

Restrained Weakly Connected Domination in the Join and Corona of Graphs

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Abstract

In this paper, we explore the concept of restrained weakly connected domination in graphs. In particular, we characterized the restrained weakly connected dominating sets in the join and corona of graphs and as a consequence, their restrained weakly connected domination numbers are obtained. A connected graph is constructed with a given order, weakly connected domination number, and restrained weakly connected domination number.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = \bigcup_{v \in X} N[v]$.

The subgraph $\langle C \rangle$ of G *induced* by C is the graph having vertex-set C and

whose edge set consists of those edges of G incident with two elements of C . A graph is called *connected* if every two vertices are joined by a path; otherwise, it is *disconnected*.

A S is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G , denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G . A dominating set $C \subseteq V(G)$ is called a *weakly connected dominating set* of G if the subgraph $\langle C \rangle_w = (N_G[C], E_w)$ weakly induced by C is connected, where E_w is the set of all edges with at least one vertex in C . The *weakly connected domination number* of G , denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set of G .

A dominating set S is called a *restrained dominating set* of G if for every $u \in V(G) \setminus S$, there exists $w \in V(G) \setminus S$ such that $uw \in E(G)$. The *restrained domination number* of G , denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G . A set S is called a *restrained weakly connected dominating set* of G if S is a weakly connected dominating set of G and for every $u \in V(G) \setminus S$, there exists $w \in V(G) \setminus S$ such that $uw \in E(G)$. The *restrained weakly connected domination number* of G , denoted by $\gamma_{sw}(G)$, is the smallest cardinality of a restrained weakly connected dominating set of G .

The concept of weakly connected domination is discussed in [2] [3, and [4]. Another domination parameter is the restrained domination which was discussed in [1] and [5]. A combination of these two concepts give rise to a new variant of domination called restrained weakly connected domination.

Remark 1.1 *Let G be a graph of order n . Then $1 \leq \gamma(G) \leq \gamma_w(G) \leq \gamma_{rw}(G)$.*

Theorem 1.2 *Let a , b , and n be positive integers such that $2 \leq a \leq b < n$. Then there exists a connected graph G such that $|V(G)| = n$, $\gamma_w(G) = a$, and $\gamma_{rw}(G) = b$.*

Proof: Consider the path $P_{2a-1} = [u_1, u_2, \dots, u_{2a-1}]$. Let G be a graph obtained from P_{2a-1} by adding the edges $v_i u_{2i}$ for $i = 1, 2, \dots, a-1$; adding the path $[u_1, v_1, u_3, v_2, \dots, v_{a-1}, u_{2a-1}]$; adding the edges $u_1 w_j$, for $j = 1, 2, \dots, b-a$; and adding the vertices $z_1, z_2, \dots, z_{n-2a-b+2}$ and forming the complete graph $K_{n-2a-b+1}$, where $V(K_{n-2a-b+1}) = \{u_{2a-1}, z_1, z_2, \dots, z_{n-2a-b+2}\}$ (see Figure 1). Then $\{u_1, u_3, \dots, u_{2a-1}\}$ is a weakly connected dominating set of G and $\{u_1, u_3, \dots, u_{2a-1}\} \cup \{w_1, w_2, \dots, w_{b-a}\}$ is a restrained weakly connected dominating set of G . Hence, $\gamma_w(G) = a$, $\gamma_{rw}(G) = b$ and $|V(G)| = (2a-1) + (a-1) + (b-a) + (n-2a-b+2) = n$. \square

Corollary 1.3 *The difference $\gamma_{rw} - \gamma_w$ can be made arbitrarily large.*

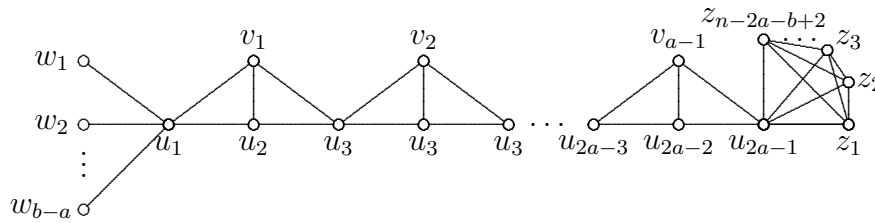


Figure 1: A graph G with $\gamma_{wcon}(G) \leq \gamma_{wcon}^r(G)$

2 Join of Graphs

The *join* of two graphs G and H , denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2.1 *Let G be a graph of order $n \geq 2$. Then $\gamma_{rw}(K_1 + G) = 1$ if and only if G has no isolated vertex.*

Proof: Suppose $\gamma_{rw}(K_1 + G) = 1$. Let $S = \{v\}$ be a restrained weakly connected dominating set of $K_1 + G$. Suppose $v \in V(K_1)$. Since S is a restrained dominating set, for each $x \in V(K_1 + G) \setminus S = V(G)$, there exists $y \in V(G)$ such that $xy \in E(G)$. This implies that G has no isolated vertex. Suppose $v \in V(G)$. Since S is a dominating set, for each $u \in V(G) \setminus \{v\}$, $uv \in E(G)$. Hence, G has no isolated vertex.

Conversely, suppose G has no isolated vertex. Let $K_1 = \langle\langle v \rangle\rangle$ and set $S = \{v\}$. Then S is a weakly connected dominating set of $K_1 + G$. Let $u \in V(K_1 + G) \setminus S$. Then $u \in V(G)$. Thus, $u \in V(\langle\langle C \rangle\rangle) \subseteq V(K_1 + G) \setminus S$ for some nontrivial component C of G . Hence, there exists $w \in V(\langle\langle C \rangle\rangle) \subseteq V(K_1 + G) \setminus S$ such that $uw \in E(G) = V(K_1 + G) \setminus S$. Therefore, S is a restrained weakly connected dominating set of $K_1 + G$. □

Theorem 2.2 *Let G be a graph of order greater than or equal to 3. Then $\gamma_{rw}(K_1 + G) = 2$ if and only if one of the following holds:*

- (i) $\gamma(G) = 2$.
- (ii) G has one isolated vertex.

Proof: Suppose $\gamma_{rw}(K_1 + G) = 2$. Let $S = \{x, y\}$ be a restrained weakly connected dominating set of $K_1 + G$. If $S \subseteq V(G)$, then S is a dominating set of $K_1 + G$. Hence, $\gamma(G) = 2$. Suppose $x \in V(K_1)$ and $y \in V(G)$. Suppose further that y is not an isolated vertex. Then y is contained in some component of G . By Theorem 2.1, $\gamma_{rw}(K_1 + G) = 1$, a contradiction. Therefore, G has one isolated vertex.

For the converse, suppose $\gamma(G) = 2$. Let $S = \{a, b\}$ be a dominating set

of G and let $K_1 = \langle \{v\} \rangle$. Clearly, S is a weakly connected dominating set of $K_1 + G$. Since $vx \in E(K_1 + G)$ for all $x \in V(G) \setminus S$, it follows that S is a restrained weakly connected dominating set of $K_1 + G$. Thus, $\gamma_{rw}(K_1 + G) = 2$. Next, suppose G has one isolated vertex, say u . Set $S = \{u, v\}$. Clearly, S is a weakly connected dominating set of $K_1 + G$. Let $x \in V(K_1 + G) \setminus S$. Then $x \in V(G) \setminus \{u\}$. Thus, $x \in V(\langle C \rangle)$ for some nontrivial component C of G . Hence, there exists $y \in V(\langle C \rangle)$ such that $xy \in E(K_1 + G)$. This means that S is a restrained weakly connected dominating set of $K_1 + G$. Hence, $\gamma_{rw}(K_1 + G) = 2$. \square

Theorem 2.3 *Let G be a connected graph of order $n \geq 3$ and let $K_1 = \langle \{v\} \rangle$. Then $S \subseteq V(K_1 + G)$ is a restrained weakly connected dominating set of $K_1 + G$ if and only if one of the following holds:*

- (i) $|S| \leq n - 1$ and S is a dominating set of G .
- (ii) $v \in S$ and $\langle V(G) \setminus (S \setminus \{v\}) \rangle$ is nontrivial connected subgraph of G .

Proof: Suppose $S \subseteq V(K_1 + G)$ is a restrained weakly connected dominating set of $K_1 + G$. Consider the following cases:

Case 1. $v \notin S$.

Then $S \subseteq V(G)$. Thus, S is a restrained weakly connected dominating set of G . Hence, $|S| \leq n - 1$, and S is a dominating set of G .

Case 2 $v \in S$.

Then $S \setminus \{v\} \subseteq V(G)$. Since S is a restrained weakly connected dominating set of $K_1 + G$, it follows that $S \setminus \{v\}$ is a restrained dominating set of G . This implies that for each $w \in V(G) \setminus (S \setminus \{v\})$, there exists $u \in V(G) \setminus (S \setminus \{v\})$ such that $uw \in E(G)$. Thus, $\langle V(G) \setminus (S \setminus \{v\}) \rangle$ is a nontrivial connected subgraph of G .

For the converse, suppose first that $|S| \leq n - 1$ and S is dominating set of G . Then S is weakly connected dominating set of $K_1 + G$. Let $u \in V(K_1 + G) \setminus S$. If $u \in V(G)$, then $uv \in E(K_1 + G)$. Suppose $u = v$. Since $|S| \leq n - 1$, there exists $w \in V(G) \setminus S$ such that $uw \in E(K_1 + G)$. Therefore, S is a restrained weakly connected dominating set of $K_1 + G$. Next, suppose that $v \in S$ and $\langle V(G) \setminus (S \setminus \{v\}) \rangle$ is a nontrivial connected subgraph of G . Then S is weakly connected dominating set of $K_1 + G$. Let $x \in V(G) \setminus (S \setminus \{v\})$. Then there exists $y \in V(G) \setminus (S \setminus \{v\})$ such that $xy \in E(K_1 + G)$. Hence, S is a restrained weakly connected dominating set of $K_1 + G$. \square

Theorem 2.4 *Let G and H be nontrivial and nonempty graphs. Then $S \subseteq V(G + H)$ is a restrained weakly connected dominating set of $G + H$ if and only if one of the following holds:*

- (i) $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ with $|S| \neq |V(G + H)| - 1$.
- (ii) S is a dominating set of G or S is a dominating set of H .

Proof: Let $S \subseteq V(G + H)$ be a restrained weakly connected dominating set of $G + H$. Suppose (i) does not hold. Then either $S \subseteq V(G)$ or $S \subseteq V(H)$. Thus, S is a dominating set of G or S is a dominating set of H .

Conversely, suppose first that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ with $|S| \neq |V(G)| - 1$. By the definition of $G + H$, it follows that S is a restrained weakly connected dominating set of $G + H$. Next, suppose that S is a dominating set of G . Then S is a weakly connected dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. If $x \in V(G) \setminus S$, then $xy \in E(G + H)$ for all $y \in V(H)$. Suppose $x \in V(H)$. Since H is nontrivial, there exists $z \in V(H)$ such that $xz \in E(G + H)$. Hence, S is a restrained weakly connected dominating set of $G + H$. Similarly, if S is a dominating set of H , then S is a restrained weakly connected dominating set of $G + H$. \square

Corollary 2.5 *Let G and H be nontrivial and nonempty graphs. Then*

$$\gamma_{rw}(G + H) = \begin{cases} 1, & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

Proof: Assume that $\gamma(G) = 1$. Let $S = \{u\}$ be a minimum dominating set of G . Then S is a restrained weakly connected dominating set of $G + H$. Hence, $\gamma_{rw}(G + H) = 1$. Suppose $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. Then $\gamma(G + H) \neq 1$. Pick $a \in V(G)$ and $b \in V(H)$. Then $ab \in E(G + H)$. Thus, $C = \{a, b\}$ is a secure weakly connected dominating set of $G + H$. Therefore, $\gamma_{rw}(G + H) = 2$. \square

3 Corona of Graphs

Let G and H be graphs of order m and n , respectively. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and m copies of H , and then joining the i th vertex of G to every vertex of the i th copy of H . For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v . Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$.

The next result characterizes the restrained weakly connected dominating set of $G \circ H$.

Theorem 3.1 *Let G and H be connected graphs of order $m \geq 2$ and $n \geq 3$, respectively. Then $S \subseteq V(G \circ H)$ is a restrained weakly connected dominating*

set of $G \circ H$ if and only if $S = S_1 \cup \left(\bigcup_{v \in S_1} S^v \right) \cup \left(\bigcup_{u \in V(G) \setminus S_1} S^u \right)$, where S_1 is a

weakly connected dominating set of G , $S^v \subseteq V(H^v)$ such that $\langle V(H^v) \setminus S^v \rangle$ is a nontrivial connected graph for each $v \in S_1$, and $S^u \subseteq V(H^u)$ is a dominating set of H^u such that $|V(H^u) \setminus S^u| \geq 1$ for each $u \in V(G) \setminus S_1$.

Proof: Suppose $S \subseteq V(G \circ H)$ is a restrained weakly connected dominating set of $G \circ H$. Let $S_1 = S \cap V(G)$. Then S_1 is a weakly connected dominating set of G . Also, S is a restrained weakly connected dominating set of $v + H^v$ for $v \in V(G)$. By Theorem 2.3, either $S \cap V(H^v)$ is a dominating set of H^v with $|V(H^v) \setminus (S \cap V(H^v))| \geq 1$ or, $v \in S \cap V(v + H^v)$ and $\langle V(H^v) \setminus (S \cap V(H^v)) \rangle$ is nontrivial and connected. Hence, $S = S_1 \cup \left(\bigcup_{v \in S_1} S^v \right) \cup \left(\bigcup_{u \in V(G) \setminus S_1} S^u \right)$.

The converse follows from Theorem 2.3. \square

Corollary 3.2 *Let G and H be connected graphs of order $m \geq 2$ and $n \geq 3$, respectively. Then $\gamma_{rw}(G \circ H) = m$.*

Proof: Let $S = V(G)$. Then S is a restrained weakly connected dominating set of $G \circ H$. Thus, $\gamma_{rw}(G \circ H) \leq |S| = m$. Next, suppose S^* is a minimum restrained weakly connected dominating set of $G \circ H$. Then $|S^* \cap V(v + H^v)| \geq 1$ for all $v \in V(G)$. Hence, $\gamma_{rw}(G \circ H) = |S^*| \geq m$. Therefore, $\gamma_{rw}(G \circ H) = m$. \square

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