AN EDGE BUT NOT VERTEX TRANSITIVE CUBIC GRAPH*

(received March 5, 1968)

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Let G be an undirected graph, without loops or multiple edges. An automorphism of G is a permutation of the vertices of G that preserves adjacency. G is vertex transitive if, given any two vertices of G, there is an automorphism of the graph that maps one to the other. Similarly, G is edge transitive if for any two edges (a, b) and (c, d) of G there exists an automorphism f of G such that $\{c, d\} = \{f(a), f(b)\}$. A graph is regular of degree d if each vertex belongs to exactly d edges.

In [1, § 3], J. Folkman describes a class of graphs which are regular of even degree, and which are edge transitive but not vertex transitive. He asks [1, p. 232 (4.5)] if there exists an edge but not vertex transitive graph which is regular of degree d, where d is prime. In this note we describe a cubic graph (that is, a graph which is regular of degree 3) which has this property. This graph has 54 vertices and it may well be the smallest cubic graph which is edge but not vertex transitive.

The graph may be described as follows: It has a Hamiltonian circuit C (that is, a simple closed curve C made up of edges and passing through all the vertices), and if the vertices are numbered from 1 to 54 in the order in which they appear on C, then the other edges are given by:

(1, 42), (2, 15), (3, 28), (4, 33), (5, 44), (6, 53), (7, 48), (8, 21), (9, 32), (10, 45), (11, 24), (12, 41), (13, 20), (14, 31), (16, 35), (17, 40), (18, 49), (19, 54), (22, 51), (23, 30), (25, 38), (26, 43), (27, 52), (29, 36), (34, 47), (37, 50), (39, 46).

A second representation of the graph is the following: Consider three copies of the Thomsen graph K(3, 3). For a particular edge y of

*The graph described in this note was discovered by Dr. Marion C. Gray in 1932. The author has independently rediscovered it and believes that it here appears in print for the first time.

Canad. Math. Bull. vol. 11, no. 4, 1968.

K(3, 3), insert a vertex in the interior of y in each of the three copies of K(3, 3), and join the resulting three vertices to a new vertex. Repeat this for each edge y of K(3, 3). (An isomorphism between the two representations of the graph can be found by letting the vertices 7, 19, 21, 49, 51 and 53 of the first representation correspond to the vertices of one of the copies of K(3, 3).)

We prove that the graph is edge transitive. The following two permutations (written in the cyclic notation) are seen to be automorphisms of the graph:

R = (2 54 42) (3 53 43) (4 6 44) (7 45 33) (8 10 32) (11 31 21)
(12 14 20) (15 19 41) (16 18 40) (22 24 30) (25 29 51)
(26 28 52) (34 48 46) (35 49 39) (36 50 38)

(vertices not mentioned are left fixed), and

S = (1 7 11 37 15 53 9 25 35) (2 6 10 38 16 54 8 24 36)
(3 5 45 39 17 19 21 23 29) (4 44 46 40 18 20 22 30 28)
(12 50 14 52 32 26 34 42 48) (13 51 31 27 33 43 47 41 49).

The automorphism R leaves fixed the vertex 1, and cyclically permutes its three neighbours 2, 42 and 54. Thus, in order to prove edge transitivity, we need only show that any odd numbered vertex can be mapped, by an automorphism of the graph, to the vertex 1. It is easy to check that this can be done by forming products of R and S (for instance, the vertex 3 can be mapped to 1 by first mapping it to 53 under R and then by mapping 53 to 1 under S^4).

The graph cannot be vertex transitive, since we note that from an odd numbered vertex it is possible for three different paths of length 4 to lead to a common final vertex (for instance, from the vertex 1 to the vertex 5), while from an even numbered vertex this is not possible. (By a path of length n we mean an ordered sequence of n + 1 distinct vertices such that each two consecutive vertices are joined by an edge.)

We conclude by stating, without proof, two further properties concerning the symmetry of the graph:

(1) For any two paths of length 3 (length 4) whose initial vertices are odd numbered (even numbered), there exists an automorphism of the graph transforming the one to the other. (The respective path lengths 3 and 4 in this statement cannot be bettered.)

(2) For any two paths of length 4 whose initial vertices are even numbered, there exist exactly two automorphisms transforming the one to the other. Using this we calculate that the order of the automorphism group of the graph is $2.27.3.2^3 = 1296$.

REFERENCE

1. J. Folkman, Regular line-symmetric graphs. J. Combinatorial theory 3 (1967) 215-232.

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