Asynchronous Games 4

A fully complete model of propositional linear logic

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An epic in 50 slides

Twenty-four seconds each
A crash course on Mazurkiewicz traces

The 2-dimensional geometry of concurrency
Trace semantics

Interpret any process

\[ \pi = a | b \]

by the sequences of actions it performs in the course of time:

\[ ab \quad \text{and} \quad ba \]
The synchronization tree of a process

\[ a \mid b \]

interpreted as
The synchronization tree of a process

\[ ab + ba \]

interpreted as

\[ \begin{array}{c}
\bullet \\
\uparrow \\
\bullet \\
\downarrow \\
\bullet \\
\end{array} \begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\uparrow \\
\bullet \\
\end{array} \]

\[ \begin{array}{c}
b \\
\downarrow \\
a \\
\uparrow \\
b \\
\end{array} \begin{array}{c}
a \\
\uparrow \\
b \\
\downarrow \\
a \\
\end{array} \]
This is a problem

Trace semantics cannot see the difference between

\[ a \mid b \quad \text{no interference} \]

and

\[ ab + ba \quad \text{interference} \]
Idea: replace synchronization trees...

\[ a \mid b \]

interpreted as

\[ \begin{array}{c}
  b \\
  a \\
\end{array} \rightarrow \\
\begin{array}{c}
  a \\
  b \\
\end{array} \]
... by Mazurkiewicz traces

\[ a \mid b \]

interpreted as

\[ \sim \]
True concurrency = homotopy

Think of this permutation as a 2-dimensional tile.
Interference = holes

\[ ab + ba \]

interpreted as

\[
\begin{array}{ccc}
  & b & \\
 a & & a \\
 & b & \\
\end{array}
\]
Asynchronous games

Games played on Mazurkiewicz traces
Game semantics

The boolean game $B$

Player in red
Opponent in blue
Traditional game semantics

The boolean game $B$
Traditional game semantics: an interleaving semantics

The tensor product of two boolean games $B_1$ and $B_2$
Bend the branches!
Tile the diagram!
Tag the positions!
A 2-dimensional space of interaction
Asynchronous games

A 2-dimensional graph equipped with tiles of the shape

in which:

- every edge is polarized Player or Opponent
- an initial position $\ast$ is distinguished
Sequential play

A **sequential play** is defined as an **alternated** path

\[
* \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \xrightarrow{m_3} \cdots x_{k-1} \xrightarrow{m_k} x_k
\]

starting by an **Opponent move**.
Strategies

A strategy is a set of **sequential plays of even length**, such that:

- $\sigma$ contains the empty play,
- $\sigma$ is closed under even-length prefix
  \[
  s \cdot m \cdot n \in \sigma \Rightarrow s \in \sigma
  \]
- $\sigma$ is deterministic
  \[
  s \cdot m \cdot n_1 \in \sigma \quad \text{and} \quad s \cdot m \cdot n_2 \in \sigma \Rightarrow n_1 = n_2
  \]

A strategy plays according to the current play.
Innocence 1994

Martin Hyland, Luke Ong, Hanno Nickau

An interactive characterization of $\lambda$-terms
Innocence: strategies with partial information

The Player view $\llbracket s \rrbracket$ : what the Player can remember of the play $s$.

An innocent strategy plays according to the current Player view.
Innocence 2004

From amnesia to positionality
Consistency (backward)

\[ \sigma \ni s_2 \quad \Rightarrow \quad s_2 \in \sigma \]

\[ \sigma \ni n_2 \quad \sim \quad m_2 \quad \sim \quad n_1 \quad \ni m_1 \quad \ni n_2 \quad \sim \quad m_1 \quad \sim \quad n_1 \quad \ni m_2 \quad \ni s_1 \]

\[ \ni s_1 \]
Consistency (forward)
Innocent strategies are positional

**Theorem [Concur 2004]** Every innocent strategy $\sigma$ is positional.

**Cor.** An innocent strategy $\sigma$ is characterized by its set of positions $\sigma^\bullet$. 
An illustration: the strategy (true $\otimes$ false)
An illustration: the strategy (true $\otimes$ false)
An illustration: the strategy (true $\otimes$ false)
An illustration: the strategy (true ⊗ false)
A quite extraordinary discovery

The concurrent nature of the λ-calculus
A sketched history of concurrency

Automata theory

Petri Nets

Synchronization trees       Mazurkiewicz traces

CCS

π-calculus     λ-calculus
A quite different topography revealed

Automata theory

Petri Nets

Synchronization trees

CCS

\(\pi\)-calculus

Mazurkiewicz traces

\(\lambda\)-calculus
What about linear logic?

Automata theory → Petri Nets

Synchronization trees → CCS → π-calculus

Mazurkiewicz traces → λ-calculus → linear logic
A serious difficulty arises
There exists no game model of linear logic!

But not that far!

We have lots of game models of intuitionistic linear logic!

In all of these game models, there exists a retraction

\[(A \rightarrow \bot) \rightarrow \bot \xrightarrow{\theta_A} A \xrightarrow{\partial_A} (A \rightarrow \bot) \rightarrow \bot\]

A phenomenon of control categories — Peter Selinger 2001
The game \((A \circ \bot) \circ \bot\)

The game \((A \circ \bot) \circ \bot\) is obtained by lifting the game \(A\) twice:

\[
\begin{array}{c}
  x \quad y \quad z \\
  * \quad \uparrow \\
  \quad \downarrow \\
  A
\end{array}
\quad
\begin{array}{c}
  x \quad y \quad z \\
  \downarrow \\
  n \quad m \\
  \quad * \\
  (A \circ \bot) \circ \bot
\end{array}
\]
$(A \rightarrow \bot) \rightarrow \bot \rightarrow (A \rightarrow \bot) \rightarrow \bot$
The identity strategy $id(A \circ \bot) \circ \bot$
The idempotent strategy $\partial_A \circ \rho_A$
Diagnosis

The two strategies $\partial_A \circ \theta_A$ and $id_{(A \rightarrow \bot) \rightarrow \bot}$ are only equal modulo "homotopy".

Idea: force the equality

$$\partial_A \circ \theta_A = id_{(A \rightarrow \bot) \rightarrow \bot}$$

in the category of games.

Leads to a classical model of linear logic
Solution

○ Assign a payoff \( \kappa(x) \in \mathbb{Z} \) to every position \( x \) of the game.

○ Consider only the \textbf{winning} strategies \( \sigma \):

\[
\forall x, \quad x \in \sigma^* \quad \Rightarrow \quad \kappa(x) \geq 0
\]

○ Call \textbf{external} a position \( x \) with null payoff : \( \kappa(x) = 0 \)

○ Identify two strategies when they play the same \textbf{external positions}.

\[
\sigma \simeq \tau
\]
Slogan

A strategy **realizes** its set of external positions.

The two strategies

\[ \text{id}(A \rightarrow \bot) \rightarrow \bot \quad \text{and} \quad \partial_A \circ \rho_A \]

are identified in this way.

The **left** and **right** implementations of **AND** are also identified.
A remark for the insider

This transforms the **non commutative** continuation monad

\[ T : A \mapsto (A \circ \bot) \circ \bot \]

into a commutative one:

\[ TA \otimes TB \rightarrow T(A \otimes TB) \rightarrow T^2(A \otimes B) \]

\[ T(TA \otimes B) \rightarrow T^2(A \otimes B) \rightarrow T(A \otimes B) \]

Property: the kleisli category is then \(*\)-autonomous.

A very nice observation by Hasegawa Masahito
Well-bracketing reduces to a winning condition

\[(\mathbb{B} \rightarrow \mathbb{B}) \rightarrow \mathbb{B} \quad \kappa(x)\]

\[q \quad -1\]
\[q \quad +1\]
\[q \quad -1\]
\[\text{true} \quad -1\]

\[(+1 \rightarrow -1) \rightarrow 0 \quad -1\]
Full completeness

- Extend payoffs to paths in order to separate the two formulas:

\[(A & B) \otimes \top \quad \text{and} \quad (A \otimes \top) \& (B \otimes \top)\]

- Extend payoffs to walks in order to reject the pseudo-proof of the sequent:

\[\vdash 1, 1, (\bot \& (\bot \otimes \bot)) \otimes \top.\]

Needs to explore several additive slices.

- Full completeness by directed proof search.
Main difficulty: garbage collect!
Exponentials

The exponential factors as

$$! A = \bigotimes_{n \in \mathbb{N}} \downarrow A$$

Uniformity is described thanks to a left and right group action on indices.

Samson Abramsky, Radha Jagadeesan, Pasquale Malacaria 1994
Work in progress

1. Our model is not the **free construction**.

2. Compare to another very nice formulation of innocence.
   (joint work with Martin Hyland and Russ Harmer)

3. Relax **alternation** and compare innocence to L-nets
   (joint work with Claudia Faggian and Samuel Mimram)

4. Add **holes** to the geometry and study languages with states
   (joint work with Nicolas Tabareau)

5. Study **first-order** and **second-order** linear logic.
Thank you!

I am currently writing the full paper.
Announcement

I organize with Anca Muscholl (LIAFA) the next session of the legendary

Spring School in Theoretical Computer Science

Monday 29 May — Friday 2 June 2006
Ile de Ré, France

Games : Semantics and Verification

whose first School was organized by Maurice Nivat in 1973. The School
meets every year since then, in a different place, and on a different subject.

PhD students, postdoctoral students, young researchers wishing to
learn more about games and their applications to Semantics and Verifica-
tion are welcome to the Spring School.
Announcement

The following people have already accepted to teach at the School:

- Samson Abramsky (Oxford, UK)
- Martin Hyland (Cambridge, UK)
- Luke Ong (Oxford, UK)
- Jacques Duparc (Lausanne, Switzerland)
- Eric Grädel (Aachen, Germany)
- Igor Walukiewicz (Bordeaux, France)
- Wieslaw Zielonka (Paris, France)
- Tom Henzinger (Lausanne, Switzerland) (to be confirmed)
- Sylvain Sorin (Ecole Polytechnique, Paris) (to be confirmed)

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The two strategies realize the same \textit{external positions} of the game.