Random Nearest Neighbor and Influence Graphs on $\mathbf{Z}^d$

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ABSTRACT: Random nearest neighbor and influence graphs with vertex set $\mathbf{Z}^\mathbf{d}$ are defined and their percolation properties are studied. The nearest neighbor graph has (with probability 1) only finite connected components and a superexponentially decaying connectivity function. Influence graphs (which are related to energy minimization searches in disordered Ising models) have a percolation transition. © 1999 John Wiley & Sons, Inc. Random Struct. Alg., 15, 262–278, 1999

1. INTRODUCTION AND MAIN RESULTS

This paper is concerned with certain random graphs $\mathscr{G}$ with vertex set $\mathbf{Z}^\mathbf{d}$ and edge set a random subset of $\mathbf{E}^\mathbf{d} = \{\{x, y\} : x, y \in \mathbf{Z}^\mathbf{d}, \|x - y\| = 1\}$, where $\|(x_1, \ldots, x_d)\| = |x_1| + \cdots + |x_d|$. Our results are mainly about the presence or absence of percolation, i.e., about the presence or absence (with probability 1) of infinite connected components of $\mathscr{G}$, and about the asymptotics of the connectivity function

$$\tau_{\mathscr{G}}(x, y) = \text{Prob}(x \leftrightarrow y) \tag{1.1}$$

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as $\|x-y\| \to \infty$. Here $x \leftrightarrow y$ denotes the event that x and y are in the same connected component (or cluster) of $\mathcal{G}$. Sometimes we will write $x \leftrightarrow_{\mathcal{G}} y$ to emphasize which graph is being used.

The simplest of our random graphs is the nearest neighbor graph, $\mathcal{N}$, whose definition is based on a random ordering of the edges in $\mathbf{E^d}$. The simplest way to construct such an ordering is to use a family $\{U_e : e \in \mathbf{E^d}\}$ of independent identically distributed (i.i.d.) random variables with a common uniform distribution on $[0, 1]$, and then order the edges according to the U_e values, noting that (with probability 1) there are no ties. Thinking of $U_{\{x,y\}}$ as the distance between x and y, we define the nearest neighbor *directed* graph $\mathcal{N}_D$ by placing a *directed* edge from each vertex x to its (unique, with probability 1) *nearest* neighbor; i.e., $\mathcal{N}_\mathrm{D}$ is the digraph with vertex set $\mathbf{Z^d}$ and edge set consisting of those directed edges (x, y) in $\mathbf{E^d_D} = \{(x, y) : \{x, y\} \in \mathbf{E^d}\}$ such that

$$U_{\{x,y\}} = \min\{U_{\{x,y'\}} : \|x-y'\| = 1\}. \tag{1.2}$$

$\mathcal{N}$ is then the (undirected) graph underlying $\mathcal{N}_\mathrm{D}$, i.e., the graph with the same vertex set $\mathbf{Z^d}$ as $\mathcal{N}_\mathrm{D}$ and edge set consisting of those $\{x, y\} \in \mathbf{E^d}$ such that (x, y) or (y, x) or both are in $\mathcal{N}_\mathrm{D}$. This graph may be thought of as a $\mathbf{Z^d}$ analogue of the nearest neighbor graph (based on the Poisson point process in $\mathbf{R^d}$) studied in [1] and shown there to be nonpercolating for any d. It is also similar to the next neighbor random graphs studied in [6]. In addition, the graphs and percolation issues we consider are similar to those treated in [2]. Our main result about $\mathcal{N}$ is the following.

Theorem 1. *For any d, the nearest neighbor graph $\mathcal{N}$ does not percolate and there exist constants $c_1, c_2, c_3, c_4 \in (0, \infty)$ depending only on d such that for all $x, y \in \mathbf{Z^d}$,*

$$\frac{c_1\, c_2^{\|x-y\|}}{\|x-y\|!} \le \tau_{\mathcal{N}}(x, y) \le \frac{c_3\, c_4^{\|x-y\|}}{\|x-y\|!}. \tag{1.3}$$

The proof of Theorem 1 will be given in Section 2 along with a description of the structure of the clusters of $\mathcal{N}$ (and of $\mathcal{N}_\mathrm{D}$). Meanwhile, we introduce influence graphs (and digraphs).

Influence graphs arise in the context of disordered Ising models. These models (see, e.g., [4]) and their influence graphs are defined in terms of a family $\{J_e : e \in \mathbf{E^d}\}$ of i.i.d. real-valued random variables. The configuration space of these models is $\mathcal{S} = \{-1, +1\}^{\mathbf{Z^d}}$ and the Hamiltonian or energy function is the formal series

$$H = -\sum_{\{x,y\}\in\mathbf{E^d}} J_{\{x,y\}} s_x s_y. \tag{1.4}$$

Regarded as a function of $s \in \mathcal{S}$, the Hamiltonian is of course only a formal object, but the change in energy, when the sign of a single s_x is changed, is well defined as a function on $\mathcal{S}$:

$$\triangle H_x(s) = 2 \sum_{y:\|x-y\|=1} J_{\{x,y\}} s_x s_y. \tag{1.5}$$

A local minimum (or metastable state) of H may then be defined as an $s' \in \mathcal{S}$ such that $\triangle H_x(s') \ge 0$ for every $x \in \mathbf{Z^d}$. In searching for local minima, one may begin with some $\tilde{s} \in \mathcal{S}$ and then successively flip the signs of single s_xs providing $\triangle H_x(s) \leq 0$. For a given realization of the couplings J_e, it is then natural to ask for each x, whether the value of s_y (with $\|x - y\| = 1$) can ever influence the decision to flip s_x, i.e., whether changing the sign of s_y can possibly change whether $\triangle H_x(s) \leq 0$. If so, we will say that y can influence x. It should be clear that y can influence x if and only if $J_{\{x,y\}} \neq 0$ and

$$\left|J_{\{x,y\}}\right| \geq \left|\sum_{\substack{z:\|x-z\|=1\\ z\neq y}} s'_z J_{\{x,z\}}\right| \tag{1.6}$$

for some choice of the s'_zs in $\{-1, +1\}$. Note that this condition only depends on the absolute values of the couplings $J_{\{x,y\}}$ and not on their signs.

We define the influence digraph $\mathcal{I}_{\mathrm{D}}$ as the directed graph with vertex set $\mathbf{Z^d}$ and edge set consisting of those (x, y) such that y can influence x. Note that the edges point in the direction where the influence comes from (i.e., toward the influencer rather than toward the influencee). The influence graph $\mathcal{I}$ is the (undirected) graph underlying $\mathcal{I}_{\mathrm{D}}$; i.e., $\{x, y\}$ is an edge of $\mathcal{I}$ if either y can influence x or x can influence y (or both). If $\mathcal{C}$ is a cluster (connected component) of $\mathcal{I}$, then in a dynamical search for local minima, as described above, the evolution of the s'_xs for $x \in \mathcal{C}$ depends only on the initial values $\tilde{s}_x$ for these same xs (and the order in which flips are attempted), but does not depend on the values of the s_xs for $x \notin \mathcal{C}$. Thus nonpercolation is an important property of $\mathcal{I}$ which, if valid, reduces the analysis of such dynamical searches from a problem on infinite graphs to one on finite graphs; i.e., it *localizes* the dynamics. Applications of nonpercolation to dynamics (at zero temperature) of disordered Ising models may be found in [3].

Before stating our main result on random influence graphs, we discuss their relation to nearest neighbor graphs. For simplicity, we (temporarily) assume that the common distribution of the J_es (and hence of the $|J_e|$s) is continuous and then (as in [5]) we embed our model into a one-parameter family of models with couplings $J_e(\lambda)$, $\lambda \geq 0$, where

$$J_e(\lambda) = \mathrm{sgn}(J_e)e^{-\lambda K_e}, \qquad K_e = -\ln|J_e|. \tag{1.7}$$

The K_es are i.i.d. continuous random variables and of course $J_e(1) = J_e$.

Note that there is always at least one neighbor y of x that can influence x, namely, the y which maximizes $|J_{\{x,y\}}(\lambda)|$ (or equivalently, minimizes $K_{\{x,y\}}$); to see that this is so, choose the s'_z values in (1.6) so that the signs of $s'_z J_{\{x,z\}}(\lambda)$ alternate as z successively takes on the values $z_2, z_3, \ldots, z_{2d}$ with $|J_{\{x,z_i\}}(\lambda)| \geq |J_{\{x,z_{i+1}\}}(\lambda)|$ for each i. This means that if we define the nearest neighbor graphs $\mathcal{N}_{\mathrm{D}}$ and $\mathcal{N}$, with the K_es replacing the U_es (which has no effect on the distribution of the ordering of the edges), then every edge in $\mathcal{N}_{\mathrm{D}}$ (respectively, $\mathcal{N}$) belongs to $\mathcal{I}_{\mathrm{D}}$ (respectively, $\mathcal{I}$): $\mathcal{N}_{\mathrm{D}} \subseteq \mathcal{I}_{\mathrm{D}}$ and $\mathcal{N} \subseteq \mathcal{I}$.

For some vertices x (and some values of λ), it will be the case that the *only* neighbor of y that can influence x is its nearest neighbor. This will be so when

$$\tilde{J}_x^{(1)}(\lambda) > \sum_{i=2}^{2d} \tilde{J}_x^{(i)}(\lambda), \tag{1.8}$$

where $\tilde{J}_x^{(1)}(\lambda) \geq \tilde{J}_x^{(2)}(\lambda) \cdots \geq \tilde{J}_x^{(2d)}(\lambda)$ are the order statistics of the $2d$ i.i.d. variables $|J_{\{x,y\}}(\lambda)|$ for y a neighbor of x. The influence graph, which we temporarily denote $\mathcal{J}(\lambda)$, depends on the parameter λ. If we set $\lambda = 0$, all $|J_e(0)|$s are $+1$ and $\mathcal{J}(0)$ is the full lattice $(\mathbf{Z^d}, \mathbf{E^d})$ which obviously percolates. In the limit $\lambda \to \infty$, it is not hard to see (as we discuss in the next paragraph) that

$$\theta_\lambda \equiv \operatorname{Prob}\left(\tilde{J}_x^{(1)}(\lambda) \leq \sum_{i=2}^{2d} \tilde{J}_x^{(i)}(\lambda)\right) \tag{1.9}$$

tends to zero and the limiting object $\mathcal{J}(\infty)$ reduces to the nearest neighbor graph $\mathcal{N}$, which does not percolate according to Theorem 1. The next theorem shows that if λ is sufficiently large, and hence θ_λ is sufficiently small, $\mathcal{J}(\lambda)$ still does not percolate.

To see that $\theta_\lambda \to 0$ as $\lambda \to \infty$, we note (as in [5]) that the inequality (1.8) would follow if $\tilde{J}_x^{(i)}(\lambda) < \tilde{J}_x^{(i-1)}(\lambda)/2$ for each i. Thus (1.8) is valid unless $1/2 \leq |J_{\{x,y\}}(\lambda)|/|J_{\{x,y'\}}(\lambda)| \leq 2$ for some distinct neighbors y and y' of x and so

$$\begin{aligned}\theta_\lambda &\leq \binom{2d}{2} \operatorname{Prob}\left(\frac{1}{2} \leq \frac{|J_{\{x,y\}}(\lambda)|}{|J_{\{x,y'\}}(\lambda)|} \leq 2\right) \\ &= \binom{2d}{2} \operatorname{Prob}\big(-\lambda^{-1}\ln 2 \leq (K_{e'} - K_e) \leq \lambda^{-1}\ln 2\big),\end{aligned} \tag{1.10}$$

where $K_{e'}$ and K_e are a pair of i.i.d. random variables, as in (1.7). Thus as $\lambda \to \infty$, θ_λ converges to $\operatorname{Prob}(K_{e'} = K_e)$, which vanishes if the common distribution of K_e and $K_{e'}$ is continuous, as we have assumed.

The next theorem also shows, as we now explain, that when λ is sufficiently small, $\mathcal{J}(\lambda)$ *does* percolate, as it does for $\lambda = 0$. This is based on an argument (see Lemma 3.5) concerning the random variable

$$\hat{J}_x(\lambda) \equiv \sum_{i=1}^{2d} (-1)^{i-1} \tilde{J}_x^{(i)}(\lambda), \tag{1.11}$$

which shows that if $\tilde{J}_x^{(2d-1)}(\lambda) > 0$ *and* $\tilde{J}_x^{(2d-1)}(\lambda) \geq \hat{J}_x(\lambda)/2$, then every neighbor can influence x except possibly its (strictly) *farthest* neighbor. Let us then define

$$\hat{\theta}_\lambda \equiv \operatorname{Prob}\big(\tilde{J}_x^{(2d-1)}(\lambda) > 0, \tilde{J}_x^{(2d-1)}(\lambda) \geq \hat{J}_x(\lambda)/2\big). \tag{1.12}$$

The next theorem states that if $\hat{\theta}_\lambda$ is sufficiently close to 1, then $\mathcal{J}(\lambda)$ percolates. We claim that $\hat{\theta}_\lambda \to 1$ as $\lambda \to 0$, and thus that $\mathcal{J}(\lambda)$ percolates for small λ. To verify this claim, note first that $\tilde{J}_x^{(2d-1)}(\lambda) > 0$ with probability 1 since $\operatorname{Prob}(|J_e| = 0) = 0$ (as we have assumed that $|J_e|$ has a continuous distribution) and second, by appropriate grouping into pairs of the form $\tilde{J}_x^{(2i)}(\lambda) - \tilde{J}_x^{(2i+1)}(\lambda) \geq 0$, that

$$-\hat{J}_x(\lambda) + 2\tilde{J}_x^{(2d-1)}(\lambda) \geq -\tilde{J}_x^{(1)}(\lambda) + \tilde{J}_x^{(2d-2)}(\lambda) + \tilde{J}_x^{(2d-1)}(\lambda) + \tilde{J}_x^{(2d)}(\lambda). \tag{1.13}$$

Thus $\tilde{J}_x^{(2d-1)}(\lambda) \geq \hat{J}_x(\lambda)/2$ would follow if

$$\frac{\tilde{J}_x^{(2d-2)}(\lambda) + \tilde{J}_x^{(2d-1)}(\lambda) + \tilde{J}_x^{(2d)}(\lambda)}{\tilde{J}_x^{(1)}(\lambda)} \geq 1 \tag{1.14}$$

and thus if $\tilde{J}_x^{(i)}(\lambda)/\tilde{J}_x^{(1)}(\lambda) \geq 1/3$ for $i \geq (2d-2)$ or more simply if for all neighbors y and y', $|J_{\{x,y\}}(\lambda)|/|J_{\{x,y'\}}(\lambda)| \geq 1/3$. Consequently, we have, analogously to (1.10), that

$$\begin{aligned}\hat{\theta}_\lambda &\geq 1 - \binom{2d}{2} \operatorname{Prob}\left(\frac{|J_{\{x,y\}}(\lambda)|}{|J_{\{x,y'\}}(\lambda)|} < \frac{1}{3}\right) \\ &= 1 - \binom{2d}{2} \operatorname{Prob}(K_{e'} - K_e < -\lambda^{-1} \ln 3). \end{aligned} \tag{1.15}$$

Thus $\hat{\theta}_\lambda \to 1$ as $\lambda \to 0$ because K_e and $K_{e'}$ are finite random variables.

We are finally ready to state our theorem on influence graphs. In its statement we do *not* assume that the common distribution of the $|J_e|$s is continuous.

Theorem 2. *For any $d \geq 2$ and i.i.d. (not necessarily continuously distributed) J_es, define θ and $\hat{\theta}$ as in* (1.9) *and* (1.12) *but with J_e in place of $J_e(\lambda)$. There exists $\hat{\epsilon}(d) > 0$ such that if $\hat{\theta} > 1 - \hat{\epsilon}(d)$, then the influence graph $\mathcal{I}$ percolates. There also exists $\epsilon(d) > 0$ such that if $0 < \theta < \epsilon(d)$, then $\mathcal{I}$ does not percolate and*

$$c_1' e^{-c_2'\|x-y\|} \leq \tau_{\mathcal{I}}(x, y) \leq c_3' \, e^{-c_4'\|x-y\|} \tag{1.16}$$

for all $x, y \in \mathbf{Z}^\mathbf{d}$ and some constants $c_1', c_2', c_3', c_4' \in (0, \infty)$ with c_3', c_4' depending only on θ and d.

The rest of the paper is organized as follows. In Section 2, we analyze the nearest neighbor graph $\mathcal{N}$ (and $\mathcal{N}_\mathrm{D}$), giving the proof that $\mathcal{N}$ does not percolate (and the rest of Theorem 1) and an analysis of the clusters of $\mathcal{N}_\mathrm{D}$. We conclude Section 2 with a discussion of the "all but farthest neighbor" graph $\mathcal{A}$. In Section 3, we analyze the influence graph $\mathcal{I}$ and prove Theorem 2. Our analysis of $\mathcal{I}$ for small θ (respectively, small $1 - \hat{\theta}$) is based on regarding $\mathcal{I}$ as a kind of perturbation of $\mathcal{N}$ (respectively, of $\mathcal{A}$).

2. NEAREST NEIGHBOR GRAPHS

Proof of Theorem 1. Since $\mathcal{N}$ is translation invariant, without loss of generality we assume y is at the origin and prove the bounds for $\tau_{\mathcal{N}}$ with 0 replacing y in (1.3). That $\mathcal{N}$ does not percolate follows easily from (1.3) because then the mean size (number of vertices) of $\mathcal{C}_x$, the cluster containing x, is

$$\sum_{y \in \mathbf{Z}^\mathbf{d}} \tau_{\mathcal{N}}(x, y) < \infty,$$

and hence each $\mathcal{C}_x$ is finite with probability 1.

We first obtain the lower bound. Without loss of generality, by performing reflections and permutations on the coordinates of x, we may assume that $x_1 > 0$ (and that $x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$). Let $y_{-1} = (-1, 0, \ldots, 0)$, $y_0 = (0, \ldots, 0), \ldots, y_{\|x\|} = x$ be the sites (in order) on a directed path r_{D} from y_{-1} to the origin to x. (In this paper, we use the terms site and vertex interchangeably and paths are always site self-avoiding.) For $i = 1, \ldots, \|x\|$, let Q_i be the set of $2d-1$ neighbors of y_{i-1} other than y_{i-2} (but including y_i); let Q_0 be the set of $2d-1$ neighbors of y_{-1} other than $(-2, 0, \ldots, 0)$. Define $U_i = U_{\{y_{i-1}, y_i\}}$ and $\tilde{U}_i = \min\{U_{\{y_{i-1}, z\}} : z \in Q_i\}$.

Then

$$\begin{aligned}\tau_{\mathcal{N}}(0, x) &\geq \operatorname{Prob}\big((y_{i-1}, y_i) \in \mathcal{N}_{\mathrm{D}} \text{ for } i = 1, \ldots, \|x\|\big)\\ &= \operatorname{Prob}(U_0 > U_1 > \cdots > U_{\|x\|} \text{ and } U_i = \tilde{U}_i \text{ for } i = 1, \ldots, \|x\|)\\ &\geq \operatorname{Prob}\big(\tilde{U}_0 > \tilde{U}_1 > \cdots > \tilde{U}_{\|x\|} \text{ and } U_i = \tilde{U}_i \text{ for } i = 0, 1, \ldots, \|x\|\big)\\ &= \operatorname{Prob}\big(\tilde{U}_0 > \tilde{U}_1 > \cdots > \tilde{U}_{\|x\|}\big) \cdot \big[\operatorname{Prob}(U_1 = \tilde{U}_1)\big]^{\|x\|+1},\end{aligned} \tag{2.1}$$

where the last equality uses independence a few times. Since the $U_{\{y_{i-1}, z\}}$s for $z \in Q_i$ are $2d-1$ i.i.d. continuous random variables,

$$\operatorname{Prob}(\tilde{U}_i = U_i) = \frac{1}{2d-1} \tag{2.2}$$

and we get

$$\begin{aligned}\tau_{\mathcal{N}}(0, x) &\geq \operatorname{Prob}\big(\tilde{U}_0 > \tilde{U}_1 > \cdots > \tilde{U}_{\|x\|}\big)\left(\frac{1}{2d-1}\right)^{\|x\|+1}\\ &= \frac{1}{(\|x\|+1)!}\left(\frac{1}{2d-1}\right)^{\|x\|+1}.\end{aligned} \tag{2.3}$$

The lower bound in Theorem 1 follows, taking $c_2 < 1/(2d-1)$ and an appropriate c_1 (depending on d and c_2).

To prove the upper bound, we use Lemma 2.2. Let r be any finite path in $(\mathbf{Z}^{\mathbf{d}}, \mathbf{E}^{\mathbf{d}})$ and let $|r|$ denote the number of edges in r. Let R_z denote the set of finite paths with one endpoint at z. We write $r \in \mathcal{N}$ to mean that *every* edge of r is in the graph $\mathcal{N}$. Then

$$\begin{aligned}\tau_{\mathcal{N}}(x, y) &\leq \sum_{r \in R_x \cap R_y} \operatorname{Prob}(r \in \mathcal{N})\\ &\leq \sum_{n \geq \|x-y\|} \sum_{r \in R_x \cap R_y : |r| = n} \frac{2^n}{n!}\\ &\leq \sum_{n \geq \|x-y\|} \frac{2d(2d-1)^n 2^n}{n!}.\end{aligned} \tag{2.4}$$

The second inequality follows from Lemma 2.2. The third inequality follows from the fact that the number of self-avoiding walks in $\mathbf{Z}^{\mathbf{d}}$ of length n and starting at a fixed vertex is bounded from above by $2d(2d-1)^n$. The upper bound in Theorem 1 now follows. ■

The following lemma is needed for Lemma 2.2 and also for Theorem 3 which gives a description of the structure of finite clusters of $\mathcal{N}$ (and of $\mathcal{N}_{\mathrm{D}}$). A closed loop (or simply a loop) is a path that is site self-avoiding except that the last site coincides with the first.

Lemma 2.1. *With probability* 1,

(1) *there is exactly one directed edge in* $\mathcal{N}_{\mathrm{D}}$ *from each site, and*
(2) $\mathcal{N}_{\mathrm{D}}$ *contains no directed closed loops with more than two edges.*

Proof. The first part follows immediately from the fact that the edges are ordered according to U_e values, which are unique almost surely. Thus from among the $2d$ edges touching each site, with probability 1, exactly one edge takes a minimum value and this becomes the directed edge from that site. For the second part, note that a closed loop of n edges ($n \geq 3$) would require $U_{e_1} < \cdots < U_{e_n} < U_{e_1}$, a contradiction. ■

Let $\tilde{V}_r$ be the event that the U_e values along the edges of a finite path r of length $|r|$ are strictly decreasing until a site z in r, and then strictly increasing after this site. We refer to $\tilde{V}_r$ as a "valley event" since first the U_e values dip down and then climb up again along the path r. Note that $\tilde{V}_r$ includes the cases where the point $z \in r$ occurs at an endpoint of r, even though this does not truly represent a valley. Thus starting at one end of r, if we proceed to the other end, there is a $z \in r$ which represents the low point in the valley. We have the following lemma.

Lemma 2.2.

$$\operatorname{Prob}(r \in \mathcal{N}) \leq \operatorname{Prob}(\tilde{V}_r) = \frac{2^{|r|}}{|r|!}. \tag{2.5}$$

Proof. The inequality in the lemma follows from the fact that if $r \in \mathcal{N}$, then $\tilde{V}_r$ must occur. This is a consequence of part (1) of Lemma 2.1, as the reader can easily verify.

We continue with the equality of the lemma. For r any path and w, w' any sites on r, let $r(w, w')$ denote the *directed* subpath of r from w to w'. Denote by $D_{r(w,w')}$ the event that the values of U_e are strictly decreasing along $r(w, w')$. Let a and b denote the two endpoints of r. Then we may partition $\tilde{V}_r$, according to which z in r is the low point in the valley, as $\tilde{V}_r = \bigcup_{z \in r} \{D_{r(a,z)} \cap D_{r(b,z)}\}$. Since $D_{r(a,z)}$ and $D_{r(b,z)}$ are independent events, $\operatorname{Prob}(D_{r(w,w')}) = 1/|r(w,w')|!$ and $|r(b,z)| = |r| - |r(a,z)|$, we have

$$\begin{aligned}
\operatorname{Prob}(\tilde{V}_r) &= \sum_{z \in r} \frac{1}{|r(a,z)|!} \frac{1}{(|r| - |r(a,z)|)!} \\
&= \sum_{k=0}^{|r|} \frac{1}{k!} \frac{1}{(|r|-k)!} \\
&= \sum_{k=0}^{|r|} \frac{1}{|r|!} \binom{|r|}{k} = \frac{2^{|r|}}{|r|!}. \qquad ■
\end{aligned} \tag{2.6}$$

It is possible that two sites, x and y, are each others' nearest neighbors. In this case (x, y) and (y, x) (both directed edges) form a directed closed loop (in $\mathcal{N}_D$). We call such a loop a *miniloop*. We say an edge (x_1, y_1), in a directed graph, is directed toward a vertex x if there is a directed path from x_1 to x, whose first edge is (x_1, y_1).

Lemma 2.3. *Let $\mathcal{G}_D$ be a finite directed graph, whose underlying graph $\mathcal{G}$ is connected. Suppose that in $\mathcal{G}_D$*

(1) *there is exactly one directed edge from every vertex x, and*
(2) *there are no directed closed loops, other than miniloops.*

Then

(i) *$\mathcal{G}$ is a tree,*
(ii) *$\mathcal{G}_D$ contains exactly one miniloop, between some vertices x and y, and*
(iii) *every edge in $\mathcal{G}_D$ (besides (x, y) and (y, x)) is directed toward both x and y.*

Proof. If $\mathcal{G}$ contains a loop, then any assignment of directions (from $\mathcal{G}_D$) to those edges would have to violate hypothesis (1) or (2) of the lemma. This proves (i).

To see that there can be no more than one miniloop (for a connected $\mathcal{G}$), note that the existence of more than one miniloop would require that for some x, y_1 and y_2, both (x, y_1) and (x, y_2) are in $\mathcal{G}_D$, contradicting hypothesis (1) of the lemma. To see that there must be at least one miniloop (for a finite $\mathcal{G}_D$), start with some directed edge (x_1, x_2) and then using hypothesis (1) continue with the unique directed edge (x_2, x_3) from x_2, etc. This process must end, because $\mathcal{G}_D$ is finite, but because of hypothesis (2), it can only end with the edge (x_n, x_{n-1}) for some n. This yields one miniloop between x_{n-1} and x_n and completes the proof of (ii). Exactly the same construction (and the uniqueness of the miniloop) implies the validity of (iii). ■

By a cluster $\mathcal{C}_D$ of $\mathcal{N}_D$, we mean a maximal subdigraph of $\mathcal{N}_D$, such that its underlying graph $\mathcal{C}$ is a cluster of $\mathcal{N}$. The following theorem is an immediate consequence of Theorem 1, Lemma 2.1, and Lemma 2.3.

Theorem 3. *Let $\mathcal{C}_D$ be any cluster of $\mathcal{N}_D$ and let $\mathcal{C}$ be the corresponding cluster of $\mathcal{N}$. With probability* 1, *$\mathcal{C}_D$ and $\mathcal{C}$ are finite and satisfy* (i), (ii), *and* (iii) *of Lemma* 2.3.

The last topic we treat in this section is the all but farthest neighbor graph $\mathcal{A}$ and its directed version $\mathcal{A}_D$. Using the same uniform variables U_e as for the nearest neighbor graph we say a neighbor y of x is the *farthest* neighbor of x if $U_{\{x,y\}} > U_{\{x,y'\}}$ for every other neighbor y'. We then include the directed edge (x, y) in $\mathcal{A}_D$ unless y is the farthest neighbor of x; thus the undirected edge $\{x, y\}$ is included in $\mathcal{A}$ *unless* y is the farthest neighbor of x, *and* x is the farthest neighbor of y. Note that an edge $\{x, y\}$ *not* in $\mathcal{A}$ is analogous to a miniloop of $\mathcal{N}_D$, if one replaces each U_e by $1 - U_e$. Our result about the graph $\mathcal{A}$ is the following.

Theorem 4. *For any $d \geq 2$, the all but farthest neighbor graph $\mathcal{A}$ percolates; indeed $\mathcal{A}$ is a connected graph, with probability* 1. *Furthermore if $e_1, \ldots, e_n$ are distinct edges* *in* $\mathbf{E^d}$, *then*

$$\text{Prob}(e_i \notin \mathscr{A} \text{ for } i = 1, \ldots, n) \le \left(\frac{1}{2d}\right)^n. \tag{2.7}$$

Proof. We will show that $\bar{\mathscr{A}}$, the restriction of $\mathscr{A}$ to $\mathbf{Z^2}$ (or more accurately to any two-dimensional coordinate hyperplane of $\mathbf{Z^d}$, such as $\mathbf{Z^2} \times (m_1, \ldots, m_{d-2})$) is connected. To show that $\bar{\mathscr{A}}$ is connected, we need only show that in the dual lattice $\mathbf{Z^{2*}} \equiv \mathbf{Z^2} + (1/2, 1/2)$, the edges e^* that are dual to (i.e., perpendicular bisectors of) edges e *not* in $\bar{\mathscr{A}}$ form neither infinite paths nor closed loops.

To see that closed loops cannot occur, we argue as follows. If e_1^* and e_2^* are two dual edges that meet at one endpoint and are perpendicular, then they are dual to two edges $e_1 = \{x, y\}$ and $e_2 = \{y, z\}$ that touch, i.e., have an endpoint in common. (They are also perpendicular, but we will not make use of that.) So that e_1 and e_2 are both *not* in $\bar{\mathscr{A}}$, it must be that both x and z are farthest neighbors of y, which is impossible. (The same argument shows that two edges in $\mathbf{E^d}$ that touch have zero probability of both not being in $\bar{\mathscr{A}}$; we need to use this below.) Thus edges in $\mathbf{Z^{2*}}$ dual to edges not in $\bar{\mathscr{A}}$ can only form paths parallel to the coordinate axes (i.e., without turns); thus they cannot form closed loops at all.

To see that there is zero probability of an infinite path in $\mathbf{Z^{2*}}$ all of whose edges are dual to edges not in $\bar{\mathscr{A}}$, we note that such an infinite path must also be parallel to a coordinate axis. There are only countably many such paths and for any such path the probability for all its edges to be dual to edges not in $\bar{\mathscr{A}}$ vanishes according to (2.7), which remains to be proved.

The proof of (2.7) is as follows. Define for each site $x \in \mathbf{Z^d}$, the set N_x of d edges between x and the vertices $x + (1, 0, \ldots, 0), x + (0, 1, \ldots, 0), \ldots, x + (0, \ldots, 0, 1)$. Note that the N_xs form a disjoint partition of $\mathbf{E^d}$. For an edge $e = \{x, y\} \in \mathbf{E^d}$, define $N_e = N_x \cup N_y$ and note that if $e_1, \ldots, e_n$ do not touch (i.e., no pairs touch), then the N_{e_i}s are disjoint. If any pair of the edges $e_1, \ldots, e_n$ does touch, then, as explained earlier in this proof, the probability in (2.7) is zero. Thus in (2.7) we may assume that none of the edges touch. So that $e \notin \mathscr{A}$, it is necessary that U_e be smaller than the $U_{e'}$ values of the $2(2d-1)$ other edges touching e and thus U_e must be the smallest of the $2d\, U_{e'}$ values for $e' \in N_e$. This yields, for non-touching e_is,

$$\text{Prob}\left(\bigcap_{i=1}^{n} \{e_i \notin \mathscr{A}\}\right) \le \text{Prob}\left(\bigcap_{i=1}^{n} \left\{U_{e_i} = \min_{e' \in N_{e_i}} U_{e'}\right\}\right) = \left(\frac{1}{2d}\right)^n. \quad \blacksquare \tag{2.8}$$

3. INFLUENCE GRAPHS

This section is devoted to proving Theorem 2. We begin with the proof of the lower bound on the connectivity function of (1.16). We state this as a lemma. Note that the exceedingly weak hypothesis of the lemma is valid if $\theta < 1$.

Lemma 3.1. *If* $\text{Prob}(|J_e| \neq 0) > 0$, *then the first inequality of* (1.16) *is valid for all* x, y *and some* $c_1', c_2' \in (0, \infty)$.

Proof. If $\text{Prob}(|J_e| = b) > 0$ for some $b > 0$, the following simple argument suffices. Choose a path r between x and y with exactly $\|x - y\|$ edges; let $\bar{r}$ denote the set of edges either in r or touching any site in r, and write $|\bar{r}|$ for the number of edges in $\bar{r}$. If all the edges $\{z, z'\}$ touching a site z have $|J_{\{z,z'\}}| = b$, then (see (1.6)) every neighbor z' can influence z. Thus we have

$$\begin{aligned}\tau_{\mathcal{I}}(x, y) &\geq \text{Prob}\left(\bigcap_{e \in r}\{e \in \mathcal{I}\}\right) \\ &\geq \text{Prob}\left(\bigcap_{e \in \bar{r}}\{|J_e| = b\}\right) \\ &= \left[\text{Prob}(|J_e| = b)\right]^{|\bar{r}|}. \end{aligned} \tag{3.1}$$

It is easy to see that $|\bar{r}| \leq C_d|r|$ for some positive integer C_d depending only on d and thus the first inequality of (1.16) is valid with $c_1' = 1$ and $\exp(-c_2') = [\text{Prob}(|J_e| = b)]^{C_d}$.

If there is no such b, the proof requires more care. If $\text{Prob}(|J_e| \neq 0) > 0$ and there is no $b > 0$ with $\text{Prob}(|J_e| = b) > 0$, then we claim that there exist positive constants b, η, δ satisfying all the following properties:

(i) $\text{Prob}(|J_e| \in (b - \eta - \delta, b - \eta + \delta)) > 0$,
(ii) $\text{Prob}(|J_e| \in (b + \eta - \delta, b + \eta + \delta)) > 0$,
(iii) $b > (2d - 1)(\eta + \delta)$,
(iv) $(2d - 2)\eta > 2d\delta$,
(v) $(b - \eta - \delta)/(b + \eta + \delta) > 1/3$.

To find such b, η, δ, we first choose $b_1 > 0$ in the support of the distribution of $|J_e|$. Since $\text{Prob}(|J_e| = b_1) = 0$, it follows that for arbitrarily small η_1 there exists $b_2 \neq b_1$ with $|b_2 - b_1| < \eta_1$ and b_2 also in the support. We will choose b and η so that $\{b - \eta, b + \eta\} = \{b_1, b_2\}$. The reader can verify that all the desired conditions are satisfied by first taking η_1 small enough and then δ small enough.

We claim that if $d + 1$ of the edges $\{z, z'\}$ touching a site z have $|J_{\{z,z'\}}| \in (b - \eta - \delta, b - \eta + \delta)$ and the remaining $(d - 1)$ edges have $|J_{\{z,z'\}}| \in (b + \eta - \delta, b + \eta + \delta)$, it will again be the case that every neighbor z' can influence z. (We will call such a site z a "special" site.) We proceed to verify the claim. Let $\tilde{J}_z^{(i)}$, $i = 1, \ldots, 2d$, denote the order statistics of the $|J_{\{z,z'\}}|$s. Then condition (v) implies (see the discussion surrounding (1.14)) that every neighbor z' can influence z except *possibly* for the neighbor that minimizes $|J_{\{z,z'\}}|$. Let us denote that neighbor by $\hat{z}$. To see that the influence criterion (1.6) is satisfied for $\hat{z}$, we must show that

$$\left|J_{\{z,\hat{z}\}}\right| \geq \sum_{z' \neq \hat{z}} t_{z'}\left|J_{\{z,z'\}}\right| \geq 0 \tag{3.2}$$

for some choice of $t_{z'}$s in $\{-1, +1\}$. We choose these signs to be $+1$ when $|J_{\{z,z'\}}| \in (b - \eta - \delta, b - \eta + \delta)$ and -1 for the other z's. Then $|J_{\{z,\hat{z}\}}| > b - \eta - \delta > 0$, while the summation in (3.2) is bounded above by $d(b - \eta + \delta) - (d - 1)(b + \eta - \delta)$ and bounded below by $d(b-\eta-\delta)-(d-1)(b+\eta+\delta)$. Thus the first inequality of (3.2) is a consequence of

$$b-\eta-\delta > d(b-\eta+\delta)-(d-1)(b+\eta-\delta)$$
$$= b-(2d-1)\eta+(2d-1)\delta, \tag{3.3}$$

which is just condition (iv), while the second inequality of (3.2) is a consequence of condition (iii).

To complete the proof of Lemma 3.1 it suffices to find a path r between x and y of length $|r| = \|x-y\|$ and show that

$$\text{Prob(every } x \in r \text{ is special)} \geq c_1' \, e^{-c_2'|r|}. \tag{3.4}$$

We leave this as an exercise for the reader, noting only that it is based on the conditions (i) and (ii). ■

We next state and prove a lemma that is a standard result. It gives sufficient conditions for nonpercolation and for the upper bound on the connectivity function of (1.16). If r is a path in $(\mathbf{Z^d}, \mathbf{E^d})$, we say $r \in \mathcal{I}$ if every edge of r belongs to $\mathcal{I}$. We also recall that $|r|$ denotes the number of edges in r.

Lemma 3.2. *Suppose there are constants* $\bar{c}_3, \bar{c}_4 \in (0, \infty)$ *such that for all finite paths* r,

$$\text{Prob}(r \in \mathcal{I}) \leq \bar{c}_3 \, \bar{c}_4^{|r|}. \tag{3.5}$$

If $\bar{c}_4 < (2d-1)^{-1}$, *then* $\mathcal{I}$ *does not percolate and the second inequality of* (1.16) *is valid for some* c_3', $c_4' \in (0, \infty)$ *depending only on* $\bar{c}_3$, $\bar{c}_4$, *and* d.

Proof. It suffices to prove the claimed upper bound on $\tau_{\mathcal{I}}$ since, by standard arguments (i.e., summing over y for fixed x, as in the beginning of the proof of Theorem 1), that implies finite expected cluster size and hence nonpercolation. As in (2.4), and using the same notation, we have

$$\tau_{\mathcal{I}}(x,y) \leq \sum_{r \in R_x \cap R_y} \text{Prob}(r \in \mathcal{I})$$
$$\leq \sum_{n \geq \|x-y\|} \sum_{r \in R_x \cap R_y : |r| = n} \text{Prob}(r \in \mathcal{I})$$
$$\leq \sum_{n \geq \|x-y\|} 2d(2d-1)^{n-1} \bar{c}_3 \, \bar{c}_4^n. \tag{3.6}$$

The claimed upper bound of (1.16) immediately follows. ■

We state and prove another lemma, a standard fact from elementary large deviation theory, that will be used when we relate $\mathcal{I}$ for small θ (respectively, for small $1-\hat{\theta}$) to $\mathcal{N}$ (respectively, to $\mathcal{A}$).

Lemma 3.3. *Let $X_{n,p}$ be a binomial (n, p) random variable (i.e., the number of successes in n i.i.d. trials with success probability p). For any $0 < p < \alpha < 1$, there exists $G(\alpha, p) > 0$, such that for all n,*

$$\text{Prob}(X_{n,p} \geq \alpha n) \leq e^{-G(\alpha, p)n}. \tag{3.7}$$

Furthermore, $G(\alpha, p)$ can be chosen so that for any fixed $\alpha > 0$, $G(\alpha, p)$ increases as p decreases and

$$G(\alpha, p) \to \infty \qquad \text{as } p \downarrow 0. \tag{3.8}$$

Proof. By the exponential Markov inequality,

$$\text{Prob}(X_{n,p} \geq \alpha n) \leq \inf_{\{u>0\}} e^{-u\alpha n} E(e^{uX_{n,p}}), \tag{3.9}$$

which yields (3.7) with

$$G(\alpha, p) = -\inf_{\{u>0\}} \left[\ln(1 - p + pe^u) - \alpha u\right]. \tag{3.10}$$

The fact that $G > 0$ for $0 < p < \alpha < 1$ and is monotonic in p easily follows. The divergence (3.8) of G as $p \downarrow 0$, follows, for example, by taking $u = \ln(1/p)$ in (3.10) yielding

$$G(\alpha, p) \geq -\ln(1 - p + 1) + \alpha \ln(1/p) \to \infty. \qquad \blacksquare \tag{3.11}$$

We can now prove the small θ half of Theorem 2.

Proof of Theorem 2 for Small θ. By Lemma 3.2, it suffices to obtain the bound (3.5) for $\theta < \epsilon(d)$ with $\bar{c}_3, \bar{c}_4$ depending only on d and θ. Let us define

$$B = \left\{ x \in \mathbf{Z^d} : \tilde{J}_x^{(1)} \leq \sum_{i=2}^{d} \tilde{J}_x^{(i)} \right\}, \tag{3.12}$$

where, as usual, the $\tilde{J}_x^{(i)}$s are the order statistics of the $|J_{\{x,y\}}|$s so that $\theta = \text{Prob}(x \in B)$. We also define $\tilde{U}_e = -\ln|J_e|$ (allowing the value $+\infty$) and then define the random digraph $\tilde{\mathcal{N}}_{\text{D}}$ (and underlying graph $\tilde{\mathcal{N}}$) by declaring that $(x, y) \in \tilde{\mathcal{N}}_{\text{D}}$ whenever $\tilde{U}_{\{x,y\}} < \tilde{U}_{\{x,y'\}}$ for every other neighbor y' of x. (We do not accept that $\infty < \infty$.) If neither of two neighboring sites x and y is in B, then $\{x, y\} \in \mathcal{J}$ if and only if $\{x, y\} \in \tilde{\mathcal{N}}$ (see the discussion surrounding (1.8) in Section 1). If the $|J_e|$s are continuously distributed, then $\tilde{\mathcal{N}}$ is equidistributed with our standard nearest neighbor graph $\mathcal{N}$. In general, one can do a coupling of $\mathcal{N}$ and $\tilde{\mathcal{N}}$ so that $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ (but we will not need to use this).

The basic idea of the proof is that when θ is small, in a long path $|r|$, there should be a small density of sites from B and hence many segments of r that belong to

$\tilde{\mathcal{N}}$ and are not too short; but such segments in $\tilde{\mathcal{N}}$ have small probability by the $\tilde{\mathcal{N}}$ analogue of Lemma 2.2. We proceed to make this argument precise.

For r a fixed (long) path, denote $\{x \in r : x \in B\}$ by B_r and its cardinality by $|B_r|$. Then

$$\text{Prob}(r \in \mathcal{I}) \leq \text{Prob}\big(|B_r| > \beta|r|\big) + \sum_{\gamma : |\gamma| \leq \beta|r|} \text{Prob}(r \in \mathcal{I}, B_r = \gamma), \tag{3.13}$$

where the sum is over subsets γ of the sites in r with $|\gamma| \leq \beta|r|$. The (small) constant β will be chosen later. The event that $x \in B$ depends on the $J_{\{x, y\}}$s for all neighbors y of x and hence a collection of these events will only be mutually independent if no pair of the xs in the collection are neighbors. Let us write $\mathbf{Z}^{\mathbf{d}}_{+}$ and $\mathbf{Z}^{\mathbf{d}}_{-}$ to denote the even and odd subsets of $\mathbf{Z}^{\mathbf{d}}$ (red and black sites in a checkerboard pattern). Then $|B_r| = |B_r \cap \mathbf{Z}^{\mathbf{d}}_{+}| + |B_r \cap \mathbf{Z}^{\mathbf{d}}_{-}|$; the two summands are *not* independent but the summands each have a binomial distribution with parameters (n_+, θ) and (n_-, θ), where $n_+ + n_- = |r| + 1$ and $|n_+ - n_-| \leq 1$. Thus, by Lemma 3.3 (and assuming $\theta < \beta$),

$$\begin{aligned}\text{Prob}(|B_r| > \beta|r|) &\leq \text{Prob}\big(X_{n_+, \theta} \geq \beta n_+\big) + \text{Prob}\big(X_{n_-,\theta} \geq \beta n_-\big)\\ &\leq 2e^{-G(\beta, \theta)|r|/2}.\end{aligned} \tag{3.14}$$

We will need to make sure that whatever our eventual choice of β, $\epsilon(d)$ is chosen small enough so that (by (3.8)) $G(\beta, \theta) > 2\ln(2d - 1)$; this is so that $\bar{c}_4 < (2d-1)^{-1}$ in (3.5).

We now need to estimate $\text{Prob}(r \in \mathcal{I}, B_r = \gamma)$ for our fixed (long) path r and some subset γ of the sites of r with $|\gamma| \leq \beta|r|$. Let us start at x and proceed along the path r toward y listing in order all the nonempty (maximal) subpaths $r_1, r_2, \ldots, r_m$ of r that contain *no* sites from γ (and thus no sites from B, since $\gamma = B_r$ in (3.13), so that $r \in \mathcal{I}$ requires that each $r_i \in \tilde{\mathcal{N}}$). We thus have, using the valley events $\tilde{V}_r$ defined as in Lemma 2.2 (but with $\tilde{U}_e$ replacing U_e),

$$\begin{aligned}\text{Prob}(r \in \mathcal{I}, B_r = \gamma) &\leq \text{Prob}\big(B_r = \gamma, r_i \in \tilde{\mathcal{N}} \text{ for } i = 1, \ldots, m\big)\\ &\leq \text{Prob}\left(B_r = \gamma, \bigcap_{i=1}^{m} \tilde{V}_{r_i}\right)\\ &= \text{Prob}(B_r = \gamma) \prod_{i=1}^{m} P(\tilde{V}_{r_i}),\end{aligned} \tag{3.15}$$

where the equality follows from the mutual independence of the events in question.

Next we bound the product in (3.15) by restricting it further to those is with $|r_i| \geq K$ (where K is a large integer to be chosen later). Because the $\tilde{U}_e$s, unlike the U_es, need not have a continuous distribution, the equation (2.6) is replaced here by an inequality, yielding

$$\prod_{i=1}^{m} P(\tilde{V}_{r_i}) \le \prod_{i=1}^{m} \frac{2^{|r_i|}}{|r_i|!}$$
$$\le 2^{|r|} \prod_{i:|r_i|\ge K} \left[\left(\frac{1}{|r_i|!}\right)^{1/|r_i|}\right]^{|r_i|}$$
$$\le 2^{|r|} \exp\left[-c(K) \sum_{i:|r_i|\ge K} |r_i|\right], \tag{3.16}$$

where

$$e^{-c(K)} \equiv \sup_{l\ge K}(1/l!)^{1/l}. \tag{3.17}$$

Note that, by Stirling's formula, $c(K) \to \infty$ as $K \to \infty$.

We now need a lower bound on the total number of edges in all the $|r_i|$s with $|r_i| \ge K$. There are $|r|$ edges in r, and we remove fewer than the K edges on each side of each site of γ; since $|\gamma| \le \beta|r|$, there remain at least $(1 - 2\beta K)|r|$ edges, and so

$$\sum_{\gamma:|\gamma|\le\beta|r|} \text{Prob}(r \in \mathcal{J}, B_r = \gamma) \le \sum_{\gamma:|\gamma|\le\beta|r|} \text{Prob}(B_r = \gamma) 2^{|r|} e^{-c(K)(1-2\beta K)|r|}$$
$$\le 2^{|r|} e^{-c(K)(1-2\beta K)|r|}. \tag{3.18}$$

Combining (3.13), (3.14), and (3.18), we see that we obtain (3.5) with $\bar{c}_4 < (2d-1)^{-1}$ by the following choices of K, β, and $\epsilon(d)$. First choose K so large (depending only on d) that

$$e^{-c(K)} < \frac{(2d-1)^{-1}}{2}. \tag{3.19}$$

Next choose β so small (depending only on d and K) that

$$e^{-c(K)(1-2\beta K)} < \frac{(2d-1)^{-1}}{2}. \tag{3.20}$$

Finally choose ϵ so small (depending only on d and β) that

$$e^{-G(\beta,\epsilon)/2} < (2d-1)^{-1}. \tag{3.21}$$

The proof of the small θ half of Theorem 2 is now complete. ∎

In the other half of Theorem 2, we will use an argument analogous to Lemma 3.2, but applied to the dual lattice (of a two-dimensional slice of $\mathbf{Z^d}$). Let us identify the slice $\mathbf{Z}^2 \times (0, \ldots, 0)$ of $\mathbf{Z^d}$ with $\mathbf{Z}^2$ and, as in the proof of Theorem 4, denote its dual lattice $\mathbf{Z}^2 + (1/2, 1/2)$ by $\mathbf{Z}^{2*}$. For a path $r^* \in \mathbf{Z}^{2*}$, we write $r^* \notin \mathcal{J}$ to mean that *every* edge e^* in r^* is the dual of an edge e with $e \notin \mathcal{J}$. The following lemma is a standard result.

Lemma 3.4. *Suppose there are constants* $c_3^*, c_4^* \in (0, \infty)$ *such that for all finite paths* r^* *in* $\mathbf{Z}^{2*}$,

$$\operatorname{Prob}(r^* \notin \mathcal{I}) \leq c_3^* \, (c_4^*)^{|r^*|}. \tag{3.22}$$

If $c_4^* < 1/3$, *then* $\mathcal{I}$ *percolates.*

Proof. We will prove that $\mathcal{I}$ restricted to the plane $\mathbf{Z}^2$ already percolates. To demonstrate this, it suffices to show that the following event has zero probability: for every n, there is a closed loop l_n^* in $\mathbf{Z}^{2*}$ surrounding the square $[-n, n]^2$ with $l_n^* \notin \mathcal{I}$. But that event requires that for infinitely many positive integers m, a certain event, L_m, occurs. L_m is the event that there is a path r_m^* in $\mathbf{Z}^{2*}$ starting at $x^*(m) \equiv (m - 1/2, -1/2)$ with $|r_m^*| \geq m$ and $r_m^* \notin \mathcal{I}$. So by the Borel–Cantelli lemma, to prove percolation, it suffices to show that $\operatorname{Prob}(L_m)$ is summable over m.

Letting R_{x^*} denote the set of paths in $\mathbf{Z}^{2*}$ starting at x^*, and using (3.22), we have (as in (3.6)) that

$$\begin{aligned}\operatorname{Prob}(L_m) &\leq \sum_{n=m}^{\infty} \sum_{r^* \in R_{x^*(m)} : |r^*| = n} \operatorname{Prob}(r^* \notin \mathcal{I}) \\ &\leq \sum_{n=m}^{\infty} 4 \cdot 3^{n-1} c_3^* (c_4^*)^n \end{aligned} \tag{3.23}$$

and the summability in m easily follows when $c_4^* < 1/3$. ■

Before finally doing the proof of the small $1 - \hat{\theta}$ half of Theorem 2, we need one more lemma that justifies the claim made after (1.11), relating $\hat{\theta}$ and $\mathcal{I}$.

Lemma 3.5. *Let* $y_1, \ldots, y_d$ *denote the neighbors of* x *ordered so that* $\tilde{J}_x^{(i)} \equiv |J_{\{x, y_i\}}|$ *satisfies* $\tilde{J}_x^{(1)} \geq \tilde{J}_x^{(2)} \geq \cdots$, *and define* $\hat{J}_x \equiv \tilde{J}_x^{(1)} - \tilde{J}_x^{(2)} + \tilde{J}_x^{(3)} - \cdots$. *Suppose* $\tilde{J}_x^{(2d-1)} > 0$ *and* $\tilde{J}_x^{(2d-1)} \geq \hat{J}_x/2$; *then* y_i *can influence* x *for* $i = 1, 2, \ldots, 2d - 1$.

Proof. We need to show that for $i \leq 2d - 1$, there are signs t_j^i so that

$$\tilde{J}_x^{(i)} \geq \pm \sum_{j \neq i} t_j^i \tilde{J}_x^{(j)}. \tag{3.24}$$

For i odd, we choose $t_j^i = (-1)^{j-1}$ as in the signs defining $\hat{J}_x$. Then

$$\tilde{J}_x^{(i)} + \sum_{j \neq i} t_j^i \tilde{J}_x^{(j)} = \hat{J}_x \geq 0, \tag{3.25}$$

while

$$\tilde{J}_x^{(i)} - \sum_{j \neq i} t_j^i \tilde{J}_x^{(j)} = 2\tilde{J}_x^{(i)} - \hat{J}_x \geq 2\tilde{J}^{(2d-1)} - \hat{J}_x \geq 0. \tag{3.26}$$

For i even with $i \neq 2d$, we choose $t_j^i = (-1)^{j-1}$ except for $j = i + 1$, where we take $t_{i+1}^i = -1$. Then

$$\tilde{J}_x^{(i)} + \sum_{j \neq i} t_j^i \tilde{J}_x^{(j)} = \hat{J}_x + 2\big(\tilde{J}_x^{(i)} - \tilde{J}_x^{(i+1)}\big) \geq 0, \tag{3.27}$$

while

$$\tilde{J}_x^{(i)} - \sum_{j \neq i} t_j^i \tilde{J}_x^{(j)} = 2\tilde{J}_x^{(i+1)} - \hat{J}_x \geq 2\tilde{J}^{(2d-1)} - \hat{J}_x \geq 0. \tag{3.28}$$

This completes the proof of the lemma. ■

Proof of Theorem 2 for Small $1 - \hat{\theta}$. By Lemma 3.4, it suffices to show that (3.22), with $c_4^* < 1/3$, is valid for $1 - \hat{\theta} < \hat{\epsilon}(d)$. We define

$$\hat{B} = \left\{x \in \mathbf{Z^d} : \tilde{J}_x^{(2d-1)} = 0 \text{ or } \tilde{J}_x^{(2d-1)} < \hat{J}_x/2\right\}, \tag{3.29}$$

so that $1 - \hat{\theta} = \text{Prob}(x \in \hat{B})$. Let us also define $\tilde{U}_e = -\ln|J_e|$ (as in the proof of the small θ part of Theorem 2) and the digraph $\tilde{\mathcal{A}}_{\text{D}}$ (and its underlying graph $\tilde{\mathcal{A}}$) by declaring that $(x, y) \in \tilde{\mathcal{A}}_{\text{D}}$ unless $\tilde{U}_{\{x,y\}} > \tilde{U}_{\{x,y'\}}$ for all other neighbors y' of x. It follows from Lemma 3.5 that y can influence x if $x \notin \hat{B}$ and $(x, y) \notin \tilde{\mathcal{A}}_{\text{D}}$. Thus $\{x, y\} \in \mathcal{I}$ if $x \notin \hat{B}$ and $y \notin \hat{B}$ and $\{x, y\} \notin \tilde{\mathcal{A}}$.

We use arguments similar to those in the proof of the small θ part of Theorem 2. To demonstrate (3.22) with $c_4^* < 1/3$, for small $1 - \hat{\theta}$, we proceed as follows. For r^* a path in $\mathbf{Z}^{2*}$, let $\hat{B}_{r^*}$ denote the random subset of edges e^* of r^*, such that e^* is dual to $\{x, y\}$ with either x or y (or both) in $\hat{B}$. Then

$$\text{Prob}(r^* \notin \mathcal{I}) \leq \text{Prob}\big(|\hat{B}_{r^*}| > \beta|r^*|\big) + \sum_{\gamma : |\gamma| \leq \beta|r^*|} \text{Prob}(r^* \notin \mathcal{I}, \hat{B}_{r^*} = \gamma), \tag{3.30}$$

where the sum is over subsets γ of edges in r^* with cardinality $|\gamma| \leq \beta|r^*|$.

Just as in the small θ part of the theorem, write $\mathbf{Z}_+^2$ and $\mathbf{Z}_-^2$ to denote the even and odd subsets of $\mathbf{Z}^2$. Each edge e^* of r^* is dual to $\{x, y\}$ with one of x and y in $\mathbf{Z}_+^2$ and the other in $\mathbf{Z}_-^2$. Let r_+^* denote the set of sites $x \in \mathbf{Z}_+^2$ such that for some $\{x, y\} \in \mathbf{E}^{2*}$, the dual edge to $\{x, y\}$ is in r^*, and define r_-^* similarly. Then $|r_+^*|, |r_-^*| \leq |r^*|$ and $|\hat{B}_{r^*}| \leq |r_+^* \cap \hat{B}| + |r_-^* \cap \hat{B}|$, where the two summands are not independent, but each has a binomial distribution with parameters $(|r_+^*|, 1 - \hat{\theta}), (|r_-^*|, 1 - \hat{\theta})$. Then by Lemma 3.3 we have

$$\text{Prob}\big(|\hat{B}_{r^*}| > \beta|r^*|\big) \leq 2e^{-G(\beta, 1-\hat{\theta})|r^*|/2}. \tag{3.31}$$

We will eventually choose $\hat{\epsilon}(d)$ and β such that $G(\beta, 1 - \hat{\theta}) > 2\ln 3$ so that we get $c_4^* < 1/3$ in (3.22).

To get an estimate on $\text{Prob}(r^* \notin \mathcal{I}, \hat{B}_{r^*} = \gamma)$, let $G(r^*, \gamma)$ denote the set of $e \in \mathbf{E}^2$ such that $e^* \in r^*$ but there is no edge $f \in \mathbf{E}^2$ touching e with $f^* \in \gamma$. Also let $G^*(r^*, \gamma)$ denote $\{e^* \in r^* : e \in G(r^*, \gamma)\}$. Then the event $r^* \notin \mathcal{I}$ is contained in the event $G(r^*, \gamma) \notin \mathcal{I}$ and so

$$\begin{aligned} \text{Prob}\big(r^* \notin \mathcal{I}, \hat{B}_{r^*} = \gamma\big) &\leq \text{Prob}\big(G(r^*, \gamma) \notin \mathcal{I}, \hat{B}_{r^*} = \gamma\big) \\ &\leq \text{Prob}\big(G(r^*, \gamma) \in \tilde{\mathcal{A}}, \hat{B}_{r^*} = \gamma\big) \\ &= \text{Prob}\big(G(r^*, \gamma) \in \tilde{\mathcal{A}}\big)\,\text{Prob}(\hat{B}_{r^*} = \gamma). \end{aligned} \tag{3.32}$$

The second inequality follows from Lemma 3.5, as explained above, and the equality uses independence of the events in question. Now using the easily verified fact that

(2.7) of Theorem 4 remains valid when $\mathscr{A}$ is replaced by $\tilde{\mathscr{A}}$, we get $\text{Prob}(G(r^*, \gamma) \in \tilde{\mathscr{A}}) \leq (1/(2d))^{\psi}$, where $\psi = |G(r^*, \gamma)|$. It is clear that for some K' (depending only on d), $|G(r^*, \gamma)| \geq |r^*| - K'|\gamma|$. Then given $|\gamma| \leq \beta|r^*|$ we have $\psi \geq |r^*| - \beta K'|r^*|$. Thus we have that

$$\sum_{\gamma: |\gamma| \leq \beta|r^*|} \text{Prob}(r^* \notin \mathscr{I}, \hat{B}_{r^*} = \gamma) \leq \sum_{\gamma: |\gamma| \leq \beta|r^*|} \hat{c}_3 \text{Prob}(\hat{B}_{r^*} = \gamma)(1/(2d))^{(1-K'\beta)|r^*|}$$
$$\leq \hat{c}_3(1/(2d))^{(1-K'\beta)|r^*|}. \tag{3.33}$$

We obtain (3.22) with $c_4^* < 1/3$ as desired, by combining (3.30), (3.31), and (3.33), and by choosing β and $\hat{\epsilon}$ carefully as follows. First choose β small enough to get $(1/(2d))^{(1-K'\beta)} < 1/3$. Then choose $\hat{\epsilon}$ (depending on d and β) so that $e^{-G(\beta, \hat{\epsilon})/2} < 1/3$. The proof is now complete. ∎

REFERENCES

[1] O. Häggström and R. Meester, Nearest neighbor and hard sphere models in continuum percolation, Random Struct Alg 9 (1996), 295–315.

[2] M. Harris and R. Meester, Nontrivial phase transition in a dependent parametric bond percolation model, Markov Processes and Related Fields 2 (1996), 513–528.

[3] S. Nanda, C.M. Newman, and D.L. Stein, "Dynamics of Ising spin systems at zero temperature," On Dobrushin's Way (from Probability Theory to Statistical Physics) R. Minlos, S. Shlosman, and Y. Suhov, (Editors), American Mathematical Society, Providence, RI, to appear.

[4] C.M. Newman, Topics in Disordered Systems, Birkhäuser, Basel, 1997.

[5] C.M. Newman and D.L. Stein, Spin-glass model with dimension-dependent ground state multiplicity, Phys Rev Lett 72 (1994), 2286–2289.

[6] B. Pittel and R. Weishaar, The random bipartite nearest neighbor graphs, Random Struct Alg, 15 (1999), 279–310