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Tracking control of mechanical systems with a unilateral position constraint inducing dissipative impacts

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Abstract

In this paper, the tracking control problem is considered for mechanical systems with unilateral constraints and dissipative impacts. In these systems, impacts are triggered at the exact moment when the constraint is closed. Hence, when a reference trajectory experiences an impact, the constraint of a nearby plant trajectory is expected to close just before or after the impact of the reference, introducing a small time mismatch between the impacts of the plant and the reference, even if this trajectory is arbitrarily close to the reference. Considering mechanical systems with a unilateral constraint and dissipative impacts, we design continuous-time controllers that can handle this impact time mismatch, and achieve accurate tracking of reference trajectories containing impacts. The behaviour of the resulting closed-loop dynamical system is illustrated with an exemplary bouncing ball system.

1 Introduction

Many mechanical systems contain unilateral position constraints, and experience impacts when a unilateral constraint is closed. In this paper, we will design tracking controllers for mechanical systems with dissipative impacts, and model these as hybrid systems, which are characterised by the combination of continuous-time dynamics and jumps, cf. [1, 2, 3]. In addition to mechanical systems with impacts, hybrid systems are used to model, switching electrical circuits, reset control systems, biological systems, etc. We will focus on mechanical systems with dissipative impacts (a restitution coefficient smaller than one), and design controllers that ensure that the plant follows a time-varying reference trajectory that contains impacts.

For mechanical systems with impacts, and more generally, for hybrid systems with state-triggered jumps, tracking controllers will encounter the “peaking phenomenon” in the Euclidean tracking error, as observed by [4, 5, 6, 7, 8, 9, 10, 11]: the plant and reference trajectory will generically show jumps with a small time mismatch, and during this time period, the Euclidean tracking error will be large. Since even for arbitrarily close initial conditions, a jump time mismatch is expected, the Euclidean tracking error behaves unstable in the sense of Lyapunov.

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For hybrid systems where jumps can be triggered by the controller, the peaking phenomenon can be avoided by forcing the jumps of the plant to coincide with those of the reference trajectory. Such controllers are designed in [4, 7, 12, 13], and observers where jumps of the observer coincide with those of the plant are designed in [8]. However, for hybrid systems with state-triggered jumps, including mechanical systems with impacts, jumps of the plant can not be forced to coincide with jumps of the reference trajectory.

Over the last several years, various tracking controller designs are proposed for mechanical systems with impacts, that deal with this peaking phenomenon. In [14, 15, 5], local controllers are presented for a class of hybrid systems with linear flow and jump dynamics, which include mechanical systems with impacts as special cases. In these references, periodic reference trajectories are considered, and the Euclidean tracking error is required to converge only away from the jump times. In addition, convergence of the jump times is ensured by evaluating the closed-loop dynamics using a return map.

In [10], a tracking problem is formulated by requiring convergence of a non-Euclidean tracking error measure, that is tailored to the specific hybrid system, and designed such that convergence of this tracking error measure corresponds to an intuitive notion of tracking. Additionally, the tracking error measure remains constant over jumps, and hence, does not exhibit the peaking phenomenon. In [11], this approach is used to design tracking controllers for mechanical systems with non-dissipative impacts. For these systems, the state vector \( x \), containing the position and velocity with respect to the impacting surface, is mapped onto \(-x\) during an impact, where, prior to the impact, the position is zero and the velocity points in the direction of the constraint. Hence, in [11], a plant trajectory \( x \) is not required to track the reference trajectory \( r \) such that \( |x-r| \to 0\), but instead, convergence of \( d(r,x) = \min(|x-r|,|x+r|) \) to zero is required. Independently, in [16, 17, 9], tracking and observer problems are considered for billiard systems, and controllers are designed that ensure asymptotic stability of a set containing the reference trajectory and its mirror images. The local tracking controller developed for this set is similar to the tracking controller designed in [11].

Both the tracking control design of [11] and the design of [16, 17, 9] exploit the property that the post-impact velocity equals minus the pre-impact velocity, and study the behaviour of \( x-r \) and \( x+r \) along closed-loop solutions (after an impact of \( x \) or \( r \), \( x+r \) equals the difference between the plant state \( x \) and the reference trajectory \( r \) before the impact). Due to this setup, the approach of [11, 16, 17, 9] is restricted to non-dissipative impacts where the restitution coefficient is equal to one. Tracking control for mechanical systems with dissipative impacts (with restitution coefficients strictly smaller than one) for general, non-periodic reference trajectories, has not been considered so far in the literature. This is highly relevant, since in physical systems dissipation will always appear to some extent.

In the current paper, we will address the tracking problem for impacts with restitution coefficient \( \epsilon \in (0,1] \) for mechanical systems with one degree of freedom. For these systems, we show that a Lyapunov function can be defined with the following behaviour. When the reference experiences an impact prior to the plant, the velocity decrease of the reference at the impact is initially ignored by a rescaling of the reference state \( r \) with a factor \( \frac{1}{\epsilon} \), such that the Lyapunov function remains constant. If, subsequently, the plant jumps, then this rescaling is undone, such that the Lyapunov function is decreasing. Effectively, in the case where the reference jumps first, the dissipative effect of the impacts of both the reference and plant trajectory is taken into account only after the jump of the plant. Using this rescaling function, a switching control law is designed that enables converging closed-loop behaviour of either \( x-r, x+\frac{1}{\epsilon}r \) or \( \frac{1}{2}x+r \), such that the plant trajectory converges to the reference away from the jump times, and an intuitively correct notion of tracking is achieved.

This paper is organised as follows. In Section 2, we introduce the class of systems under study and define the corresponding solution concept. In Section 3, we recall the tracking problem definition from [10] and define a tracking error for mechanical systems with dissipative impacts. Controllers solving the tracking problem are designed in Section 4, and are illustrated in Section 5 with an example. Section 6 presents a discussion of the main result. A technical lemma that is used to proof our main result is given in the Appendix.
Notation

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; $\mathbb{R}$ the set of real numbers; $\mathbb{N}$ the set of natural numbers including 0. Let $\mathbb{R}(S)$ denote the smallest closed convex hull containing a set $S \subset \mathbb{R}^n$, and $S^2 = S \times S$. Given vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $|x|$ denotes the Euclidean vector norm, and $\col(x,y)$ denotes $[x^\top y^\top]^\top$. A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class-$K_\infty$ (denoted $\alpha \in K_\infty$) if it is continuous, zero at zero, strictly increasing and unbounded. For symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \prec 0$ ($A \succ 0$) when $A$ is negative definite (positive definite) and $A \prec B$ ($A \succ B$) when $A - B \prec 0$ ($A - B \succ 0$). Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of a symmetric matrix $A$, respectively. Finally, if $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and $A \succ 0$ then $|x|_A^2 = x^\top A x$.

2 Mechanical systems with impacts

2.1 Modelling

In this paper, we consider mechanical systems with one degree of freedom (1DOF) and a single unilateral position constraint with impact, as depicted in Figure 1. As shown in [11], trajectories of such systems can be modelled with

$$\dot{x} = \begin{bmatrix} x_2 \\ f(t, x) + u + \lambda(x_1, x_2) \end{bmatrix}, \quad x \in C := [0, \infty) \times \mathbb{R},$$

$$x^+ = g(x) = \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix}, \quad x \in D := \{0\} \times (-\infty, 0),$$

where the constraint is positioned at $x_1 = 0$. Here, (1a) describes the flow of trajectories with control signal $u$, which is the controller force when the system has unit mass, $\lambda$ is the contact force when the constraint is active, and $f(t, x)$ represents possible other forces. Equation (1b) models impacts in which the velocity changes sign. Energy is dissipated according to the restitution coefficient $\epsilon \in (0, 1]$. The contact force $\lambda(x_1, x_2)$ satisfies

$$\lambda(x_1, x_2) \in \begin{cases} 0, & (x_1, x_2) \neq (0, 0), \\
(0, \infty), & (x_1, x_2) = (0, 0), \end{cases}$$

and ensures that the unilateral contact constraint $x \geq 0$ is not violated when $x = [x_1 x_2]^\top = [0 0]^\top$.

Figure 1: Example of a mechanical system described by (1), (2).

In this paper, we present controllers that solve a local tracking problem near a reference trajectory $r$, that is a solution of (1) for a given feedforward signal $u = u(t)$, where $r$ is non-Zeno and, at any time instant, bounded away from the origin. For all trajectories near this reference trajectory, the contact force $\lambda$ vanishes, such that nearby trajectories are described by the simpler hybrid system

\[ f(t, x) + u + \lambda(x_1, x_2) \]
\[ \dot{x} = F(t, x, u) := \begin{bmatrix} x_2 \\ f(t, x) + u \end{bmatrix}, \quad x \in C := [0, \infty) \times \mathbb{R} \]  \\
\[ x^+ = g(x) := \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix}, \quad x \in D := \{0\} \times (-\infty, 0). \]  

Throughout this paper, we assume that \( u \) is bounded and \( f \) is continuous in \( x \) and locally essentially bounded in \( t \).

In order to define solutions of the hybrid system (3), we assume that the input \( u \) satisfies \( u(t) \in \mathcal{U} \) for a compact set \( \mathcal{U} \subset \mathbb{R} \). Using the framework of [1], solutions \( \varphi \) of the hybrid system (3) are defined on a hybrid time domain \( \text{dom} \varphi \subset [0, \infty) \times \mathbb{N} \). A hybrid time instant is given as \( (t, j) \in \text{dom} \varphi \), where \( t \) denotes the continuous time lapsed, and \( j \) denotes the number of experienced jumps. The arc \( \varphi \) denotes a solution of (3) when, for all \( (t, j) \in \text{dom} \varphi \) such that \( (t, j+1) \in \text{dom} \varphi \), \( \varphi(t, j) \in D \) and \( \varphi(t, j+1) = g(\varphi(t, j)) \), and, for almost all \( t \in I_j := \{t \mid (t, j) \in \text{dom} \varphi\} \) and all \( j \) such that \( I_j \) has non-empty interior, \( \varphi(t, j) \in C \) and \( \frac{d}{dt} \varphi(t, j) = F(t, \varphi(t, j), u(t, \varphi(t, j))) \). In other words, \( \varphi \) is a solution of (3a) during flow, and jumps satisfy (3b). In this paper, we only consider maximal solutions, i.e., solutions that can not be continued towards a larger time domain. A solution \( \varphi \) is said to be non-Zeno if \( \text{dom} \varphi \) is unbounded in the \( t \)-direction.

### 3 Tracking control problem

#### 3.1 Tracking problem formulation

Tracking controllers for hybrid systems with state-triggered jumps, such as mechanical systems with impacts, will generically show the following “peaking phenomenon” in the Euclidean tracking error, cf. [4, 5, 6, 7, 8, 9, 10], which we illustrate in Figure 3a) for an example. If jumps are state-triggered, i.e., they occur when the state reaches a certain surface in the state space, then, generically, a reference and plant trajectory that are initially close will not reach this surface exactly at the same time, but shortly after each other. Hence, in the intermediate time period, the Euclidean distance between the plant and reference trajectory will be approximately equal to the Euclidean distance of the jump. Consequently, if the Euclidean distance between plant and reference trajectory is considered as a tracking error, then, for a small time interval near the impacts, the error will be large, even when the initial error was arbitrarily small. Hence, this error behaves unstable in the sense of Lyapunov. Due to this peaking phenomenon, a tracking problem formulation that requires asymptotic stability of the Euclidean tracking error is not feasible for hybrid systems with state-triggered jumps, such as mechanical systems with impacts.

To resolve this problem, in [11], the present authors designed a local tracking controller for the case of ideal, non-dissipative impacts, i.e., \( \epsilon = 1 \), by requiring asymptotic convergence of the function \( d_{\text{ideal}}(r, x) = \min(\|x - r\|, \|x + r\|) \) along closed-loop trajectories, where \( d_{\text{ideal}} \) is considered as the tracking error measure. In the present paper, we formulate a tracking problem by requiring convergence of a tracking error definition \( d_e \) tailored to the dissipative impact law (1b), which depends explicitly on the restitution coefficient \( \epsilon \in (0, 1] \).

We will design the function \( d_e \) such that this tracking error does not change at jumps, i.e., \( d_e(r, g(x)) = d_e(r, x) \) for each \( x \in D \) and \( d_e(g(r), x) = d_e(r, x) \) for each \( r \in D \). Additionally, we design \( d_e \) to be a continuous function on \( (C \cup D)^2 \). Consequently, when evaluated along closed-loop trajectories, the function \( d_e(r, x) \) is a continuous function of \( t \), and independent of \( j \).

To construct a tracking error function \( d_e \), with the properties given above, we will adapt \( d_{\text{ideal}} \) using the following coordinate transformation:

\[ M_e(x) := \begin{bmatrix} x_1 \\ \alpha(x)x_2 \end{bmatrix}, \quad \text{with } \alpha(x) := \begin{cases} \frac{1}{\epsilon} & x_2 > 0 \\ 1 & x_2 \leq 0. \end{cases} \]  

We can use \( d_{\text{ideal}} \) in this new coordinate system, which yields a distance function \( d_e(r, x) \) given as:

\[ d_e(r, x) := \min(\|M_e(x) - M_e(r)\|, \|M_e(x) + M_e(r)\|). \]
This tracking error measure $d_e$ fulfills the requirements just mentioned as it is continuous and insensitive to impacts, such that $d_e(r, x)$ remains constant when either $x$ or $r$ experiences an impact.

According to Theorem 1 of [10], convergence of this tracking error to zero ensures that for all $\delta > 0$, after a sufficiently long transient time, $|x(t) - r(t)| < \delta$ whenever the reference position $r_1$ satisfies $r_1(t) > \delta$. To illustrate the consequences of this statement, assume that the reference $r$ does not converge asymptotically to $D$ without attaining $r(t) \in D$, and let $\delta > 0$ be sufficiently small. In that case, for all times $t$ where $r$ does not experience an impact in the near future or past, we find that $r_1(t) > \delta$. Hence, after a transient time, convergence of the tracking error $d_e$ to zero implies that $|x(t) - r(t)|$ is small when impacts of the reference do not occur in the near past or future.

Analogous to the common approach in tracking control for ODEs, we consider reference trajectories $r$ that are solutions to (3) for a given feedforward signal $u = u_0(t)$ and design a state- and time-dependent control law $u = u_d(t, r, x)$. To study the stability of the closed-loop system, we combine the dynamics of the reference trajectory with the dynamics of the plant and evaluate the tracking error $d_e(r, x)$ along trajectories. For this purpose, we create an extended hybrid system with state $q = \text{col}(r, x)$. The dynamics of this hybrid system is given by

$$
\begin{align*}
\dot{q} &= F_e(t, q), & q \in C^2 \\
q^+ &= \text{col}(q_1, -q_2, q_3, q_4), & q \in D \times (C \cup D) \\
q^+ &= \text{col}(q_1, q_2, q_3, -q_4), & q \in (C \cup D) \times D,
\end{align*}
$$

where

$$
F_e(t, q) := \begin{bmatrix}
f(t, \text{col}(q_1, q_2)) + u_0(t) \\
q_4 \\
f(t, \text{col}(q_3, q_4)) + u_d(t, \text{col}(q_1, q_2), \text{col}(q_3, q_4))
\end{bmatrix}.
$$

We define $\bar{r}(t, j) := \text{col}(q_1, q_2)(t, j)$ and $\bar{x}(t, j) = \text{col}(q_3, q_4)(t, j)$, such that $\bar{r}, \bar{x} : \text{dom } q \to C \cup D$ are reparameterisations of $r : \text{dom } r \to C \cup D$ and $x : \text{dom } x \to C \cup D$ on the combined hybrid time domain $\text{dom } q$.

From [11, 10], we adopt the following stability definition and tracking problem formulation.

**Definition 1** (Stability with respect to distance $d_e$). Let $d_e$ be given in (5). A reference trajectory $r(t, j)$ of system (3) is called

- stable with respect to $d_e$ if for all $t_0, j_0 \geq 0$ and $\delta_1 > 0$ there exists a $\delta_2(t_0, j_0, \delta_1) > 0$ such that $\forall t \geq t_0, \forall j \geq j_0$:

$$
d_e(\bar{r}(t_0, j_0), \bar{x}(t_0, j_0)) < \delta_2(t_0, j_0, \delta_1) \Rightarrow d_e(\bar{r}(t, j), \bar{x}(t, j)) < \delta_1;
$$

- locally asymptotically stable with respect to $d_e$ if it is stable with respect to $d_e$ and for any $t_0, j_0 \geq 0$ there exists a $\delta_3(t_0, j_0) > 0$ such that:

$$
\lim_{t+j \to \infty} d_e(\bar{r}(t, j), \bar{x}(t, j)) \to 0.
$$

Using this definition, the tracking problem is formalised as follows.

**Definition 2** (Tracking problem). Given a hybrid system (3) with reference trajectory $r$, design a control law $u_d(t, r, x)$ such that the trajectory $r$ is locally asymptotically stable with respect to $d_e$ given in (5).
3.2 Sufficient conditions for stability

In order to guarantee that trajectories of (6) have hybrid time domains that are unbounded in \( t \)--direction, we require that \( r \) is non-Zeno unique and bounded, as formalised in the following assumption.

**Assumption 1.** The reference trajectory \( r = \text{col}(r_1, r_2) \) is non-Zeno, bounded, and the unique solution of (3) with a bounded feedforward signal \( u_f \) and initial condition \( r(0,0) \).

Sufficient conditions for the uniqueness of solutions to hybrid systems are given in [1, Proposition S5, page 47]. In our case, the required uniqueness of the reference trajectory implies that \( \text{col}(q_1(t,j), q_2(t,j)) \) is equal to \( \bar{r}(t,j) \) when the initial condition \( \text{col}(q_1, q_2) = r(0,0) \) is chosen. The reference trajectory \( r \) is assumed to be both non-Zeno and unique, such that, since (3) will show Zeno-behaviour near the origin, the reference \( r \) is bounded away from the origin, i.e., \( \min_{(t,j) \in \text{dom } r} |r(t,j)| > 0 \).

The following theorem of [11] provides sufficient conditions for the asymptotic stability of a reference trajectory \( r \) using a control Lyapunov function \( V \).

**Theorem 1** ([11]). Consider a hybrid system (3), distance \( d \) given in (5), reference trajectory \( r \), and feedforward signal \( u_f \) satisfying Assumption 1. Let the control law \( u_d(t,r,x) \) be given, and let \( F_c(t, \text{col}(r,x)) \) be defined in (7). If there exist functions \( \alpha_1, \alpha_2 \in K_{\infty} \), a continuously differentiable function \( V(r,x) \) and scalars \( \epsilon, \delta_1 > 0 \) such that

\[
\alpha_1(d_c(r(t,j),x)) \leq V(r(t,j),x) \leq \alpha_2(d_c(r(t,j),x)) \tag{10a}
\]

holds for all \( x \in C \cup D, (t,j) \in \text{dom } r, \) and

\[
\begin{align*}
V(g(r(t,j)),x) & \leq V(r(t,j),x), \quad \text{for } r(t,j) \in D \tag{10b} \\
V(r(t,j),g(x)) & \leq V(r(t,j),x), \quad \text{for } x \in D \tag{10c} \\
\nabla_{\text{col}(r,x)} F_c(t, \text{col}(r(t,j),x)) & \leq -cV(r(t,j),x), \quad \text{for } x, r(t,j) \in C \tag{10d}
\end{align*}
\]

hold for all \( (t,j) \in \text{dom } r \) and all \( x \in C \cup D \) such that \( d_c(r(t,j),x) < \delta_1 \), then the reference trajectory \( r \) is asymptotically stable with respect to \( d_c \) for the system (6). \( \triangle \)

4 Controller design

In this section, we design a state feedback \( u = u_d(t,r,x) \) for mechanical systems (1) with restitution coefficient \( \epsilon \in [0,1] \). The rationale behind the controller design is, roughly speaking, that the controller makes \( M_c(x) - M_e(r) \) converge to zero away from the jump instances, and ensures convergence of \( M_c(x) + M_e(r) \) to zero near the jump instances.

We design a tracking controller that switches based on three continuous functions \( V_a, V_b, V_c : (C \cup D)^2 \to \mathbb{R}_{\geq 0} \) given as

\[
V_a(r,x) = |x - r|^2_p, \quad V_b(r,x) = |x + \frac{1}{\epsilon} r|^2_p, \quad V_c(r,x) = \frac{1}{\epsilon^2} |x + r|^2_p, \quad \tag{11}
\]

where \( P > 0 \).

Let the controller \( u_d \) be designed as follows:

\[
u_d(t,r,x) = \begin{cases} 
-f(t,x) + (u_f(t) + f(t,r)) - \left[k_p k_d\right] (x - r) & \text{if } V_a(r,x) \leq V_b(r,x) \land r_2 \geq 0 \text{ or } V_a(r,x) < V_c(r,x) \land r_2 < 0 \\
-f(t,x) - (u_f(t) + f(t,r)) - \left[k_p k_d\right] (x + \frac{1}{\epsilon} r) & \text{if } V_b(r,x) < V_a(r,x) \land r_2 \geq 0, \\
-f(t,x) - \epsilon (u_f(t) + f(t,r)) - \left[k_p k_d\right] (x + \epsilon r) & \text{if } V_c(r,x) < V_a(r,x) \land r_2 < 0,
\end{cases} \tag{12}
\]
where the vector $[k_p, k_d]$ contains controller parameters with $k_p, k_d > 0$.

**Remark 1.** This controller design has the controller introduced in [11] as a special case ($\epsilon = 1$), since in that case, $V_b = V_c$.

The following theorem provides conditions on the parameters $k_p, k_d$ and $P$, that guarantee that this controller solves the local tracking problem as in Definition 2.

**Theorem 2.** Consider system (1) with $\epsilon \in (0, 1]$ and reference trajectory $r$ and feedforward signal $u_f$ satisfying Assumption 1. If the controller parameters $P$ and $k_p, k_d > 0$ of (11), (12) satisfy $P = P^T > 0$ and

$$P \begin{bmatrix} 0 & 1 \\ -k_p - k_d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -k_p - k_d & 0 \end{bmatrix}^T P < 0,$$  \hspace{1cm} (13)

then the controller (12) solves the local tracking problem given in Definition 2.

**Proof.** We will design a control Lyapunov function that satisfies the conditions of Theorem 1 for $d_e$ given in (5). In order to ensure (10a), this function is constructed to ensure that $V(r, x) = 0$ if and only if $d_e(r, x) = 0$. If $r_2 > 0$, then $d_e(r, x) = 0$ if and only if $x = r$ or $x = -\frac{1}{2}r$, such that the control Lyapunov function is designed to be zero if $x = r$ or $x = -\frac{1}{2}r$, and for this reason, min$(V_a(r, x), V_c(r, x))$ is used. If $r_2 \leq 0$, then $d_e(r, x) = 0$ if and only if $x = r$ or $\frac{1}{2}x = -r$, such that min$(V_a(r, x), V_c(r, x))$ is employed.

To construct a continuous control Lyapunov function using this rationale, we will use Assumption 1 to define a linear interpolation between $V_b$ and $V_c$ as follows.

By Assumption 1, the reference trajectory $r$ is assumed to be both non-Zeno and unique, such that, since (3) will show Zeno-behaviour near the origin, the reference $r$ is bounded away from the origin. Consequently, we can select a scalar $\delta > 0$ such that $|r(t, j)| > 2\delta$ for all $(t, j) \in \text{dom } r$.

Define the function $V_{bc}(r, x) := \frac{\delta + r}{2\delta} V_b(r, x) + \frac{\delta - r}{2\delta} V_c(r, x)$ that linearly interpolates between $V_b$ and $V_c$, such that $V_{bc}(r, x) = V_b(r, x)$ when $r_2 = \delta$ and $V_{bc}(r, x) = V_c(r, x)$ when $r_2 = -\delta$. Now, we define the continuous function:

$$V_{bc}(r, x) := \begin{cases} V_b(r, x), & r_2 \geq \delta \\ V_{bc}(r, x), & r_2 \in (-\delta, \delta) \\ V_c(r, x), & r_2 \leq -\delta. \end{cases} \hspace{1cm} (14)$$

Let the control Lyapunov function $V$ be given by

$$V(r, x) = \min(V_a(r, x), V_{bc}(r, x)).$$  \hspace{1cm} (15)

**Existence of functions $\alpha_1, \alpha_2$ satisfying** (10a). The control Lyapunov function $V(r, x)$ is zero if and only if either $V_a(r, x) = 0$ (which implies $x = r$ and hence $M_e(r) = M_e(x)$), or $V_{bc}(r, x) = 0$. In order to prove the existence of functions $\alpha_1, \alpha_2$ that satisfy condition (10a), we first, will prove that

$$V_{bc}(r, x) = 0 \Leftrightarrow \begin{cases} r_2 \geq \delta & \land & x = -\frac{1}{2}r \\ r_2 \leq -\delta & \land & \frac{1}{2}x = r \end{cases} \hspace{1cm} (16)$$

For $r_2 \geq \delta$ or $r_2 \leq -\delta$, the equivalences of this statement follow directly from (14) and (11), and for this reason, we will prove that $V_{bc}(r, x) = 0$ implies that either $r_2 \geq \delta$ or $r_2 \leq -\delta$, by deriving the equivalent statement that $|r_2| < \delta$ implies $V_{bc}(r, x) \neq 0$.

If $|r_2| < \delta$ then $|r| > 2\delta$ implies $r_1 > 0$. Consequently, using $x_1 \geq 0$, we find that $V_b(r, x) = 0$ and $V_c(r, x) = 0$ are unfeasible for $|r_2| < \delta$, as can be seen from the definition in (11). Since $V_{bc} \geq \min(V_b, V_c)$, we find $V_{bc}(r, x) > 0$ when $|r_2| < \delta$, such that (16) is proven.

Using the definition of $M_e$ in (4), equation (16) can be rewritten as

$$V_{bc}(r, x) = 0 \Leftrightarrow |r_2| \geq \delta \land M_e(x) = -M_e(r),$$  \hspace{1cm} (17)
where we note that, for \( r, x \in C \cup D \), the relation \( M_s(x) = -M_s(r) \) can only hold true if \( x_1 = r_1 = 0 \). However, using \(|r| > 2\delta\), this directly implies that \(|r_2| \geq \delta\), and we obtain
\[
V_{bc}(r, x) = 0 \Leftrightarrow M_s(x) = -M_s(r). \tag{18}
\]
Using (5) and the fact that \( V_s(r, x) = 0 \) if and only if \( x = r \), which is equivalent to \( M_s(x) = M_s(r) \), we obtain
\[
d_s(r, x) = 0 \Leftrightarrow V(r, x) = 0. \tag{19}
\]
In order to apply Lemma 3, which is given in the Appendix, we introduce functions \( \beta_1 \) and \( \beta_2 \) that coincide with \( d_s \) and \( V \) in the region of interest, but ensure that the set \( \{ \text{col}(r, x) \mid \beta_1(r, x) = 0 \} \) is compact. For this reason, first, observe that the required boundedness of \( r \) in Assumption 1 implies that there exists an \( R > 0 \) such that \( |(t, j)| < R \), \( \forall (t, j) \in \text{dom } r \). Now, let \( \beta_1(\text{col}(r, x)) = \max(d_s(r, x), |r - 2R|) \) and \( \beta_2(\text{col}(r, x)) = \max(V(r, x), |r - 2R|) \), and observe that these functions coincide with \( d_s \) and \( V \), respectively, along trajectories of (6), since in that case, \( |r| - 2R < 0 \). However, by construction, the set \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid \beta_1 = 0 \} \) is compact, since \( |r| \leq 2R \) and, using the definition of \( d_s \), \( |x| \leq \frac{1}{2}|r| \leq \frac{3}{2}R \) if \( d_s(r, x) = 0 \). Hence, we can apply Lemma 3 and find that there exist functions \( \alpha_1, \alpha_2 \in K_{\infty} \) such that
\[
\alpha_1(\beta_1(\text{col}(r, x))) \leq \beta_2(\text{col}(r, x)) \leq \alpha_2(\beta_2(\text{col}(r, x))). \tag{20}
\]
Since, for each \( r = (t, j), (t, j) \in \text{dom } r \), the functions \( \beta_1 \) and \( \beta_2 \) satisfy \( \beta_1(\text{col}(r, x)) = d_s(r, x) \) and \( \beta_2(\text{col}(r, x)) = V(r, x) \), (10a) is obtained.

**Local differentiability of \( V \).** To prove that \( V \) is locally differentiable, first, we use that \( V_s(r, x) = 0 \) if and only if \( x = r \), and note that (18) states that \( V_s(r, x) = 0 \) if and only if \( x = M_s^{-1}(-M_s(r)) \), since the mapping \( M_s \) given in (4) is invertible. Since \( r \neq M_s^{-1}(-M_s(r)) \) for all \( r \neq 0 \), the 2-dimensional set \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid V_s(r, x) = 0, 2\delta \leq |r| \leq 2R \} \) is compact and has an empty intersection with the compact set \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid V_s(r, x) = 0, 2\delta \leq |r| \leq 2R \} \). Using continuity of \( V_s \) and \( V_{bc} \), for sufficiently small \( K \in (0, \lambda_{\min}(P)\delta^2) \), we find \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid V_s(r, x) \leq K, 2\delta \leq |r| \leq 2R \} \cap \{ \text{col}(r, x) \in (C \cup D)^2 \mid V_{bc}(r, x) \leq K, 2\delta \leq |r| \leq 2R \} = \emptyset \), where we select \( K < \lambda_{\min}(P)\delta^2 \) for reasons that become apparent in the next paragraph. Hence, we obtained
\[
V_s(r, x) \neq V_{bc}(r, x), \text{ if } r = (t, j), V(r, x) \leq K, \tag{21}
\]
which directly implies that the minimum function in (15) does not introduce non-smoothness of the control Lyapunov function \( V \) in the domain \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid V(r, x) \leq K \} \).

Now, we will prove that
\[
|r_2| > \delta, \text{ if } V_{bc}(r, x) \leq K, \tag{22}
\]
to show that the linear interpolation given in (14), at least locally, is not introducing non-smoothness of \( V \). For the sake of contradiction, assume \( V_{bc}(r, x) \leq K \) and \(|r_2| \leq \delta \). Since \(|r| > 2\delta\), it follows that \( r_1 > \delta \). In addition, \( x_1 \geq 0 \), such that \(|\frac{1}{2}x + r| \geq |\frac{1}{2}x_1 + r_1| \geq \delta \) and \(|x + \frac{1}{2}r| \geq |x_1 + \frac{1}{2}r_1| > \frac{1}{2}\delta \geq \delta \). Now, we can use \( V_s(r, x) \geq \lambda_{\min}(P)|x + \frac{1}{2}r|^2 \) and obtain \( V_s(r, x) \geq \lambda_{\min}(P)\delta^2 \), and, analogously, we find \( V_s(r, x) \geq \lambda_{\min}(P)\delta^2 \). Due to the definition of \( V_{bc} \) in (14), for \( r_2 \in (-\delta, \delta) \), it satisfies \( V_{bc} \geq \min(V_s, V_c) \), and, therefore, \( V_{bc}(r, x) \geq \lambda_{\min}(P)\delta^2 \). However, \( K \) is designed to satisfy \( K \in (0, \lambda_{\min}(P)\delta^2) \), such that \( V_{bc}(r, x) \leq K \) implies \( V_{bc}(r, x) < \lambda_{\min}(P)\delta^2 \), and we obtain a contradiction. Hence, we have proven (22), and for all \( (r, x) \) such that \( V(r, x) < K \), we can distinguish the three cases \( V(r, x) = V_s(r, x) < V_{bc}(r, x), V(r, x) = V_b(r, x) < V_{bc}(r, x) \) and \( V(r, x) = V_s(r, x) < V_b(r, x) \). Consequently, the control Lyapunov functions \( V \) is differentiable in the set \( \{ \text{col}(r, x) \in (C \cup D)^2 \mid V(r, x) \leq K \} \).

**Impact conditions.** Now, we will study the behaviour of \( V(r, x) \) along jumps and prove that \( V(r, x) \leq V(r, g(x)) \) for \( x \in D \), provided that \( V(r, x) < K \). Observe that \( y \in D \cup g(D) \) implies \( g_1 = 0, g_2 \neq 0 \), such that \( V_a(r, y) = 0 \) if \( r = y \), and subsequently \( r_2y_2 > 0 \). Since \( V_{bc}(r, y) = 0 \) if \( r = M_s^{-1}(-M_s(y)) \), this would imply \( r_2y_2 > 0 \). Using continuity of \( V_a \) and \( V_{bc} \), selecting \( K \) sufficiently small therefore yields
\[
V(r, y) = V_a(r, y) \Leftrightarrow r_2y_2 > 0, \text{ for } y \in D \cup g(D), \tag{23}
\]
when \( V(r, y) < K \).

If \( x \in D \), and hence, \( x_2 < 0 \), we distinguish the cases \( r_2 \geq 0 \) and \( r_2 < 0 \). Observe that the second element of \( g(x) \) is positive, since this is the post-impact velocity. If \( r_2 < 0 \), we can select \( y = g(x) \) and apply (23) to observe that \( V(r, g(x)) = V_e(r, g(x)) \), and using \( r_2 < 0 \), (14) and (22), we find \( V(r, g(x)) = V_e(r, g(x)) = \frac{1}{2}(-x^2 + r)^2 = |x - r|^2_p = V_e(r, x) \). Subsequently, if we use \( y = x \), then combination of \( r_2 x_2 > 0 \) and (23) yields \( V(r, x) = V_e(r, x) \). Hence, we obtain \( V(r, g(x)) = V(r, x) \). If \( r_2 \geq 0 \), then selection of \( y = g(x) \) in (23) yields \( V(r, g(x)) = V_e(r, g(x)) \).

One finds that \( V(r, g(x)) = V_e(r, g(x)) = \frac{1}{2}(-x^2 + r)^2 = c^2|x + \frac{1}{2}r|^2_p = c^2V_e(r, x) \), and, since \( r_2 \geq 0, x_2 < 0 \) prior to the jump, we find \( V(r, x) = V_e(r, x) \) by application of (23) with \( y = x \), and (22). Hence, using \( c \leq 1 \), we obtain \( V(r, g(x)) = c^2V(r, x) \leq V(r, x) \). Analogously, one finds \( V(g(r), x) \leq V(r, x) \) for \( r \in D \), such that (10b) and (10c) are satisfied.

**Flow condition.** It remains to be shown that there exists a \( c > 0 \) such that \( \nabla V(r, x) F_c(t, \col(r, x)) \leq -cV(r, x) \), restricting our attention to the set \( V(r, x) < K \). As shown above, in this case, \( V(r, x) \) is continuously differentiable, and, locally, given by either \( V_e \), \( V_h \), or \( V_c \). Hence, we can study \( \nabla V F_c(t, \col(r, x)) \) separately for the three cases \( V(r, x) = V_e(r, x) \), \( V(r, x) = V_h(r, x) \) and \( V(r, x) = V_c(r, x) \).

To do so, we introduce \( L := P = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}^T \), and observe that (13) implies the existence of a scalar \( c > 0 \) such that \( L \leq -c P \).

If \( V(r, x) = V_e(r, x) = |x - r|^2_p \), from (7) and (12), we obtain

\[
\dot{x} + \dot{r} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} (x - r),
\]

such that

\[
\nabla V(r, x) \begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} = 2(x - r)^T P(x - r) = (x - r)L(x - r) \leq -cV(r, x).
\]

Similarly, if \( V(r, x) = V_h(r, x) = |x + \frac{1}{2}r|^2_p \), then \( r_2 \geq 0 \), such that \( \ddot{x} + \dot{r} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} (x + \frac{1}{2}r) \),

and we obtain

\[
\nabla V(r, x) \begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} = 2(x + \frac{1}{2}r)^T P(x + \frac{1}{2}r) = (x + \frac{1}{2}r)L(x + \frac{1}{2}r) \leq -cV(r, x).
\]

Finally, if \( V(r, x) = V_c(r, x) = |\frac{1}{2}x + r|^2_p \), then \( \ddot{x} + \dot{r} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} (\frac{1}{2}x + r) \), yielding

\[
\nabla V(r, x) \begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} = 2(\frac{1}{2}x + r)^T P(\frac{1}{2}x + r) = (\frac{1}{2}x + r)L(\frac{1}{2}x + r) \leq -cV(r, x).
\]

Hence, (10d) is satisfied. Using Theorem 1, this implies that \( r \) is asymptotically stable with respect to \( d_r \), which completes the proof.

**Remark 2.** In this proof, it is shown that \( V(r, x) \) is locally differentiable for all trajectories where \( V(r, x) \leq K \), and additionally, \( V(r, x) = V_e(r, x) \) if \( r_2 = 0 \). Hence, for such trajectories, switches of the controller (12) can only be triggered by impacts of \( x \) or \( r \). For this reason, the three cases of (12) correspond to the cases where either the reference and plant states are close to each other, the reference trajectory experienced an impact and the plant trajectory did not, or the plant trajectory did experience an impact and the reference trajectory did not.

\[\Box\]

## 5 Illustrative example

To illustrate our results in an exemplary bouncing ball system, we consider the system (1) with \( f(t, x) = -G \), with gravitational acceleration \( G = 10 \) and \( \epsilon = \frac{1}{2} \). To induce a reference trajectory
which does not converge to zero and is non-Zeno, we design the periodic feedforward signal

$$u_{ff}(t) = \begin{cases} 
(1 - \epsilon^2)G, & \tau(t) < \frac{v}{G\epsilon^2}, \\
0, & \tau(t) \geq \frac{v}{G\epsilon^2},
\end{cases} \quad (27)$$

with parameter $v = 5$ and $\tau(t) := t \mod \frac{v}{G\epsilon^2}(1 + \frac{1}{\epsilon})$. For this input, the following reference trajectory satisfies Assumption 1:

$$r(t) = \begin{cases} 
\left[ \frac{v}{G\epsilon^2} \tau(t) - \frac{v^2}{G^2} \sigma(t)^2 \right], & \tau(t) < \frac{v}{G\epsilon^2} \\
\left[ \frac{v^2}{G^2} \sigma(t)^2 - \frac{v}{G\epsilon^2} \sigma(t) - \frac{v^2}{G^2} \right], & \tau(t) \geq \frac{v}{G\epsilon^2}.
\end{cases} \quad (28)$$

In Figure 2, the reference trajectory $r$ is shown.

Choosing $(k_p, k_d) = (0.2, 0.4)$ and $P = \begin{bmatrix} 1.25 & 1.25 \\ 1.25 & 3.75 \end{bmatrix}$, Theorem 2 ensures that the controller (12) asymptotically stabilises the reference trajectory $r$ in the sense of Definition 1, as illustrated in Figure 2 for a plant trajectory $x$ with initial condition $x(0,0) = (2 10)^T$. The Euclidean tracking error $|x - r|$ and the distance $d_F(r,x)$ are shown in Figure 3. As shown in this figure, the Euclidean tracking error $|x - r|$ shows the peaking phenomenon. In contrast, the tracking error measure $d_F$ converges asymptotically to zero. As shown in Figure 2, this corresponds to an intuitive notion of tracking: the impact times of the plant converge to those of the reference trajectory, and away from the impact times, after a transient period, the distance between the reference and plant trajectory becomes arbitrarily small.

## 6 Discussion

In this paper, tracking controllers are designed for mechanical systems with a unilateral position constraint and dissipative impact law, which, although a case of significant practical relevance, was up to now not studied in the literature. The controller design ensures that, despite the peaking phenomenon occurring in the Euclidean tracking error, the tracking error measure introduced in
this paper behaves in an asymptotically stable fashion, and an intuitively correct notion of tracking is achieved. For the design of a suitable controller, we employ a control Lyapunov function, that is based on distinguishing three different cases, which correspond to the situations where either the reference and plant states are close to each other, the reference trajectory jumped and the plant trajectory did not yet experience a jump, or the plant trajectory jumped and the reference trajectory did not. In the latter two cases, the control Lyapunov function is based on a re-scaled version of the reference or plant trajectory, respectively. Using this control Lyapunov function, a control law is designed that ensures accurate tracking.

Although the focus in this paper is on mechanical systems with dissipative impacts, we believe that the controller design, based on control Lyapunov functions using the distinction between the three mentioned cases, will enable tracking controller design procedures for a larger class of hybrid systems with state-triggered jumps, which is the subject of future research.

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A Existence of lower and upper bounds

In order to prove Theorem 2, we present the following technical lemma.

**Lemma 3.** Let the continuous functions $\beta_1, \beta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be such that
\[
\beta_1(q) = 0 \Leftrightarrow \beta_2(q) = 0.
\] (29)
In addition, assume that the set $\mathcal{A} := \{q \in \mathbb{R}^n \mid \beta_1(q) = 0\}$ is compact. If $\beta_1(q) \to \infty$ and $\beta_2(q) \to \infty$ when $\inf_{y \in \mathcal{A}} \|q - y\| \to \infty$, then there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that
\[
\alpha_1(\beta_1(q)) \leq \beta_2(q) \leq \alpha_2(\beta_1(q)).
\] (30)

**Proof.** Let $\mathcal{A} = \{q \in \mathbb{R}^n \mid \beta_1(q) = 0\}$ and let $d_{A}(q) = \inf_{y \in \mathcal{A}} \|q - y\|$. First, we will obtain an upper bound on $\beta_1$. Similar to the proof of Lemma 4.3 of [18], we define the function
\[
\phi(s) := \sup_{q \in \mathbb{R}^n, d_{A}(q) \leq s} \beta_1(q), \quad s \in [0, \infty),
\] (31)
and observe that $\phi(s)$ is continuous since $\mathcal{A}$ is compact, increasing (possibly not strictly increasing), $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$. Hence, defining the continuous, strictly increasing function $\tilde{\alpha}_2(s) := s + \phi(s)$, we observe that

$$
\beta_1(q) \leq \phi(d_\mathcal{A}(q)) \leq \tilde{\alpha}_2(d_\mathcal{A}(q)).
$$

(32)

By construction, $\tilde{\alpha}_2(s) \to \infty$ as $s \to \infty$, such that $\tilde{\alpha}_2 \in \mathcal{K}_\infty$.

To obtain a lower bound on $\beta_1$, let

$$
\psi(s) := \inf_{q \in \mathbb{R}^n, d_\mathcal{A}(q) \geq s} \beta_1(q), \ s \in [0, \infty),
$$

(33)

and observe that $\psi(s)$ is continuous since $\mathcal{A}$ is compact, increasing (possibly not strictly increasing), $\psi(0) = \psi(s) > 0$ for $s > 0$. Hence, one can select a continuous and strictly increasing function $\tilde{\alpha}_1(s)$ that satisfies $\frac{1}{2} \psi(s) \leq \tilde{\alpha}_1(s) \leq \psi(s)$ and obtain

$$
\beta_1(q) \geq \psi(d_\mathcal{A}(q)) \geq \tilde{\alpha}_1(d_\mathcal{A}(q)).
$$

(34)

By assumption, $\beta_1(q) \to \infty$ as $d_\mathcal{A}(q) \to \infty$, such that we conclude that $\tilde{\alpha}_1(s) \to \infty$ as $s \to \infty$, such that $\tilde{\alpha}_1 \in \mathcal{K}_\infty$. Hence, we obtain

$$
\tilde{\alpha}_1(d_\mathcal{A}(q)) \leq \beta_1(q) \leq \tilde{\alpha}_2(d_\mathcal{A}(q)),
$$

(35)

with $\mathcal{K}_\infty$ functions $\tilde{\alpha}_1, \tilde{\alpha}_2$. Using (29) and the definition of the set $\mathcal{A}$, we observe that $\mathcal{A} = \{ q \in \mathbb{R}^n \mid \beta_2(q) = 0 \}$. Hence, analogous arguments as given above imply that there exist $\mathcal{K}_\infty$-functions $\tilde{\alpha}_1, \tilde{\alpha}_2$ such that

$$
\tilde{\alpha}_1(d_\mathcal{A}(q)) \leq \beta_2(q) \leq \tilde{\alpha}_2(d_\mathcal{A}(q)).
$$

(36)

Since $\mathcal{K}_\infty$ functions are invertible, we can define the functions $\alpha_2(s) := \tilde{\alpha}_2(\tilde{\alpha}_1^{-1}(s))$ and $\alpha_1(s) := \tilde{\alpha}_1(\alpha_2^{-1}(s))$. For these functions, (30) is satisfied, which follows directly from the combination of (35) and (36). Since $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, the result is obtained.

References


