ix_i for $i = 1, 2, ..., a$; adding the edges u_av_j for $j = 0, 1, ..., b-a$; and adding the vertices w_k for $k = 0, 1, ..., n-a-b$ and forming the complete graph K_r, where $V(K_r) = \{u_1, x_1, w_1, ..., w_{n-a-b}\}$ (see Figure 1). Then $S_1 = \{x_1, x_2, ..., x_{a-1}, u_a\}$ is a minimum independent dominating set

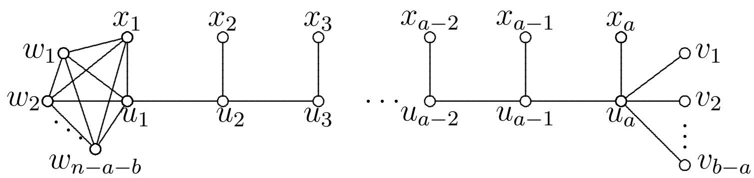

Figure 1: A graph G with $i(G) \leq i_{ns}(G)$

of G and $S_2 = \{x_1, x_2, ..., x_a\} \cup \{v_1, v_2, ..., v_{b-a}\}$ is a minimum independent non-split dominating set of G. Thus, $i(G) = |S_1| = a$, $i_{ns}(G) = |S_2| = b$, and $|V(G)| = n$. □

2 Main Results

Theorem 2.1 *Let G be a graph. Then $i_{ns}(G) = 1$ if and only if $G = K_1 + H$, where H is a connected graph.*

Proof: Suppose $i_{ns}(G) = 1$. Let $S = \{x\}$ be an independent non-split dominating set of G. Then $\langle V(G)\backslash S\rangle$ is connected. Set $K_1 = \langle\{x\}\rangle$ and $H = \langle V(G)\backslash S\rangle$. Then $G = K_1 + H$.

The converse is clear. □

Corollary 2.2 *$i_{ns}(K_1 + G) = 1$ if and only if G is a connected graph.*

Theorem 2.3 *$i_{ns}(K_1+G) = 2$ if and only if G is not connected and $i(G) = 2$.*

Proof: Suppose $i_{ns}(K_1 + G) = 2$. Let $K_1 = \langle\{v\}\rangle$ and $S = \{u, w\}$ be an independent non-split dominating set of $K_1 + G$. Since G is not connected, v is a cut-vertex of $K_1 + G$. By Remark 1.1(i), $v \notin S$ and so, $S \subseteq V(G)$. Thus, S is an independent dominating set of G. Hence, $i(G) \leq |S| = 2$. Since G is not connected, $i(G) \neq 1$. Therefore, $i(G) = 2$.

Conversely, suppose $i(G) = 2$. Let $S = \{x, y\}$ be an independent dominating set of G. Then S is an independent dominating set of $K_1 + G$. If $K_1 = \langle\{v\}\rangle$, then $vz \in E(K_1 + G)$ for all $z \in V(G)\backslash S$. This implies that $\langle V(K_1 + G)\backslash S\rangle$ is comnected. Hence, S is an independent non-split dominating set of K_1+G. Thus, $i_{ns}(K_1 + G) \leq |S| = 2$. Since G is not connected, $i_{ns}(K_1 + G) \neq 1$ by

Corollary 2.2. Therefore, $i_{ns}(K_1 + G) = 2$. □

The next result characterizes the independent non-split dominating sets of $K_1 + G$.

Theorem 2.4 *Let $K_1 = \langle \{v\} \rangle$ and G a nonconnected graph. Then $S \subseteq V(K_1 + G)$ is an independent non-split dominating set of $K_1 + G$ if and only if S is an independent dominating set of G.*

Proof: Suppose S is an independent non-split dominating set of $K_1 + G$. Since G is not connected, v is a cut-vertex of $K_1 + G$. By Remark 1.1(i), $v \notin S$. Thus, $S \subseteq V(G)$ and hence, S is an independent dominating set of G.

Conversely, suppose S is an independent dominating set of G. Then S is an independent dominating set of $K_1 + G$. Thus, $V(K_1 + G) \backslash S \neq \varnothing$ and by definition of $K_1 + G$, $\langle V(K_1 + G) \backslash S \rangle$ is connected. Therefore, S is an independent non-split dominating set of $K_1 + G$. □

Corollary 2.5 *Let $K_1 = \langle \{v\} \rangle$ and G a nonconnected graph. Then $i_{ns}(K_1 + G) = i(G)$.*

Proof: Suppose S is a minimum independent non-split dominating set of $K_1 + G$. By Theorem 2.4, S is an independent dominating set of G. Thus, $i(G) \leq |S| = i_{ns}(K_1 + G)$. On the other hand, suppose S' is a minimum independent dominating set of G. By Theorem 2.4, S' is an independent non-split dominating set of $K_1 + G$. Hence, $i(G) = |S'| \geq i_{ns}(K_1 + G)$. Therefore, $i_{ns}(K_1 + G) = i(G)$. □

Theorem 2.6 *Let G and H be graphs, both not isomorphic to $\overline{K_n}$. Then $S \subseteq V(G + H)$ is an independent non-split dominating set of $G + H$ if and only if either S is an independent dominating set of G or S is an independent dominating set of H.*

Proof: Suppose S is an independent non-split dominating set of $G + H$. Then either $S \subseteq V(G)$ or $S \subseteq V(H)$. Thus, either S is an independent dominating set of G or S is an independent dominating set of H.

For the converse, suppose S is an independent dominating set of G. Then $V(G) \backslash S \neq \varnothing$. Let $x \in V(G) \backslash S$. Then $xy \in E(G + H)$ for all $y \in V(H)$. This implies that $\langle V(G+H) \backslash S \rangle$ is connected. Hence, S is an independent non-split dominating set of $G + H$. Similarly, if S is an independent dominating set of H, then S is an independent non-split dominating set of $G + H$. □

Corollary 2.7 *Let G and H be graphs, both not isomorphic to $\overline{K_n}$. Then*

$$i_{ns}(G + H) = \min\{i(G), i(H)\}.$$

Proof: Suppose that $i(G) \leq i(H)$. Let S be a minimum independent dominating set of G. By Theorem 2.6, S is an independent non-split dominating set of $G + H$. Thus, $i(G) = |S| \geq i_{ns}(G + H)$. On the other hand, let S' be a minimum independent non-split dominating set of $G + H$. By Theorem 2.6, S' is an independent dominating set of G. Hence, $i(G) \leq |S'| = i_{ns}(G + H)$. Therefore, $i_{ns}(G+H) = i(G)$. Consequently, $i_{ns}(G+H) = \min\{i(G), i(H)\}$.□

Corollary 2.8 *Let G be a connected graph and n a positive integer greater than or equal to 2. Then $i_{ns}(G + \overline{K_n}) = \min\{i(G), n\}$.*

The next result characterizes the independent non-split dominating set of $G \circ H$.

Theorem 2.9 *Let G be a connected graph and let H be any graph. Then $C \subseteq V(G \circ H)$ is an independent non-split dominating set of $G \circ H$ if and only if $C = \bigcup_{v \in V(G)} S^v$, where S^v is an independent dominating set of H^v for all $v \in V(G)$.*

Proof: Suppose $C \subseteq V(G \circ H)$ is an independent non-split dominating set of $G \circ H$. Let $v \in V(G)$. Suppose $v \in C$. Then $v \notin V(G \circ H) \backslash C$. Since v is a cut-vertex of $G \circ H$, it follows $\langle V(G \circ H) \backslash C \rangle$ is disconnected. This contradicts the assumption that C is an independent non-split dominating set of $G \circ H$. Hence, $v \notin C$ for all $v \in V(G)$. Thus, $C \cap V(H^v) \neq \varnothing$ and $C \cap V(H^v)$ is a dominating set of H^v for all $v \in V(G)$. Set $S^v = C \cap V(H^v)$ for all $v \in V(G)$. Then $C = \bigcup_{v \in V(G)} S^v$.

Conversely, suppose $C = \bigcup_{v \in V(G)} S^v$, where S^v is an independent dominating set of H^v for all $v \in V(G)$. Clearly C is an independent dominating set of $G \circ H$. Let $x, y \in V(G \circ H) \backslash C$. If $xy \in E(G \circ H)$, then we are done. Suppose $xy \notin E(G \circ H)$. Consider the following cases:

Case 1. $x, y \in V(G)$.

Since G is connected, there exists an $x - y$ path in $\langle V(G \circ H) \backslash C \rangle$.

Case 2. $x \in V(G)$ and $y \in V(H^v) \backslash S^v$ for some $v \in V(G)$.

Then $vy \in E(G \circ H)$. by Case 1, there exists an $x - v$ path in $\langle V(G \circ H) \backslash C \rangle$. Thus, there exists an $x - y$ path in $\langle V(G \circ H) \backslash C \rangle$.

Case 3. $x \in V(H^u) \backslash S^u$ and $y \in V(H^v) \backslash S^v$ for some $u, v \in V(G)$.

Then $xu, vy \in E(G \circ H)$. By Case 1, there is a $u - v$ path in $\langle V(G \circ H) \backslash C \rangle$. Hence, there is an $x - y$ path in $\langle V(G \circ H) \backslash C \rangle$.

Therefore, $\langle V(G \circ H) \backslash C \rangle$ is connected. Accordingly, C is is an independent non-split dominating set of $G \circ H$. □

Corollary 2.10 *Let G be a connected graph and let H be any graph. Then $i_{ns}(G \circ H) = i(H)|V(G)|$.*

Proof: Suppose C is a minimum independent non-split dominating set of $G \circ H$. By Theorem 2.9, $C = \bigcup_{v \in V(G)} S^v$, where S^v is an independent dominating set of H^v for all $v \in V(G)$. Thus, $i_{ns}(G \circ H) = |C| = |S^v||V(G)| \geq i(H)|V(G)|$. On the other hand, let D be a minimum independent dominating set of H. For each $v \in V(G)$, let $S^v \subseteq V(H^v$ such that $\langle S^v \rangle \cong \langle D \rangle$. Then $C = \bigcup_{v \in V(G)} S^v$ is an independent non-split dominating set of $G \circ H$ by Theorem 2.9. Hence, $i_{ns}(G \circ H) \leq |C| = |S^v||V(G)| = i(H)|V(G)|$. Therefore, $i_{ns}(G \circ H) = i(H)|V(G)|$. □

Corollary 2.11 *Let G be a connected graph of order m. Then*
(i) $i_{ns}(G \circ K_n) = m$.
(ii) $i_{ns}(G \circ \overline{K_n}) = mn$.

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