

# THE DISTRIBUTION OF VALUES OF ANALYTIC FUNCTIONS ON CONVEX SETS

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## Abstract

Proceeding the study of local properties of analytic functions started in [Br] we prove new dimensionless inequalities for such functions in terms of their Chebyshev degree. As a consequence we obtain the reverse Hölder inequalities for analytic functions with absolute (i.e., independent of dimension) constants. For polynomials such inequalities were recently proved by Bobkov who sharpened and generalized the previous Bourgain result and by Sodin and Volberg.

## 1. Introduction.

**1.1.** In order to formulate the main result we first recall the definition of the *Chebyshev degree* for analytic functions.

Let  $B_c(0, 1) \subset B_c(0, r) \subset \mathbb{C}^n$  be the pair of open complex Euclidean balls of radii 1 and  $r$  centered at 0. Denote by  $\mathcal{O}_r$  the set of holomorphic functions defined on  $B_c(0, r)$ . Our definition is motivated by the following result (see [Br, Th. 1.1]).

**Theorem 1.1** *Let  $f \in \mathcal{O}_r$ ,  $r > 1$ , and  $I$  be a real interval situated in  $B_c(0, 1)$ . (Hereafter we identify  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$ .) There is a constant  $d = d(f, r) > 0$  such that for any  $I$  and any measurable subset  $\omega \subset I$*

$$\sup_I |f| \leq \left( \frac{4|I|}{|\omega|} \right)^d \sup_\omega |f|. \quad (1.1)$$

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The optimal constant in this inequality is called the *Chebyshev degree* of  $f \in \mathcal{O}_r$  in  $B_c(0, 1)$  and is denoted by  $d_f(r)$ . According to the classical Remez inequality  $d_f(r)$  does not exceed the (total) degree of  $f$ , provided  $f$  is a polynomial. Example 1.14 in [Br] shows that even in this case it can be essentially smaller than degree.

Further, let us recall that a function  $h : \mathbb{R}^k \rightarrow \mathbb{R}_+$  is *log-concave* if its support

$$K = \{x \in \mathbb{R}^k : h(x) > 0\}$$

is convex and  $\log h$  is a concave function on  $K$ .

Examples of log-concave functions. (a) A nonnegative function which is concave on a convex body in  $\mathbb{R}^k$  and is zero outside. In particular, the indicator function of a convex body is log-concave.

(b) Many density functions of statistics, e.g.,  $e^{-|x|^\alpha}$ ,  $\alpha > 0$ .

(c)  $\max(0, l^k)$  where  $l$  is linear and  $k$  is a positive integer.

Let now  $\mu_h$  be a measure on  $\mathbb{R}^k$  with density  $h$ . For a convex body  $V \subset \mathbb{R}^k$  set

$$|V| := \mu_h(V), \quad f_V := \exp\left(\frac{1}{|V|} \int_V \log |f| d\mu_h\right).$$

Clearly,

$$f_V \leq \frac{1}{|V|} \int_V |f| d\mu_h.$$

In the formulation of the main result we assume without loss of generality that

$$\mu_h(V) = 1.$$

**Theorem 1.2** *Let  $\mathbb{R}^k \subset \mathbb{C}^n (\cong \mathbb{R}^{2n})$  be a  $k$ -dimensional affine subspace,  $V \subset B_c(0, 1) \cap \mathbb{R}^k$  be a  $k$ -dimensional convex body and  $h : \mathbb{R}^k \rightarrow \mathbb{R}_+$  be a log-concave function supported on  $V$ . There are absolute constants  $c, C > 0$  (i.e. independent of dimensions  $k, n$ ) such that for every  $r > 1$  and  $f \in \mathcal{O}_r$*

$$(1) \quad \mu_h\{x \in V : |f(x)| > \alpha f_V\} \leq C \exp(-c\alpha^{1/d_f(r)})$$

and (1.2)

$$(2) \quad \mu_h\{x \in V : |f(x)| \leq \alpha f_V\} \leq C(c\alpha)^{1/d_f(r)} (\log \alpha)^{1/2}, \alpha \leq e^{-1}.$$

**Corollary 1.3** *Under the assumptions of Theorem 1.2*

$$\frac{1}{|V|} \int_V |f|^p d\mu_h \leq (cpd_f(r))^{pd_f(r)} (f_V)^p \leq (cpd_f(r))^{pd_f(r)} \left(\frac{1}{|V|} \int_V |f| d\mu_h\right)^p \quad (p > 1)$$

with an absolute constant  $c > 0$ .

In particular, if  $f_V \leq 1$ , then the Orlicz norm of  $f$  defined by the Orlicz function  $\exp(1/d_f(r)) - 1$  on  $(V, d\mu_h)$  is bounded by an absolute constant.

**Corollary 1.4** *Under assumptions of Theorem 1.2*

$$\frac{1}{|V|} \int_V |\log |f| - C_V(f)| d\mu_h \leq C d_f(r) .$$

Here  $C > 0$  is an absolute constant and  $C_V(f) := \frac{1}{|V|} \int_V \log |f| d\mu_h$ .

A similar result for analytic functions comparing  $\log |f|$  with  $\sup_V \log |f|$  was obtained in [Br]. In this case the constant in the inequality is equivalent to  $\log k$  (for  $k = \dim V$ ). In the case of polynomials Corollary 1.3 implies the fundamental Bourgain inequality [B] (with  $h \equiv 1$ ) and its generalizations proved by Bobkov [Bo] and by the author in [Br, Th.1.11]. Inequality (2) implies a similar inequality for plurisubharmonic functions on  $\mathbb{C}^n$  of a logarithmic growth recently proved in [SV]. This paper contains also Corollary 1.4 for such functions.

**1.2.** As in the above cited papers [Bo] and [SV] our main tool is a remarkable result of Kannan, Lovász and Simonovits ([KLS, Cor. 2.2]) which reduces estimation of a multidimensional integral to the corresponding one-dimensional ones. Using this we establish the following basic inequality which gives Theorem 1.2 as a simple consequence.

**Theorem 1.5** *Let  $f, V, r > 1$  and  $\mu_h$  be as in Theorem 1.2. Then*

$$\left( \frac{1}{|V|} \int_V |f|^m d\mu_h \right)^n \left( \frac{1}{|V|} \int_V |f|^{-p} d\mu_h \right)^q \leq (2e)^{n+q} 4^{(mn+pq)d_f(r)} \frac{\Gamma(md_f(r) + 1)^n}{(1 - pd_f(r))^q}$$

provided  $m, n, p, q > 0$  satisfy

$$mn = pq, \quad p < \frac{1}{d_f(r)} .$$

Here, as usual,  $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ .

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## 2. Proof of Theorem 1.5.

**2.1.** In this section we collect the results used in the proof of the theorem. First introduce the following definition (see [KLS]).

By an *exponential needle* we mean a segment  $I = [a, b]$  in  $\mathbb{R}^n$ , together with a real constant  $\gamma$ . If  $(E, \gamma)$  is an exponential needle and  $f$  is an integrable function defined on  $I$ , then we set

$$\int_E f = \int_0^{|b-a|} f(a + tu) e^{\gamma t} dt,$$

where  $u = (1/|b - a|)(b - a)$ .

**Theorem 2.1 [KLS]** *Let  $f_1, f_2, f_3, f_4$  be four nonnegative continuous functions defined on  $\mathbb{R}^n$ , and  $\alpha, \beta > 0$ . Then the following are equivalent:*

(a) *For every log-concave function  $F$  defined on  $\mathbb{R}^n$  with compact support,*

$$\left( \int_{\mathbb{R}^n} F(t) f_1(t) dt \right)^\alpha \left( \int_{\mathbb{R}^n} F(t) f_2(t) dt \right)^\beta \leq \left( \int_{\mathbb{R}^n} F(t) f_3(t) dt \right)^\alpha \left( \int_{\mathbb{R}^n} F(t) f_4(t) dt \right)^\beta .$$

(b) *For every exponential needle  $E$*

$$\left( \int_E f_1 \right)^\alpha \left( \int_E f_2 \right)^\beta \leq \left( \int_E f_3 \right)^\alpha \left( \int_E f_4 \right)^\beta .$$

**Remark 2.2** The above theorem is also valid for nonnegative  $f_1, f_2, f_3, f_4$  such that  $f_1, f_2$  are the limits of monotone increasing sequences of continuous functions defined on  $\mathbb{R}^n$  and  $f_3, f_4$  are the limits of monotone decreasing sequences of continuous functions defined on  $\mathbb{R}^n$  (see Remark 2.3 in [KLS]). In particular, we can apply this theorem in the case if  $K$  is a closed convex body,  $f_1, f_2$  are nonnegative continuous functions defined on  $K$  which are 0 outside  $K$  and  $f_3, f_4$  are nonnegative functions which are constant on  $K$  and 0 outside.

We also use the following distributional inequality that follows directly from inequality (1.1) (see [Br]).

Let  $I \subset B_c(0, 1) \subset \mathbb{C}^n$  be a real segment and  $f \in \mathcal{O}_r$ . For the distribution function  $D_{f_I}(t) := |\{x \in I : |f(x)| \leq t\}|$  (with respect to the usual Lebesgue measure on  $I$ ) define  $(f_I)_*(t) = \inf\{s : D_{f_I}(s) \geq t\}$ . Then

$$(f_I)_*(t) \geq \left( \frac{t}{4|I|} \right)^{d_f(r)} \sup_V |f| . \quad (2.1)$$

**2.2. Proof of Theorem 1.5.** Let  $f \in \mathcal{O}_r$  and  $I \subset B_c(0, 1)$  be a real interval. Then the functions  $f_\epsilon := (|f| + \epsilon)|_I$ ,  $\epsilon > 0$ , and  $f_{\epsilon, a, b}(t) = f_\epsilon(at + b)$ ,  $t \in I$ ,  $a, b \in \mathbb{R}$ , also satisfy inequality (1.1). We must apply the KLS theorem to functions  $f_1 := (|f| + \epsilon)^m$ ,  $f_2 := (|f| + \epsilon)^{-p}$  (continuous on  $V$ ) and  $f_3 := 2e \cdot 4^{md_f(r)} \Gamma(md_f(r) + 1)$ ,  $f_4 := 2e \cdot 4^{pd_f(r)} / (1 - pd_f(r))$  on  $V$  and 0 outside  $V$  and then take the limit when  $\epsilon \rightarrow 0$ . To avoid abuse of notation and because our estimates below do not depend on  $\epsilon$  we may assume without loss of generality that  $|f|$  itself has no zeros on  $B_c(0, 1)$ .

According to the KLS theorem and Remark 2.2 the theorem follows from the inequality

$$\left( \int_E |f|^m \right)^n \left( \int_E |f|^{-p} \right)^q \leq (2e)^{n+q} 4^{(mn+pq)d_f(r)} \frac{\Gamma(md_f(r) + 1)^n}{(1 - pd_f(r))^q} \left( \int_E 1 \right)^{n+q}$$

for an exponential needle  $E \subset V$ . Making an affine change of variables in the above integrals we reduce the problem to the following inequality

$$\left( \int_0^s |\tilde{f}(x)|^m e^{-x} dx \right)^n \left( \int_0^s |\tilde{f}(x)|^{-p} e^{-x} dx \right)^q \leq (2e)^{n+q} 4^{(mn+pq)d_f(r)} \frac{\Gamma(md_f(r) + 1)^n}{(1 - pd_f(r))^q} (1 - e^{-s})^{n+q} .$$

Here  $\tilde{f}$  is a function obtained from  $f$  by this change of variables. As we already mentioned  $\tilde{f}$  satisfies (1.1). Below we denote  $\|\tilde{f}\|_I := \sup_I |\tilde{f}|$ .

First, let  $0 \leq s \leq 1$ . Then

$$\begin{aligned} \left( \int_0^s |\tilde{f}(x)|^m e^{-x} dx \right)^n \left( \int_0^s |\tilde{f}(x)|^{-p} e^{-x} dx \right)^q &\leq \left( \int_0^s \left( \frac{|\tilde{f}(x)|}{\|\tilde{f}\|_{[0,s]}} \right)^m dx \right)^n \left( \int_0^s \left( \frac{\|\tilde{f}\|_{[0,s]}}{|\tilde{f}(x)|} \right)^p dx \right)^q \\ &\leq s^n \left( \int_0^s \left( \frac{\|\tilde{f}\|_{[0,s]}}{f_*(t)} \right)^{-p} dt \right)^q \leq s^n \left( s \int_0^1 \left( \frac{4}{t} \right)^{pd_f(r)} dt \right)^q \leq 4^{pqd_f(r)} s^{n+q} \left( \frac{1}{1-pd_f(r)} \right)^q \leq \\ &4^{pqd_f(r)} (2(1-e^{-s}))^{n+q} \left( \frac{1}{1-pd_f(r)} \right)^q. \end{aligned}$$

Here we applied inequality (2.1) to the lower distribution function  $\tilde{f}_*$  of  $\tilde{f}$  and used the inequality  $1 - e^{-s} > s/2$  for  $0 < s \leq 1$ . Observe that the obtaining constant is even less than the required one.

Assume now that  $s > 1$ . We estimate each of two factors of the given expression. Without loss of generality we may assume that  $s$  is an integer. Then

$$\begin{aligned} \int_0^s |\tilde{f}(x)|^m e^{-x} dx &= \sum_{i=0}^{s-1} \int_i^{i+1} |\tilde{f}(x)|^m e^{-x} dx \leq \sum_{i=0}^{s-1} \left( \int_i^{i+1} |\tilde{f}(x)|^m dx \right) e^{-i} \leq \\ &\sum_{i=0}^{s-1} \left( \int_i^{i+1} \left( \frac{|\tilde{f}(x)|}{\|\tilde{f}\|_{[i,i+1]}} \right)^m dx \right) e^{-i} \leq \sum_{i=0}^{s-1} \|\tilde{f}\|_{[0,i+1]}^m e^{-i} \leq \\ &\sum_{i=0}^{\infty} (4(i+1))^{md_f(r)} e^{-i} \|\tilde{f}\|_{[0,1]}^m \leq 4^{md_f(r)} e \int_0^{\infty} x^{md_f(r)} e^{-x} dx \|\tilde{f}\|_{[0,1]}^m = \\ &4^{md_f(r)} e \Gamma(md_f(r) + 1) \|\tilde{f}\|_{[0,1]}^m. \end{aligned}$$

We used here inequality (1.1) to estimate  $\sup_{[0,i+1]} |\tilde{f}|$  by  $\sup_{[0,1]} |\tilde{f}|$ . Similarly,

$$\begin{aligned} \int_0^s |\tilde{f}(x)|^{-p} e^{-x} dx &\leq \sum_{i=0}^{s-1} \left( \int_i^{i+1} |\tilde{f}(x)|^{-p} dx \right) e^{-i} \leq \\ &\sum_{i=0}^{s-1} \left( \int_0^{i+1} \left( \frac{\|\tilde{f}\|_{[0,i+1]}}{|\tilde{f}(x)|} \right)^p \frac{1}{\|\tilde{f}\|_{[0,i+1]}^p} dx \right) e^{-i} \leq \\ &\sum_{i=0}^{s-1} \left( \int_0^{i+1} \left( \frac{\|\tilde{f}\|_{[0,i+1]}}{f_*(t)} \right)^p \frac{1}{\|\tilde{f}\|_{[0,1]}^p} dt \right) e^{-i} \leq \\ &\left( \sum_0^{s-1} \int_0^{i+1} \left( \frac{4(i+1)}{t} \right)^{pd_f(r)} \frac{1}{\|\tilde{f}\|_{[0,1]}^p} dt \right) e^{-i} \leq \\ &\sum_{i=0}^{s-1} \frac{4^{pd_f(r)} (i+1)}{1-pd_f(r)} \frac{1}{\|\tilde{f}\|_{[0,1]}^p} e^{-i} \leq \frac{e 4^{pd_f(r)}}{1-pd_f(r)} \frac{1}{\|\tilde{f}\|_{[0,1]}^p}. \end{aligned}$$

Using that  $pq = mn$  and  $1 - e^{-s} \geq 1/2$  for  $s \geq 1$  we get from these inequalities

$$\begin{aligned} \left( \int_0^s |\tilde{f}(x)|^m e^{-x} dx \right)^n \left( \int_0^s |\tilde{f}(x)|^{-p} e^{-x} dx \right)^q &\leq \\ &4^{(mn+pq)d_f(r)} \cdot (2e)^{n+q} \frac{\Gamma(md_f(r) + 1)^n}{(1-pd_f(r))^q} (1 - e^{-s})^{n+q}. \end{aligned}$$

This completes the proof of the theorem.  $\square$

### 3. Proof of Theorem 1.2 and Corollaries.

**Proof of Theorem 1.2.** (1) We apply Theorem 1.5 to  $g := |f|^{1/d_f(r)}$  with  $n = 1$ ,  $p = 1/2$ ,  $q = 2m$  and  $m$  a positive integer. Assume without loss of generality that  $g_V = 1$  and set  $E_\alpha := \{x \in V : g(x) > \alpha\}$ ,  $|E_\alpha| := \mu_h(E_\alpha)$ . Then from Theorem 1.5 we obtain

$$\alpha^m |E_\alpha| \left( \int_V g^{-1/2} d\mu_h \right)^{2m} \leq \left( \int_V g^m d\mu_h \right) \left( \int_V g^{-1/2} d\mu_h \right)^{2m} \leq 4^{2m} (2e)^{2m+1} 2^{2m} (m!)$$

which is equivalent to

$$\begin{aligned} \alpha^m |E_\alpha| &\leq \frac{2^{8m+1} e^{2m+1} (m!)}{\left( \int_V g^{-1/2} d\mu_h \right)^{2m}} \leq 2^{8m+1} e^{2m+1} (m!) \exp \left( -2m \log \left( \int_V g^{-1/2} d\mu_h \right) \right) \\ &\leq 2^{8m+1} e^{2m+1} (m!) (g_V)^m = 2^{8m+1} e^{2m+1} (m!) \leq e^{10m} (m!) . \end{aligned} \quad (3.1)$$

We used here the Jensen inequality

$$\int_V g^{-1/2} d\mu_h \geq \exp \left( \frac{-1}{2} \int_V \log g d\mu_h \right) .$$

Since  $|V| = 1$ , we also have

$$|E_\alpha| \leq 1 .$$

Dividing both sides of (3.1) by  $e^{11m} (m!)$  and summing by  $m$  from 0 to  $\infty$  we get

$$\exp(\alpha/e^{11}) |E_\alpha| \leq 2,$$

or

$$|E_\alpha| \leq 2 \exp(-\alpha/e^{11}).$$

Since  $g := |f|^{1/d_f(r)}$ , the required inequality follows from here.

This proves part (1).

(2) Recall that  $C_V(f) := \frac{1}{|V|} \int_V \log |f| d\mu_h$ . We will estimate the measure  $|F_\gamma| := \mu_h(F_\gamma)$  of the set  $F_\gamma := \{x \in V : |\log |f| - C_V(f)| \geq \gamma\}$ ,  $\gamma \geq 1$ . We apply Theorem 1.5 to  $f$  with  $m = p = (1 - 1/\gamma)/d_f(r)$ ,  $n = q = 1$ . Then we have

$$\begin{aligned} e^{\frac{2\gamma(1-1/\gamma)}{d_f(r)}} |F_\gamma|^2 &\leq \left( \int_V e^{\frac{1-1/\gamma}{d_f(r)} (\log |f| - C_V(f))} d\mu_h \right) \left( \int_V e^{\frac{-(1-1/\gamma)}{d_f(r)} (\log |f| - C_V(f))} d\mu_h \right) \\ &\leq 4^{2(1-1/\gamma)} (2e)^2 \frac{\Gamma(2 - 1/\gamma)}{1 - (1 - 1/\gamma)} \leq 2^6 e^2 \gamma \end{aligned}$$

Hence

$$|F_\gamma| \leq 2^3 e^{1+1/d_f(r)} e^{-\gamma/d_f(r)} \gamma^{1/2} . \quad (3.2)$$

This, in particular, gives an estimate of  $\mu_h\{x \in V : \log |f| - C_V(f) \leq -\gamma\}$  which, in turn, gives the required result

$$\mu_h\{x \in V : |f(x)| \leq \alpha f_V\} \leq 2^3 e (e\alpha)^{1/d_f(r)} (\log \alpha)^{1/2}, \alpha \leq e^{-1} .$$

with  $\alpha = e^{-\gamma}$ .

The proof of Theorem 1.5 is complete.  $\square$

**Proofs of Corollaries.** Corollary 1.3 follows directly by integration of inequality (1) of Theorem 1.2 and Corollary 1.4 is a simple consequence of inequality (3.2).  $\square$

## 4. Concluding Remarks.

The basic point of the above proof is, together with the KLS theorem, the inequality (1.1). This type of inequalities hold for a more general class of functions introduced as follows.

Let  $B_c(0, 1) \subset B_c(0, r_1) \subset B_c(0, r) \subset \mathbb{C}^n$ ,  $1 < r_1 < r$ , be open complex Euclidean balls. Further, let  $l \subset \mathbb{C}^n$  be a complex straight line which intersects all the balls and  $l_1 := l \cap B_c(0, 1)$ ,  $l_{r_1} := l \cap B_c(0, r_1)$ . Let  $f$  be a plurisubharmonic function defined on  $B_c(0, r)$ . We set

$$b_f(l, r_1) = \sup_{l_{r_1}} f - \sup_{l_1} f .$$

Then the Bernstein index  $b_f(r_1)$  is defined as supremum of  $b_f(l, r_1)$  taken over all lines  $l$  (see also Definition 2.1 in [Br]). The argument for the proof of Th.1.1 in [Br] leads to the following result.

**Theorem 4.1** *Let  $f$  be as above and  $b_f(r_1) < \infty$ . Then the function  $F := e^f$  satisfies inequality (1.1) with  $cb_f(r_1)$  instead of  $d_f(r)$  where  $c \geq 1$  depends on  $r_1$  and  $r$  only.*

**Corollary 4.2** *All results of the presents paper are valid for  $e^f$  with  $cb_f(r)$  instead of  $d_f(r)$ .*

**Examples.** (a) Let  $f \in \mathcal{O}_r$  then we proved in [Br] that  $b_{\log|f|}((1+r)/2) < \infty$ . This motivates our definition of the Chebyshev degree.

Note also that  $d_f(r)$  can be estimated by the general valency of  $f$  defined as maximum of valency of  $f$  restricted to each complex disk  $l_{(1+r)/2}$ .

(b) Let  $f_1, \dots, f_k \in (\mathbb{C}^n)^*$  be complex linear functionals. A quasipolynomial with the spectrum  $f_1, \dots, f_k$  is a finite sum  $q(z) = \sum_{i=1}^k p_i(z)e^{f_i(z)}$  where  $p_i \in \mathbb{C}[z_1, \dots, z_n]$  are holomorphic polynomials. Expression  $m = \sum_{i=1}^k (1 + \deg p_i)$  is said to be degree of  $q$ . We set

$$a(q) := \max_{z \in B_c(0,1), 1 \leq i \leq k} |f_i(z)|.$$

From the results of [Br1] (see also [Br, Prop. 1.4]) it follows.

*All results of the present paper are valid for  $q|_V$  ( $V \subset \mathbb{R}^n$  is a convex body) with  $c_1 + mc_2 + c_3a(q)\text{diam}(V)$  instead of  $d_f(r)$ , where  $c_2, c_3$  are absolute positive constants. The constant  $c_1 \leq (m+1)\log km + (2k-1)a(q)$  for generic  $q$ , and  $c_1 \leq (m+1)\log km + 3a(q)$  if restriction of functionals  $f_s$  to any complex straight line passing through 0 generates a one-dimensional vector space over  $\mathbb{R}$ .*

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